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## The Underestimated Virtues of the Two-sector AK Model

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# The Underestimated Virtues of the Two-sector AK Model\*

Gabriel J. Felbermayr and Omar Licandro

## Abstract

This paper analyzes some unnoticed predictions of the two-sector AK model in line with the recent literature on embodied technical change. Firstly, by confining constant returns to capital to the investment sector, the AK model generates endogenously the secular downward trend of the relative price of equipment investment and the rising real investment rate observed in US NIPA data. Secondly, Jones' (1995) claim that the AK model fails to reconcile the empirical facts of trending real investment rates and stationary output growth vanishes in the two-sector version. Thirdly, consistent with the evidence from cross-country studies, the model predicts a negative relation between GDP per capita and the relative price of equipment. Hence, in spite of its overly simplistic structure, the two-sector AK model provides important intuition on the implications of a trending relative price of equipment investment in endogenous growth environments.

**KEYWORDS:** AK model, embodiment, endogenous growth, obsolescence, productivity slow-down

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# 1 Introduction

The standard workhorses of modern growth theory typically build on the famous Kaldor (1957) stylized facts, which report, among other things, the stationarity of aggregate ratios such as the investment to output ratio or the capital to output ratio. Recent evidence from the US National Income and Product Accounts (NIPA) suggests that these ratios are not constant over time. Based on data for the post World War II period, Whelan (2003) documents the following facts:<sup>1</sup>

- (i) The price of equipment investment relative to the price of consumer nondurables and services has been declining permanently.
- (ii) Nominal series of consumption and investment share a common stochastic trend with nominal output so that the consumption and investment shares in nominal output are stationary.
- (iii) The ratio of real equipment investment to real output is non-stationary. Indeed, the growth rate of real equipment investment has been larger than that of real nondurable consumption, reconciling facts (i), (ii) and (iii).

Recent research addresses the inconsistency of facts (i) and (iii) with standard models of exogenous and endogenous growth, in which relative prices are not allowed to change along the balanced growth path. Clearly, to reconcile the new evidence the appropriate framework needs more than just one sector and has to carefully model the sectoral incidence of technological change. Greenwood et al. (1997), henceforth abbreviated GHK, based on the seminal contribution of Solow (1960), were the first to make an attempt in this direction. Their model consists of a consumption goods sector and an investment goods sector. Sectoral production functions are Cobb-Douglas in capital and labor and identical up to a Hicks-neutral factor. In order to reflect the stylized fact (i) above, the rate of technical progress is larger in the investment sector. Technical progress in the consumption sector is disembodied. Advances in the investment sector affect the consumption sector to the extent that firms acquire new and more efficient capital goods: advances in technology are embodied. In this way, GHK are able to generate a permanent decline in the

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<sup>1</sup>See also Greenwood, Hercowitz and Krusell (1997), p. 342. Similar trends are observed in most OECD countries (see OECD, 2000a).

relative price of investment and a rising ratio of real equipment investment to real output, keeping nominal shares constant, as required by the stylized facts cited above. However, GHK take growth to be exogenous.

This paper studies the two-sector AK model originally proposed by Rebelo (1991), section II, where the consumption sector features decreasing returns to capital whilst the investment sector is described by an AK technology. The two-sector AK model has not achieved the same reputation as its one-sector brother, being almost ignored by today's major growth textbooks.<sup>2</sup> It is interesting to notice that section II in Rebelo (1991) is tuned towards demonstrating that even models with convex production possibility sets can generate endogenous growth if at least the investment sector features constant returns to capital. The main contribution of this paper is to emphasize that the two-sector AK model has some interesting features that have hitherto not been fully recognized. In particular, it is the simplest possible endogenous growth model compatible with the new growth evidence cited above. In that sense, this paper complements recent research on endogenous embodied technical change in R&D based models of endogenous growth, see Boucekkine et al. (2005[5]), Hsieh (2001) or Krusell (1998), and the model by Boucekkine et al. (2003) where technical change stems from learning-by-doing.

