Reputation in a model with a limited debt structure

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Abstract

This paper studies the optimal management of the maturity of government debt in an economy without commitment. We consider a reputation where any deviation triggers reversion to the worst sustainable equilibrium. We obtain two results. First, contrary to earlier literature, we show that a very rich debt structure is not a necessary condition to solve the time-inconsistency problem. Second, we learn how to allocate the outstanding debt into short and long-term bonds to enhance the credibility of the government policy.

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1. Introduction

This paper studies the role of the maturity structure of government debt in an economy without commitment. We consider a government that can manipulate the interest rate on debt. In this context, we learn which debt structures can sustain the Ramsey outcome in a reputational model in which any deviation from a policy announcement triggers infinite reversion to the worst sustainable equilibrium.

We analyze the properties of the best sustainable equilibrium to answer two questions. First, earlier literature has established the general principle that a very rich debt structure is
necessary in order to solve the time-inconsistency problem. This paper examines whether the debt structure is relevant and, if so, whether a very rich structure is still a requirement in the presence of a reputation mechanism. Second, given a menu of bonds with different maturities, we study how the government should restructure the outstanding debt. Should the government issue more short-term debt? or rather long-term?

Under full-commitment and no uncertainty, the standard result is that the maturity structure of government debt is irrelevant. However, without commitment, Lucas and Stokey (1983) find that the maturity of debt recovers an important role. Their paper studies the government’s problem of how to finance public services through taxes on labor and through debt to maximize the welfare of the individuals. They show that the careful selection of the maturity of debt can make the optimal policy time-consistent as long as debt is honored and can be issued with as many maturities as periods in the economy. If the latter is not possible, Rogers (1989) shows through backward induction that increasing the available maturities of debt helps attain time-consistent policies that provide more welfare. All in all, and as Alvarez et al. (2004) point out, the general principle of this literature is that a very rich maturity structure of government debt is necessary in order to solve or reduce the time-inconsistency problem.

In the presence of reputation mechanisms different papers have analyzed the management of government debt. Among them, Chari and Kehoe (1993), Cole and Kehoe (2000) and Phelan (2004) study the problem of default on debt in economies similar to that of Lucas and Stokey (1983). In this setup Chari and Kehoe (1993) allow for a very rich debt structure to eliminate the incentives to manipulate the interest rate on government debt. Then they define a trigger mechanism that specifies reversion to a Markov equilibrium. Since governments have incentives to default on positive debt and to honor negative debt payments, they obtain that the reputation cannot support positive government debt.

The study of the maturity structure of government debt is not the focal point in Chari and Kehoe (1993), but it is the main question in both Cole and Kehoe (2000) and Phelan (2004). Cole and Kehoe (2000) study the level and maturity of government debt in financial crises where the government cannot sell new bonds and defaults on debt. They show that the government can exit or eliminate these financial crises by reducing the amount of debt or by lengthening the maturity. Phelan (2004) re-examines this problem and argues that the origin of financial crises lies in the large debt rollovers rather than the maturity. He shows that, if debt auctions and repayments can occur sufficiently frequent to avoid large debt rollovers, the maturity structure of government debt is irrelevant.

This paper considers an economy similar to those in the above mentioned papers. We focus, however, on the time-inconsistency problem of manipulation of the interest rate on government bonds. This paper is the first to study this issue in a reputational model with competitive individuals and a benevolent government. We consider a government that can issue bonds indexed to consumption with different maturities. We assume that a default on debt payments would entail a large cost to the economy. This assumption is made in order to allow for positive debt because, as in Chari and Kehoe (1993), a reputation cannot

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1 This result holds under uncertainty if state-contingent debt can be issued. Otherwise, Angeletos (2002) shows that the maturity of government debt becomes instrumental in insuring the economy against shocks.
sustain positive debt in our environment.\(^2\)\(^3\) The government does not commit to follow any policy announcement. Moreover, the government has an incentive to deviate from the announced policy by lowering the tax rate on labor at the dates in which the outstanding debt matures. This increases the private consumption in those periods, reduces the interest rate on debt and, in turn, “devaluates” the government debt.