Harrison (2003) provides econometric evidence on plant-level and sector-level returns to scale. She cannot reject constant returns to scale at the plant-level in both the consumption and the investment sector, but finds robust evidence for positive externalities at the investment sector-level. This evidence supports the more informal view that spillovers of many sorts are particularly important in investment goods industries such as the information and communication technologies, car and aeronautic, or industrial equipment (see the surveys by OECD, 2000a, and OECD, 2000b). Thus, it is likely that social returns to capital are larger in the investment sector than in the consumption sector. Taking these observations to their extreme, the two-sector AK model assumes constant returns to capital in the investment sector, but decreasing returns in the consumption sector.

Consistent with the evidence in points (i) to (iii) above, the two-sector AK model predicts that the relative price of investment trends downward, the saving rate is constant and real investment growth outpaces consumption growth. The underlying mechanism is different from the one in GHK. In their paper, the price trend depends on exogenous sectoral differences in the rate

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<sup>2</sup>See, for example, Aghion&Howitt (1998), Barro&Sala-i-Martin (1995) and Jones (2001).

of technical progress, whereas the two-sector AK model relates the movement in relative prices to the asymmetric sectoral impact of capital accumulation.

The two-sector AK model has other interesting empirical implications. Firstly, Jones (1995) criticizes the one-sector AK model, since it fails to reconcile the empirical facts of trending real investment rates and stationary output growth. In the two-sector version, however, the growth rate is constant but the real investment rate is growing. Restricting constant returns to capital to the investment sector, improves the empirical relevance of AK-type endogenous growth models and fends off the Jones critique. Secondly, the two-sector AK model has predictions consistent with the finding in cross-country studies of a negative correlation between GDP per capita and the relative price of investment goods (see Restuccia and Urrutia, 2001) for a recent survey of this literature. Hence, while the two-sector AK model is incompatible with several important facts on growth, such as the existence of transitional dynamics, it provides us with important intuition about how endogenous growth can generate a balanced growth path consistent with a trending relative price of equipment investment.

In contrast to its one-sector version, in the two-sector AK model, the user cost of capital is augmented by capital losses, due to a trending relative price of investment. Capital losses are interpreted as obsolescence costs in models of embodied technical change. In the two-sector AK model, the larger the decline rate of the relative price of investment, the larger capital losses are. This lowers the real interest rate perceived by consumers and depresses consumption growth. However, whether the lower interest rate encourages or discourages capital accumulation depends on how the income effect associated with a change in the interest rate relates to the substitution effect, that is, whether the elasticity of intertemporal substitution is smaller or greater than unity. Finally, capital losses reduce the growth rate of real output if the elasticity of intertemporal substitution is larger than the saving rate.

Exploiting the isomorphism of the two-sector AK framework with the social planner version of an economy featuring learning-by-doing in both the investment and the consumption sector, the two-sector AK model can be interpreted as a model of *embodied technical change*. With this interpretation in mind, the arrival of a new general purpose technology with a higher relative efficiency of learning in the investment sector boosts *obsolescence* costs and may generate a simultaneous increase in the decline rate of the relative price of investment and a reduction in the growth rate of aggregate output, see also Boucekkine et al. (2003). In the two-sector AK model, this exercise

is equivalent to a reduction of the output elasticity in the consumption sector.

The remainder of the paper is organized as follows. Section 2 sets out the analytical framework and derives the main propositions; section 3 provides a discussion of the results and compares them with the existing literature; finally, section 4 concludes.

## 2 The Model

In this section, we analyze the two-sector, closed economy version of the AK model introduced by Rebelo (1991). The labor force is constant, and all quantities are in per capita terms. In the consumption sector, capital is combined with labor in a constant returns to scale production function. As in the models of Boucekkine et al. (2003, 2005[5]), Hsieh (2001) and Krusell (1998), the only source of endogenous growth lies in the investment sector. It is modelled in the simplest possible way by assuming that the technology features constant returns and capital is the only factor of production.

For ease of exposition, we break down the general equilibrium analysis into two parts. Firstly, we characterize the behavior of final output firms, who rent capital to produce consumption or investment goods. Finally, the picture is completed by the optimal savings decision of the representative household.