We consider a structure of debt that is limited and, thus, debt restructuring cannot solve the time-inconsistency. In this environment we introduce a reputation mechanism where any deviation from the policy plan implies reversion to the worst sustainable equilibrium. We obtain that autarky, i.e. a balanced budget, where government bonds are not longer traded, is the worst sustainable equilibrium. That is so because if there were an equilibrium with a lower payoff, the government would always deviate to the balanced budget. The worst sustainable equilibrium depends on the level and maturity of the inherited debt. Using this equilibrium as a punishment after a deviation, we study the maturity structure of government debt in the best sustainable equilibrium.

Our first result is that the irrelevance of the debt structure never holds for economies with a sufficiently rich maturity structure of debt. An explanation for this result is that if the government deviates, then the economy reverts to a balanced budget. The government must pay back the outstanding debt at the corresponding maturities and cannot sell new bonds to smooth taxes over time. This cost of deviating becomes smaller if the inherited debt is spread over all the maturities and there are as many maturities as possible.

Our second result is that a government can minimize the incentives to deviate by concentrating the debt in few maturities or by reducing the amount sold at some maturities. The intuition behind is that, by doing so, the costs from the inability to smooth consumption are intensified and, therefore, a deviation is less appealing. Moreover, we learn that the maturities in which the government should concentrate debt are characterized by a low marginal utility of consumption. In periods with lower marginal utility of consumption (higher consumption), a lower than announced labor tax rate would “devaluate” the government debt but would also entail the cost of increasing even more the consumption in those periods and differing further from a smooth consumption pattern.

Finally, we have identified the maturity schedule of government debt, i.e. the initial debt structure, as the determinant of the dynamics of the marginal utility of consumption. In particular, we have found that it is lower in periods with a higher concentration of maturing debt. Therefore, a government that cannot commit to future policies may be able to sustain the Ramsey policy by issuing bonds with a maturity with an already high concentration of

\(^2\) The reason behind the non-sustainability of positive debt in our model is different from that in Chari and Kehoe (1993). In Chari and Kehoe (1993), the result lies in the properties of the Markov equilibrium that follows after a deviation. In our model, if there were no exogenous costs of default, as Phelan (2004) shows, the worst sustainable equilibrium would be a balanced budget with default, i.e., with zero outstanding debt, and the government would not be able to borrow again. This threat could sustain positive debt if the cost from not being able to borrow is higher than the gain from defaulting, in other words, if there is a great need to smooth taxes over time. In our economy, because there are no shocks, the necessity of smoothing taxes over time comes only from the level of outstanding debt and disappears if the government defaults completely on debt. Therefore, we assume that a direct cost of default allows sustaining positive debt. Some of these arguments are present in Stockman (2004).

\(^3\) This result is also obtained in models of international borrowing as in Bulow and Rogoff (1989).
Looking at the data, we observe that government debt is largely concentrated in the short-term for many countries. Following the reasoning above, we could conclude that shortening the debt would alleviate their time-inconsistency problem.

The two main contributions on the empirical evidence on public debt management and credibility are Missale and Blanchard (1994) and Missale et al. (2002). The first article shows that debt and maturity have moved in opposite directions in Ireland, Italy and Belgium over the last 30 years. They develop a model that shows that governments may keep a low-inflation target credible by decreasing maturity as debt increases. Missale et al. (2002) examine 72 episodes of fiscal stabilization in OECD countries over the last two decades. They show that the credibility of the program, measured as the change in long-term interest rates at the start of a stabilization, is an important determinant on the choice of public debt. They find that governments tend to issue a larger share of short maturity debt the less credible is the program. The present paper develops a model that captures this empirical evidence and argues why short-term debt may be the optimal response to credibility problems.