**Final output producers and the optimal allocation of capital.** The capital stock per capita  $k_t$  is perfectly mobile across sectors and can be employed either in the investment or the consumption sector. Using superscripts  $c$  and  $i$  to denote the respective sectors, full employment implies  $k_t = k_t^i + k_t^c$ . In the investment sector, per capita output is given by an AK technology

$$i_t = Ak_t^i. \quad (1)$$

In the consumption sector, capital and labor are combined following a Cobb-Douglas production function so that

$$c_t = (k_t^c)^\alpha, \quad (2)$$

with  $\alpha \in (0, 1)$  denoting the output elasticity of capital. Under perfect competition, firms in each sector rent capital up to the point where the value of its marginal product equals the rental rate  $R_t$ . Choosing the consumption good as the numéraire and writing  $p_t$  for the relative price of investment goods, the allocation of capital is determined by the condition

$$R_t = p_t A = \alpha (k_t^c)^{\alpha-1}. \quad (3)$$

Writing  $g_{kt}$  for the growth rate of  $k_t^c$ , the above condition can be written, after time differentiation, as

$$\frac{\dot{p}_t}{p_t} = (\alpha - 1) g_{kt}. \quad (4)$$

For condition (3) to hold at every point in time, the relative price  $p_t$  has to decrease at the rate at which the marginal productivity of capital in the consumption sector falls.

The rental rate of capital follows the standard condition

$$R_t = p_t \left( r_t + \delta - \frac{\dot{p}_t}{p_t} \right), \quad (5)$$

where  $\delta \in [0, 1]$  is the rate of physical depreciation, and  $r_t$  is the interest rate in terms of consumption goods. This last equation is central for the understanding of the mechanics of the model. In contrast to models where the relative price of investment goods is constant over time, in the two-sector model the user cost of capital is augmented by capital losses, as evidenced by the term  $\dot{p}_t/p_t$ .<sup>3</sup>

**Household behavior.** The representative household solves

$$\max \int_0^\infty \frac{c_t^{1-\sigma}}{1-\sigma} e^{-\rho t} dt \quad (6)$$

$$\text{s.t. } \dot{a}_t = r_t a_t + w_t - c_t, \quad (7)$$

where  $a_t$  is per capita financial wealth, with  $a_0 > 0$  given. As usual,  $\sigma$  (with  $\sigma > 0$ ,  $\sigma = 1$  corresponds to log utility) is the inverse of the intertemporal elasticity of substitution, and  $\rho > 0$  denotes the subjective discount rate of the infinitely lived representative household. Application of the maximum principle delivers the familiar Euler equation

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\sigma} (r_t - \rho) \quad (8)$$

and the transversality condition  $\lim_{t \rightarrow \infty} a_t \mu_t e^{-\rho t} = 0$ . The costate variable  $\mu_t$  is equal to  $c_t^{-\sigma}$  and denotes the marginal utility of consumption.

**Equilibrium.** In general equilibrium, financial wealth has to be equal to the value of capital, i.e.  $a_t = p_t k_t$ . The evolution of per capita consumption

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<sup>3</sup>See Jorgenson (1963) for the original formulation of the user cost of capital theory.  
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and the amount of capital invested in the consumption sector can be found by substituting (3) in the rental rate (5). This delivers the equilibrium interest rate

$$r_t = A - \delta + \frac{\dot{p}_t}{p_t}. \quad (9)$$

By substitution in (8), the Euler equation becomes

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\sigma} \left( A - \delta + \frac{\dot{p}_t}{p_t} - \rho \right). \quad (10)$$

Since the relative price of investment goods falls over time, capital losses appear in the rental rate of capital. The larger this term, the lower the interest rate and the flatter the desired consumption profile of households. Substituting (4) into (9) and given that technology in the consumption sector implies  $g_c = \alpha g_k$ ,  $g_c \equiv \frac{\dot{c}_t}{c_t}$ , the growth rates of capital and consumption are

$$g_k = \frac{1}{\omega} (A - \delta - \rho), \quad (11)$$

$$g_c = \frac{\alpha}{\omega} (A - \delta - \rho), \quad (12)$$

where  $\omega = 1 - \alpha(1 - \sigma) > 0$ . Both,  $g_k$  and  $g_c$  are constant from  $t = 0$  on. Moreover,  $g_c = \alpha g_k$ . Since  $\alpha \in ]0, 1[$ , consumption grows at a slower pace than investment and capital. Substituting  $a_t = k_t p_t$ , the transversality condition associated to the household's problem becomes

$$\lim_{t \rightarrow \infty} \lambda_t k_t e^{-\rho t} = 0, \quad (13)$$

where the shadow value of capital  $\lambda_t \equiv p_t c_t^{-\sigma}$ . From (10),  $\dot{\lambda}_t / \lambda_t = -(A - \delta - \rho)$ .