The rest of the paper is organized as follows. Section 2 presents the model. Sections 3 and 4 study the policy under full-commitment and under reputation, respectively. Section 5 concludes. Appendix A contains the proofs.

2. The model

We consider an economy similar to that of Lucas and Stokey (1983), where a benevolent government must decide how to finance the government spending through taxes on labor income and through the issue of public debt. We depart from their model in that the economy is deterministic, the government spending is endogenous and the composition of debt is such that debt restructuring cannot solve the time-inconsistency problem.

The government debt consists of bonds indexed to consumption that can be positive or negative and that mature at given future dates. More precisely, the government can offer a price \( q_{B,t}^S(q_{S,t}^S) \) for a bond \( b_{B,t}^S \geq 0 (b_{S,t}^S \leq 0) \) to be bought (sold) by the individuals at date \( t \) for any future date \( s \geq t + 1 \), and, if accepted, the individual will receive (pay) a unit of consumption at date \( s \). We denote by \( t_{BS} \) the amount of government debt existing at date \( t \) that matures at date \( s \), i.e., \( t_{BS} = \sum_{k=S}^t b_{k}^{S,t} \). We allow the government to default on all debt obligations but assume that such a default would bring about a very large but finite direct cost to the economy. Since the paper focuses on situations where default does not occur, we can focus on situations where default does not occur.
not occur in equilibrium, we describe the environment and define a competitive equilibrium when governments choose to honor debt payments.\footnote{A general setup would include as policy variable a default rate $\delta_t \in \{0, 1\}$ on all outstanding debt into the government’s and individuals’ budget constraints, as in Chari and Kehoe (1993).}

The economy is populated by identical infinitely-lived individuals. Each individual is endowed with $d > 0$ units of time per period that can be devoted to either work or leisure $x_t \in (0, d)$. We assume that firms are identical, competitive and with a linear production technology; one unit of labor yields one unit of output. Since output is non-storable and can be used for either private consumption $c_t$ or public consumption $g_t$, the resource constraint is

$$c_t + x_t + g_t \leq d. \quad (1)$$

The representative individual cares about private consumption, public consumption, and leisure so that his discounted life-time utility can be written as

$$\sum_{t=0}^{\infty} \beta^t (u(c_t) + v(x_t) + z(g_t)), \quad (2)$$

with $\beta \in (0, 1)$, where the functions $u, v$ and $z$ are strictly increasing, strictly concave and continuously differentiable. Taking the government policy and initial debt holdings as given, the representative individual chooses consumption, leisure and bonds $\{b_{B,t}^k, b_{S,t}^k\}_{t=0}^{\infty}$ to maximize his welfare (2) subject to the budget constraint

$$c_t + \sum_{k=S,B} \sum_{s=t+1}^{\infty} q_{k,t}^k b_{k,t}^s \leq t (1 - \tau_t) (d - x_t), \quad (3)$$

the non-negativity constraints

$$c_t \geq 0, \quad d \geq x_t \geq 0, \quad \text{and} \quad b_{B,t}^k \geq 0, \quad b_{S,t}^k \leq 0, \quad (4)$$

and the no-Ponzi-game condition

$$\lim_{t \to \infty} \sum_{k=S,B} \sum_{s=t+1}^{\infty} \eta_t q_{k,t}^k b_{k,t}^s = 0. \quad (5)$$

Here $(d - x_t)$ is labor income,\footnote{Since wages are equal to 1.} $\tau_t$ is the tax rate on labor income and $\eta_t$ is the Lagrange multiplier on constraint (3). The first-order conditions for this optimization problem are