Notice that from equation (11),  $1/\omega$  is the intertemporal elasticity of substitution associated to  $k^c$ . Given our assumptions on technology, we refer to  $1/\omega$  as to the *elasticity of intertemporal substitution in foregone investment*. It measures the degree of substitution between using capital to produce consumption goods today and producing investment goods today to produce consumption goods in the future.

Equations (11), (12) and (13) are the key equations of the two-sector AK model.

**Assumption 1.** *Let the following parameter restriction hold:*

$$A - \delta > \rho > \alpha(1 - \sigma)(A - \delta).$$

The first inequality in Assumption 1 is required for the growth rate of consumption, as given by equation (12), to be strictly positive. The second inequality ensures that the utility representation in equation (6) remains bounded in equilibrium.

**Proposition 1.** *Under Assumption 1, for all  $t \geq 0$ , the growth rate of capital  $g_k$  is constant from  $t = 0$  onwards so that  $k_t = k_0 e^{g_k t}$ . Moreover, a closed form for the consumption function exists and can be written as*

$$c_t = \left(1 - \frac{\delta + g_k}{A}\right)^\alpha k_t^\alpha. \quad (14)$$

**Proof.** See the appendix. ■

In line with the one-sector AK model, the economy is on its balanced growth path from  $t = 0$ ; i.e., there are no transitional dynamics.

As stated in the Introduction, the post world war II U.S. growth experience features three important secular trends. Firstly, the relative price of investment exhibits a secular downward trend. Secondly, the share of nominal investment in nominal output is constant. Finally, the ratio of equipment investment to real output is steadily increasing. The standard one-sector AK growth model cannot account for these facts. The following proposition shows that a two-sector model with endogenous AK-type growth in the investment sector is consistent with these empirical regularities.<sup>4</sup>

**Proposition 2.** *In the two-sector AK model, (i) the relative price of investment  $p_t$  is decreasing, (ii) the nominal saving rate is constant and (iii) the ratio of real investment to real output is increasing.*

**Proof.**

(i) This just restates equation (4).

(ii) The saving rate is defined as  $s_t \equiv p_t i_t / (c_t + p_t i_t)$ . Using the results derived above, the saving rate is constant and can be written as

$$s = \frac{\alpha (\delta + g_k)}{A - (1 - \alpha) (\delta + g_k)}. \quad (15)$$

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<sup>4</sup>Some of these results are shown in Rebelo (1991). See also Greenwood et al. (1997).  
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(iii) The growth rate of real output is defined by a Divisia index. It amounts to writing the growth rate,  $g$ , as a weighted average of the growth rates of consumption and investment. Their weights are given by the current shares in nominal output,  $1 - s$  and  $s$ , respectively:

$$g = (1 - s) g_c + s g_k. \tag{16}$$

By Assumption 1,  $s \in (0, 1)$  and hence  $g \in (g_c, g_k)$ . Consequently, the ratio of real investment to real output is permanently growing. ■

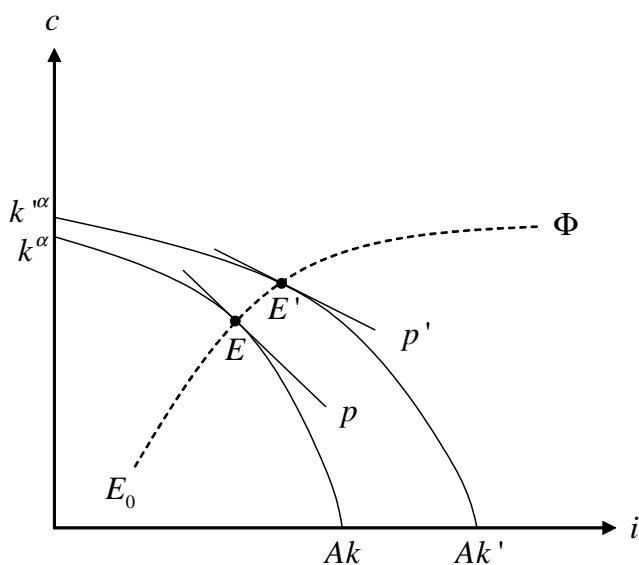


Figure 1: The economy on its expansion path  $\Phi$  from  $E$  to  $E'$  ( $k' > k$ ).