$$u'(x_t) = (1 - \tau_t) u'(c_t), \quad (6)$$

$$[u'(c_t) q_{B,t}^t - \beta^{s-t} u'(c_t)] b_{B,t}^t = 0, \quad (7)$$

$$[u'(c_t) q_{S,t}^t - \beta^{s-t} u'(c_t)] b_{S,t}^t = 0, \quad (8)$$

for all dates $t \geq 0$ and maturities $s \geq t + 1$. For a sufficiently high price $q_{B,t}^t$ of a bond $b_{B,t}^t$, Eq. (7) implies $b_{B,t}^t = 0$. Similarly, a very low price $q_{S,t}^t$ dictates $b_{S,t}^t = 0$ through condition (8). Otherwise, for an intermediate price $q_{k,t}^k$, we find\footnote{Note that, for a given maturity $s$, if $b_{B,t}^s > 0$, then $b_{S,t}^s = 0$, and the other way around.}

$$u'(c_t) q_{k,t}^k = \beta^{s-t} u'(c_t). \quad (9)$$
An allocation for consumers is a sequence \( \{a_t\}_{t=0}^{\infty} \), where \( a_t = (c_t, x_t, \{b_{b,t}^{S,t}, b_{s,t}^{S,t}\}_{s=t+1}^{\infty}) \).

The government policy consists of a choice of public spending, a flat-tax rate on labor income \( \tau_t \leq 1 \), and a list of prices for government bonds for all different maturities that satisfy the government budget constraint
\[
g_t + i_{b,t} \leq \tau_t (d - x_t) + \sum_{k=S,B} \sum_{s=t+1}^{\infty} q_{s}^{k,t} b_{s}^{k,t}.
\]
We denote the government policy at date \( t \) by \( \pi_t = (g_t, \tau_t, \{q_{s}^{B,t}, q_{s}^{S,t}\}_{s=t+1}^{\infty}) \).

**Definition 1.** Given a policy \( \{\pi_t\}_{t=0}^{\infty} \) and initial debt \( \{0b_{s}\}_{s=0}^{\infty} \), a competitive equilibrium allocation is a sequence \( \{a_t\}_{t=0}^{\infty} \) such that:

(i) the representative individual maximizes his welfare (2) subject to the budget constraint (3), the non-negativity constraints (4) and the no-Ponzi-game condition (5);
(ii) labor is paid its marginal product;
(iii) all markets clear (the resource constraint (1) is satisfied with equality).

We will assume throughout the paper that the competitive equilibrium is never on the wrong side of the Laffer curve. We now consider the economy under full-commitment and then we model the economy without commitment.

### 3. The full-commitment policy

In this section we assume that future governments commit to follow the policy plan chosen by the government at date 0. This assumption is equivalent to a full-commitment among the successive governments that makes the optimal policy planned at 0 sustainable. Formally, under full-commitment, the government at date 0 selects a policy plan \( \pi = \{\pi_t\}_{t=0}^{\infty} \). Next, taking into account \( \pi \), the individuals choose an allocation rule at each date \( t \). An allocation rule is a sequence of functions \( f = \{f_t\}_{t=0}^{\infty} \) that map policies \( \pi_t \) into allocations \( a_t \).

We follow the primal approach to solve for the policy plan and allocation rule. First, an allocation is chosen to maximize the welfare of the representative individual subject to the competitive equilibrium conditions. Then, given these conditions, the allocation determines the optimal taxes and debt prices. To do so, we plug the first-order conditions (6)–(8) into the budget constraint (3) to obtain the implementability condition
\[
u'(c_t)(c_t - i_{b,t}) - v'(x_t)(d - x_t) + \sum_{k=S,B} \sum_{s=t+1}^{\infty} \beta^{s-t} u'(c_s)b_{s}^{k,t} = 0.
\]
We then define the government’s problem as follows. Given an initial debt \( \{0b_s\}_{s=0}^{\infty} \), the government chooses \( \{c_t, x_t, g_t, \{b_{b,t}^{R,t}, b_{s,t}^{S,t}\}_{s=t+1}^{\infty}\}_{t=0}^{\infty} \) to maximize the welfare of the

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11 Since the budget constraint of the individual (3) and the resource constraint (1) hold with equality, the government budget constraint (10) is also satisfied in a competitive equilibrium.
representative individual (2) subject to the resource constraint (1), the implementability condition (11), the non-negativity constraints (4), and the transversality condition

\[ \sum_{t \to \infty} \lim_{x \to S} \sum_{s = t+1}^{\infty} \beta^s u'(c_s) b^{k,t} = 0. \] (12)

The solution to this problem is characterized by constraints (1), (4), (11), and (12), and the following first-order conditions for consumption, leisure, public spending, and government bonds, respectively:\textsuperscript{12}

\[ \mu_t = u'(c_t) + \lambda_t [u'(c_t) + u''(c_t)(c_t - r b_t)] + \sum_{k=S,B}^{t-1} \sum_{r=0}^{s} \lambda_r u''(c_t) b^{r,t}, \] (13)

\[ \mu_t = v'(x_t) + \lambda_t [v'(x_t) - v''(x_t)(d - x_t)], \] (14)

\[ \mu_t = z'(g_t), \] (15)

\[ \lambda_t = \lambda_s, \quad \text{for all } s \geq t + 1, \] (16)

for all dates \( t \geq 0 \), where \( \mu_t \) and \( \lambda_t \) are the Lagrange multipliers associated with constraints (1) and (11), respectively. This characterization is only necessary for a solution. As usual in this literature, we assume that an optimal interior solution exists.

We obtain the optimal labor tax rates and bond prices from the equilibrium conditions (6)–(8). The optimal management of debt results from Eq. (16) and is summarized in the next lemma:

**Lemma 1.** The maturity structure of government debt is irrelevant under full-commitment.

**Proof.** See Appendix A. \( \Box \)

As expected we obtain the standard result of the irrelevance of the government debt. In a world with full-commitment and no uncertainty, the government cares about the amount of debt to issue but it is indifferent about its composition. The government finds optimal a complete smoothing of the excess burden of taxation over time and can achieve this result independently of the debt structure. We now study the structure of the government debt in an economy without commitment.

### 4. The policy without commitment

From this section on we assume that future governments can reconsider the policy. In particular, a government must weigh the benefits of deviating from the announced policy against the loss of reputation that triggers a reversion to the worst sustainable equilibrium. In this section we follow Chari and Kehoe (1990) to define a sustainable equilibrium, find

\textsuperscript{12} Recall that a very large cost of default makes the government choose to honor the initial debt.
the worst and characterize the set of sustainable equilibria. We then focus on the best sustainable equilibrium.

4.1. The set of sustainable equilibria

We now describe how decisions are taken in an economy without commitment. At the beginning of date $t$, the government chooses the current policy as a function of the history $h_{t-1} = (\pi_s \mid s = 0, \ldots, t - 1)$, denoted $\sigma_t(h_{t-1})$, and a plan for future policies for all possible future histories; $\sigma_t(h_{t-1})$ defines a strategy for the government. Given a history $h_{t-1}$, the policy plan $\sigma$ induces future histories by $h_t = (h_{t-1}, \sigma_t(h_{t-1}))$. A continuation policy of $\sigma$ is $(\sigma_t(h_{t-1}), \sigma_{t+1}(h_{t-1}, \sigma_t(h_{t-1})), \ldots)$. After the government sets the current policy, individuals choose consumption, leisure, and bonds as a function of history $h_t$, denoted $f_t(h_t)$, and a plan for future allocations. Given a history $h_t$ and a policy plan $\sigma_t$, a continuation allocation of $f$ is $(f_t(h_t), f_{t+1}(h_t, \sigma_{t+1}(h_t)), \ldots)$; $f_t(h_t)$ is a strategy for the individual. We assume that individuals are anonymous, i.e. individual decisions cannot be observed, and we restrict attention to symmetric strategy equilibria where all individuals make the same choices along the equilibrium. Now we define a sustainable equilibrium:

Definition 2. A strategy profile $(\sigma, f)$ is a sustainable equilibrium if satisfies the following conditions:

(i) given the allocation rule $f$, the continuation payoff for the government is higher than the payoff from any deviation to a different strategy $\sigma'$ for every history $h_{t-1}$;
(ii) given the policy plan $\sigma$, the continuation payoff for the individual is higher than the payoff from any deviation to a different strategy $f'$ for every history $h_t$.