Figure 1 provides a graphical illustration of the equilibrium in the space  $(i, c)$ . At every point in time, the sectoral production functions shape a strictly concave production possibility frontier for a given stock of capital  $k_t$ . The expansion path  $\Phi$  shows all pairs  $(i_t, c_t)$  compatible with the equilibrium in Proposition 1.  $\Phi$  is found by combining the policy functions  $c_t = \phi_c(k_t)$  and  $i_t = \phi_i(k_t)$ , and substituting  $k_t$  out. In the  $(i, c)$  – space,  $\Phi$  can easily be shown to be strictly concave and increasing in  $i_t$ . As the economy accumulates capital, it moves north-east along  $\Phi$ . Equilibrium is identified at the intersection of the transformation curve and the expansion path and is denoted by

*E*. The relative price of investment goods is found as the slope of the transformation curve at point *E*. Due to the assumed differences in the sectoral production functions, capital accumulation affects sectors asymmetrically and the transformation curve shifts out in an uneven way. This is nothing else than a dynamic version of the familiar Rybczynski effect. Consequently, on the way from *E* to *E'*, the relative price of investment falls from  $p$  to  $p'$  at a rate proportional to the rate at which the economy accumulates capital.

As Whelan (2003) points out, chained-ideal-Fisher-quantity indexes are used in NIPA's methodology to measure the growth rate of GDP. Every year, a geometric mean of a Paasche and a Laspeyres index is computed using data from the current and previous years. Annual growth rates are then chained forward from an arbitrary base year, in which nominal magnitudes are equal to real magnitudes. This index can be accurately approximated by the so-called Divisia index, which weights the growth rate of each component of output by its current share in the corresponding nominal aggregate (see Deaton and Muellbauer (1980), pp. 174-5 for more details). Moreover, Licandro et al. (2002) provide theoretical support for the use of such a chained index in the framework of a two-sector exogenous growth model with embodied technical change (the GHK model). They compute a true quantity index, use official NIPA data to calibrate it for different parameter values, and show that their results come very close to what is obtained by applying NIPA's methodology.

The observation, replicated by the two-sector AK model, that the ratio of real equipment investment to real nondurable consumption and services is permanently growing is a salient feature of the recent literature on embodied technical progress.

## 3 Discussion

### 3.1 Additional empirical implications

Proposition 2 has shown that the two-sector AK model fits the three empirical facts pointed out in the introduction. However, the model has two interesting additional empirical implications. First, the model is consistent with Jones' (1995) time-series evidence. Second, it generates the negative correlation between the relative price of investment goods and the level of per capita income that we observe in cross-country data.

**The Jones critique.** Unlike the one-sector version, the two-sector model

survives Jones (1995) critique. Using data from 1950 to 1987 for 15 OECD countries, Jones criticizes that the standard AK model cannot account for the observed coincidence of stationary growth rates and upward-trending real investment rates. However, his critique is valid only for the one-sector model. The two-sector version reconciles stationary output growth with trending investment rates. Thus, the inconsistency detected by Jones cannot discredit the use of an AK specification in the core investment sector, but merely questions the use of models which keep the price of investment relative to consumption goods constant over time.<sup>5</sup>

**Cross country evidence.** Restuccia and Urrutia (2001) argue that cross-country data show a negative correlation between the price of equipment investment relative to consumption goods and per capita GDP. To see whether the two-sector AK model reproduces this fact, we first combine equations (3), (14), and the definition of GDP measured in PPP,  $y_t = p_t i_t + c_t$ , to find

$$y_t = Q_0 k_t^\alpha, \quad (17)$$

where  $Q_0$  is a constant.<sup>6</sup> Unlike the one-sector variant, the two-sector AK model features an elasticity of GDP with respect to capital equal to  $\alpha$ , just like a one-sector economy with a Cobb-Douglas production function. This is because in the two-sector AK model, the equilibrium capital allocation implies that, in terms of the numéraire, the output elasticities of capital are equalized across sectors. Combining (3) with (17), the relative price of investment goods can be expressed as a function of GDP:

$$p_t = Q_1 y_t^{-\frac{1-\alpha}{\alpha}}, \quad (18)$$

where  $Q_1$  is another constant.<sup>7</sup> Hence, the relative price of investment declines with per capita GDP at the proportional rate of  $-(1-\alpha)/\alpha$ . As claimed, the two-sector model is in line with cross-country evidence on the price of equipment investment relative to consumption goods and per capita GDP.