The conditions behind this definition imply sequential rationality. The first condition says that the government will not find profitable to deviate from the announced policy. The second states that individuals respond optimally to the government policy, which in turn implies a competitive equilibrium.

In this environment, it is quite immediate to identify the worst sustainable equilibrium. We demonstrate that since the government can always impose a balanced budget (with or without default) unilaterally, other equilibria that provide less welfare are not sustainable. To show this formally, let us first define a balanced budget:

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13 Chari and Kehoe (1990) study the time-inconsistency problem of capital taxes in an infinitely repeated one period economy. We follow Chari and Kehoe (1990) rather than Chari and Kehoe (1993) because the former specifies the worst sustainable equilibrium as a punishment after a deviation and the latter a Markov equilibrium, which is not the worst sustainable equilibrium.

14 Anonymity rules out the possibility of strategic behavior from the part of the individuals and, thus, allows us to restrict attention to competitive equilibria. See Phelan and Stacchetti (2001).
Definition 3. A balanced budget equilibrium with no default from date $t$ is an optimal policy and a competitive equilibrium, where $b_{t+1}^k = 0$, for $k = S, B$, for all dates $r \geq t$ and all maturities $s \geq t + 1$, and therefore, the budget of the government is balanced, i.e.,

$$g_r + r b_r = \tau_r(d - x_r), \quad \text{for all dates } r.$$  \hfill (17)

Given an initial debt $\{b_s\}_{s=t}^\infty$, a balanced budget without default solves the following problem: to maximize welfare (2) by choice of $\{c_r, x_r, g_r\}_{r=t}^\infty$ subject to the resource constraint (1), the implementability condition under balanced budget

$$z'(g_r) = u'(c_r)(c_r - r b_r) - v'(x_r)(d - x_r),$$  \hfill (18)

and $c_r \geq 0, d \geq x_r \geq 0$. This problem yields the necessary conditions

$$z'(g_r) = u'(c_r) + \lambda_r u''(c_r)(c_r - r b_r),$$  \hfill (19)

$$z'(g_r) = v'(x_r) + \lambda_r v''(x_r)(d - x_r),$$  \hfill (20)

where $\lambda_r$ is the Lagrange multiplier on constraint (18). Similarly, a balanced budget equilibrium with default is defined and computed by setting $r b_r = 0$ for all maturities and bearing the cost of default. Notice that a balanced budget with default always exists, but there are initial debt positions for which a balanced budget without default does not exist. We now show that a balanced budget yields the lowest sustainable payoff:

**Proposition 1.** The worst sustainable equilibrium is a balanced budget equilibrium. This balanced budget is an equilibrium with no default, if one exists, otherwise with default.

**Proof.** See Appendix A. \hfill \Box

A balanced budget is clearly a sustainable equilibrium. A balanced budget with or without default makes the economy static and the dynamic inconsistency disappears. Furthermore, a balanced budget is the worst sustainable equilibrium because for any equilibrium to be sustainable the continuation payoff for the government must be higher than the payoff from any possible deviation. If there exists a competitive equilibrium worse than a balanced budget, then the government can always deviate by choosing prices $\{q_{t+1}^B, q_{t+1}^S\}_{s=t+1}^\infty$ such that the individuals find optimal not to buy or sell any government bonds, i.e. the government can impose a balanced budget. If the initial debt allows, the balanced budget is without default to avoid the implicit high costs, otherwise the government defaults.\footnote{This result depends on the absence of restrictions on debt quantities and prices. If there were an upper limit on the amount of debt that can be sold, a government could also deviate by offering a sufficiently low price on debt to make the individuals buy as much debt as possible. The worst sustainable equilibrium would be the best of the equilibrium just described and a balanced budget. If there were limits on debt combined with restrictions on the bonds’ prices $\{q_{t+1}^B, q_{t+1}^S\}_{s=t+1}^\infty$, the government might not be able to impose a balanced budget without default. We have applied the APS method developed by Abreu et al. (1990) and extended to dynamic policy games by Phelan and Stacchetti (2001) and Sleet (1997) to our economy with only one period debt and obtained the following results. The worst sustainable equilibrium is characterized by purchases of high amounts of debt (as high as possible) whenever the individual receives little or zero debt payments and no purchases of debt whenever the individual receives high debt payments. This generates a very irregular path of consumption that yields less welfare than the balanced budget without default.}


The worst sustainable equilibrium depends on the initial debt structure \( \{t_s b_s\}_{s=1}^{\infty} \). We denote by \( V^D(\{t_s b_s\}_{s=1}^{\infty}) \) the welfare from the worst sustainable equilibrium (the deviation value) and characterize the set of sustainable equilibria as follows:

**Proposition 2.** A pair \((\pi, a)\) is a sustainable equilibrium if and only if

(i) \((\pi, a)\) is a competitive equilibrium at date 0.

(ii) For all dates \( t \), the following inequality holds:

\[
\sum_{s=t}^{\infty} \beta^{s-t} \left[ u(c_s) + v(x_s) + z(g_s) \right] \geq V^D(\{t_s b_s\}_{s=1}^{\infty}).
\]  

(21)

**Proof.** See Appendix A. \( \square \)

In what follows, we analyze the worst and the best sustainable equilibrium in order to characterize the optimal management of the maturity structure of government debt. Since we focus on the time-inconsistency problem of manipulation of interest rate of debt, we will restrict our analysis to non-negative debt positions \( t_s b_s \geq 0 \), for which a balanced budget without default exists. We will later discuss an extension to the case of default.

### 4.2. The worst sustainable equilibrium

The welfare from the worst sustainable equilibrium plays a crucial role in characterizing the set of sustainable equilibria. This welfare depends on the inherited debt structure. Consider the government at date 0. This government cannot change the past issues of debt but can choose the new issues \( \{b^0_s\}_{s=1}^{\infty} \) so as to affect the incentives to deviate (the deviation value) at date 1. Suppose that the government has optimally selected a menu of maturities \( \Omega \). For that set of maturities \( \Omega \), there is a composition of this debt \( \{b^0_s\}_{s=1}^{\infty} \) that makes the deviation value maximum, denoted \( V^D_{\max}(\Omega) \), and a composition that makes the deviation value minimum, denoted \( V^D_{\min}(\Omega) \). We next show how the level and the set of maturities of debt affect the deviation value:

**Lemma 2.** The deviation value satisfies the following conditions:

(i) For all maturities \( s \geq t + 1 \),

\[
\frac{\partial V^D(\{t_s b_s\}_{s=1}^{\infty})}{\partial t_s} < 0.
\]

(ii) For \( \Omega \subseteq \Omega' \), \( V^D_{\max}(\Omega') \geq V^D_{\max}(\Omega) \) and \( V^D_{\min}(\Omega') \leq V^D_{\min}(\Omega) \).

**Proof.** See Appendix A. \( \square \)

Lemma 2 summarizes two results. First, the deviation value increases with the amount of inherited debt. In other words, an initial debt brings along the need for distortionary taxation and, thus, a welfare cost. Moreover, we assume the following:
Assumption 1. For all maturities $s \geq t + 1$,
\[
\frac{\partial V^D(\{b_s\}_{s=t+1}^\infty)}{\partial b_s \partial b_t} < 0.
\]
This means that the welfare cost of debt increases with the amount of debt. This assumption is equivalent to that of convex costs of distortionary taxation by Barro (1979). Here we show an example for which this assumption holds:

Example 1. Let the contemporaneous utility function be
\[
u(c_t) + \psi(x_t) + z(g_t) = \theta \ln c_t + (1 - \theta) \ln x_t + \gamma \ln g_t,
\]
with $\theta \in (0, 1)$ and $\gamma > 0$. The condition $\theta \geq \gamma$ is sufficient to guarantee
\[
\frac{\partial V^D(\{b_s\}_{s=t+1}^\infty)}{\partial b_s \partial b_t} < 0.
\]

Proof. See Appendix A.