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<sup>5</sup>McGrattan (1998) and Li (2002) are recent attempts to challenge Jones' critique within the standard AK model. A defense of the AK model in the framework of vintage capital models is in Boucekine et al. (2005[3]).

<sup>6</sup> $Q_0 \equiv A^{-\alpha} (A - \delta - g_k)^{\alpha-1} [A - (1-\alpha)(\delta + g_k)]$ .

<sup>7</sup> $Q_1 \equiv \alpha Q_0^{1/\alpha} [A - (1-\alpha)(\delta + g_k)]^{-1}$

### 3.2 Alternative interpretations

**A model of learning-by-doing.** Next, we show that the two-sector AK model is isomorphic to the social planner version of Boucekkine et al. (2003), where endogenous growth is due to learning-by-doing in both the consumption and the investment sectors. The technological description of their model is given by

$$c_t + x_t = z_t (k_t)^\eta, \tag{19}$$

$$\dot{i}_t = q_t x_t, \tag{20}$$

where the efficiency of production increases with cumulated net investment so that  $z_t = k_t^\gamma$  and  $q_t = A k_t^\lambda$ . Parameters  $\gamma > 0$  and  $\lambda > 0$  denote the efficiency of learning in the consumption and investment sectors, respectively. In order to generate sustained balanced growth, a knife edge condition needs to be imposed:  $\lambda + \gamma + \eta = 1$ . It implies that the learning efficiencies in both sectors are negatively related. In this sense, an increase in  $\lambda$ , letting  $\eta$  unchanged, may be interpreted as a *reallocation of learning efficiency* from the consumption to the investment sector.

After the change  $\alpha = \gamma + \eta = 1 - \lambda$ , and some algebra, equations (19) and (20) become

$$c_t = (k_t^c)^\alpha$$

$$\dot{i}_t = A k_t^i,$$

where  $k_t^c$  and  $k_t^i$  are the optimal allocation of capital to the production of consumption and investment goods, respectively. Thus, the two-sector AK model can be seen as a reduced form of a learning-by-doing model where firms internalize the learning externalities.

**A model of technical change.** GHK (1997), p. 349, take the observed downward trend of the price of equipment investment relative to the price of nondurable consumption as evidence for investment specific technical change. In their theoretical model, this price trend is generated by sectoral differences in the exogenous rates of technical progress. Boucekkine et al. (2005[5]), Hsieh (2001) or Krusell (1998) take up this idea and write up models where R&D driven endogenous technical progress is confined to the investment goods sector.

In the two-sector AK model, instead, capital accumulation plays a key role. As the economy becomes more capital abundant, the capital intensive

investment sector expands overproportionally, leading to a decrease in the relative price of investment goods: along the balanced growth path, the marginal rate of transformation falls at a steady rate, as Figure 1 makes clear. This mechanism can be interpreted in two ways. Taking the AK technology in the investment sector literally, the transformation curve changes over time due to capital accumulation and not to technical progress. However, interpreting the AK model as a reduced form of the learning-by-doing model in Boucekkine et al. (2003), as discussed above, embodied technical change occurs due to the larger efficiency of learning in the investment sector, represented by the parameter  $\lambda$ .

Note that along the balanced growth path the marginal productivity of capital in the consumption sector falls, while that of the investment sector remains constant at  $A$ . Thus, the trend in  $p_t$  is driven not by decreasing marginal costs in the investment goods sector, but by increasing marginal costs in the consumption sector. However, in the learning-by-doing interpretation, along the balanced growth path, productivity rises in both sectors as a by-product of capital accumulation, implying that one may interpret the permanent reduction in the relative price of investment as embodied technical progress.

**Comparative statics with respect to  $\alpha$ .** The major difference to the one-sector model lies in the fact that the marginal return to capital in the consumption sector is not constrained to unity. If  $\alpha = 1$ , the two-sector AK model collapses to the standard one-sector AK model. Then  $\frac{\dot{p}_t}{p_t}$  is clearly zero and there are no capital losses. Here, we investigate the relation between  $\alpha$ , the decline rate of the relative price of investment goods, and the growth rates of consumption, investment and output.

**Proposition 3.**

$$(i) \frac{d|\dot{p}_t/p_t|}{d\alpha} < 0, \quad (ii) \frac{1}{\sigma} \geq 1 \iff \frac{dg_k}{d\alpha} \geq 0, \quad (21)$$

$$(iii) \frac{dg_c}{d\alpha} > 0, \quad \text{and} \quad (iv) \frac{1}{\sigma} > s \implies \frac{dg}{d\alpha} > 0. \quad (22)$$

**Proof.** (i) follows from equation (4) and (11), (ii) and (iii) are derived from (11) and (12). To show (iv), from (16)

$$g = [(1 - s)\alpha + s]g_k. \quad (23)$$

Differentiating this equation with respect to  $\alpha$  delivers

$$dg/d\alpha = g_k \left[ \frac{1 - s\sigma}{\omega} + (1 - \alpha) \frac{ds}{d\alpha} \right]. \quad (24)$$

As shown in the appendix,  $ds/d\alpha > 0$  so that  $\sigma^{-1} > s$  is indeed a sufficient condition for  $dg/d\alpha > 0$ . Obviously, the saving rate  $s$  depends also on  $\sigma$ . The appendix shows that there are parameter values for which the condition  $\sigma^{-1} > s$  holds. ■

**The growth effects of embodied technical change.** Once the two-sector model is interpreted as a model of technical change, we can inquire about the growth effects of the embodiment hypothesis. In the present context, the growth rate of real output and the speed of embodied technical change are not necessarily positively related. The reason is that growth rates depend on the difference between the interest rate and the subject discount rate. As cheaper investment goods keep coming into the market, due for example to a reduction in  $\alpha$  or an increase in  $\lambda$ , the value of installed capital depreciates due to capital losses. These obsolescence costs drive up the rental rate of capital, and –in face of a constant marginal productivity of capital –reduce the equilibrium interest rate (see equation (10)). While the typical intertemporal substitution effect of a lower interest rate unambiguously leads to a flatter consumption path, the growth rate of the capital stock may fall or rise, depending on whether the intertemporal elasticity of substitution is larger or smaller than unity. As a consequence, an increased rate of embodied technical change reduces the growth rate of consumption but has ambiguous effects on the growth rate of investment and thus on the growth rate of real GDP.

## 4 Conclusion

In this paper, we analyze the two-sector AK model proposed by Rebelo (1991), section II, where constant aggregate returns to capital are confined to the investment goods sector. This setup, an endogenous growth extension to the model of Greenwood et al. (1997), fits the following empirical observations. First, it provides a simple endogenous growth rationale for the secular downward trend of the price of equipment investment relative to consumption and the increasing ratio of real investment to real output observed in US NIPA data. Second, again in line with evidence, real output grows faster than consumption but more slowly than equipment investment. Third, since the model is compatible at the same time with stationary output growth and upward-trending real investment, it overcomes Jones' (1995) well-known critique of the AK model. Fourth, the two-sector AK model predicts a negative relation between GDP per capita and the relative price of equipment, which is consis-

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tent with the evidence from cross-country studies as in Restuccia and Urrutia (2001).

There is broad agreement amongst growth theorists that AK-type models suffer from several shortcomings. They exaggerate the role of capital accumulation over innovative activities and do not display any transition dynamics. However, it is worth noting that the two-sector AK model captures some salient features of the recent empirical evidence. Most importantly, it provides us with intuition on the endogenous growth mechanisms which can generate a balanced growth path consistent with a trending relative price of investment goods.

## A Proof of proposition 1

Substituting  $a_t = p_t k_t$ ,  $w_t = (1 - \alpha) (k_t^c)^\alpha$  and  $c_t = (k_t^c)^\alpha$  into equation (7), the law of motion of the per capita capital stock turns out to be

$$\dot{k}_t = (A - \delta) k_t - A k_t^c.$$

Using  $e^{-(A-\delta)t}$  as the integrating factor, substituting  $k_t^c = k_0^c e^{g_k t}$  and rearranging terms yields

$$e^{-(A-\delta)t} \left[ \dot{k}_t - (A - \delta) k_t \right] = -e^{-(A-\delta-g_k)t} A k_0^c.$$

The LHS can easily be recognized as  $\frac{d}{dt} [e^{-(A-\delta)t} k_t]$  and the RHS as  $\frac{d}{dt} \left[ \frac{1}{(A-\delta-g_k)} e^{-(A-\delta-g_k)t} A k_0^c \right]$ . Integrating and dividing by  $e^{-(A-\delta)t}$  gives

$$k_t = \frac{1}{(A - \delta - g_k)} e^{g_k t} A k_0^c + C e^{(A-\delta)t} \quad (\text{A1})$$

where  $C$  and  $k_0^c$  are constants which can be determined using the initial condition  $k_0 > 0$  and the transversality condition (13). Using expression (A1) and the law of motion for the state variable in the transversality condition it follows that

$$\lim_{t \rightarrow \infty} \left\{ \frac{A}{(A - \delta - g_k)} e^{-(A-\delta-g_k)t} + C \right\} = 0. \quad (\text{A2})$$

The first term in the braced brackets converges towards zero since  $A - \delta - g_k > 0$ . Therefore, the TVC requires the constant  $C$  to be zero. From (A1) we get  $k_0 = A k_0^c / (A - \delta - g_k)$ . ■

## B Proof of proposition 3(iii)

The derivative of the saving rate with respect to  $\alpha$  is

$$\frac{ds}{d\alpha} = \frac{g_k \frac{1-\sigma}{\omega} \alpha A + (\delta + g_k)(A - \delta + g_k)}{[A - (1 - \alpha)(\delta + g_k)]^2}. \quad (\text{A5})$$

Using this result, together with the growth rate  $g_k$  in expression (24) a necessary and sufficient condition for the sign of  $dg/d\alpha$  can be derived. This condition can be stated as

$$\frac{1}{\sigma} \geq s + (1 - \alpha)\phi \iff dg/d\alpha \geq 0 \quad (\text{A6})$$

where

$$\phi = \frac{(\delta + g_k)(A - \delta + g_k)}{[A - (1 - \alpha)(\delta + g_k)]} + (\delta + g_k)(A - \delta + g_k) \quad (\text{A7})$$

Assumption 1 implies that  $\phi > 0$ . In the body of the paper we focus on sufficiency, hence  $\frac{1}{\sigma} > s \iff dg/d\alpha > 0$ .

Since  $s$  depends  $\sigma$ , we need to check, whether there are indeed parameter constellations for which the above inequality can hold. One only has to show that the equation  $\sigma^{-1} = s(\sigma)$  has a unique solution  $\sigma^*$ . Clearly,  $s$  is a continuous function of  $\sigma$ . Moreover, it is decreasing in  $\sigma$  since

$$\frac{\partial s}{\partial \sigma} = - \left[ \frac{\alpha}{A - (1 - \alpha)(\delta + g_k)} \right]^2 \frac{A}{\omega} g_k < 0. \quad (\text{A8})$$

Additionally,  $\partial^2 s / \partial \sigma^2 > 0$ , so that the saving rate is strictly convex in  $\sigma$  (strictly concave in  $1/\sigma$ ). As  $\sigma$  tends towards infinity,  $s$  converges towards the constant

$$s^- = \frac{\alpha\delta}{A - \delta + \alpha\delta} \quad (\text{A9})$$

strictly larger than zero (by Assumption 1) and as  $\sigma$  tends towards zero,  $s$  converges towards the constant

$$s^+ = \frac{\alpha}{1 - \alpha} \frac{A - (\rho + \alpha\delta)}{\rho + \alpha\delta} > s^-. \quad (\text{A10})$$

Therefore, there is a unique solution  $\sigma^*$  to the equation  $\sigma^{-1} = s(\sigma)$  and this solution is bounded below by  $(s^+)^{-1}$ . Hence, the set of parameters that satisfies the above inequality is not empty. ■

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