Second, Lemma 2 shows how the set of maturities affects the deviation value. It is obvious that the maximum (minimum) deviation value for a given set of maturities must be equal or higher (lower) than that for a more limited set. However, to understand this better, we next describe the debt structure that maximizes and the one that minimizes the deviation value:

Lemma 3. Let the full-commitment be sustainable. For a given set of maturities $\Omega$, we obtain the following results:

(i) The issues of debt at date $t$, $\{b_s\}_{s \in \Omega}$, that yield $V^D_{\text{max}}(\Omega)$ satisfy $t+1b_s = \bar{b}$ at steady state for all maturities $s \in \Omega$.
(ii) The issues of debt at date $t$, $\{b_s\}_{s \in \Omega}$, that yield $V^D_{\text{min}}(\Omega)$ satisfy $t+1b_s > 0$, for maturity $s$, and $t+1b_r = 0$ for all other maturities.

Proof. See Appendix A.

The economy after a deviation describes a government under a balanced budget. In this context, the government cannot use debt to smooth consumption across periods. We know that the maximum deviation value is not decreasing with the number of maturities. Moreover, Lemma 3 shows that, at steady state, the debt structure that maximizes the deviation value is characterized by an identical level of debt at all maturities and, therefore, $V^D_{\text{max}}(\Omega') > V^D_{\text{max}}(\Omega)$ for $\Omega \subset \Omega'$. In other words, the potential losses from the inability to smooth consumption are smaller when the outstanding debt can be spread along a larger set of maturities. We also know that the minimum deviation value is not increasing with the set of maturities. From Lemma 3, we learn that the minimum deviation value is characterized by concentrating all debt in only one maturity and, thus, limiting further the possibilities of smoothing consumption.
4.3. The best sustainable equilibria

From the set of sustainable equilibria, we analyze the best sustainable equilibrium. In order to choose the best sustainable policy, the government must now take into account the period-by-period incentive compatibility constraint (21) satisfying that the welfare value of continuing with the announced policy must be higher than the welfare value of deviating and reverting to the worst sustainable equilibrium.

Formally, the government’s problem is to choose \( \{c_t, x_t, g_t, [b_{t+1}^{S,t}, b_{t+1}^{B,t}]_{t=0}^\infty \} \) to maximize welfare (2) subject to the resource constraint (1), the implementability constraint (11), the incentive compatibility constraint (21), the non-negativity constraints (4) and the transversality condition (12), given the initial debt \( \{\delta b_s\}_{s=0}^\infty \). If, for some debt structure, the incentive compatible constraints (21) never bind, then the best sustainable equilibrium is the full-commitment. However, if there is not such a debt structure, the Lagrangian of the government’s problem fails to be concave because \( \frac{\partial}{\partial b_t} \frac{\partial}{\partial b_t} < 0 \). Given this limitation, we will focus on economies where the full-commitment can be sustained.

In contrast to the full-commitment, the dependence of the incentive compatible constraints (21) on the debt structure suggests that the government may care not only about the amount of debt to issue but also about the maturity. Let us then study the relevance of the debt structure:

**Proposition 3.** Consider an initial debt \( \delta b_s > 0 \) for some \( s \geq 0 \), and a finite set of maturities \( \Omega \). If \( V_{FC}^{\Omega} \geq V_{\Omega}^{\max} \) for all dates \( t \geq 0 \), then we obtain the following results:16

(i) The debt structure is irrelevant for all set of maturities \( \Omega' \subset \Omega \).

(ii) There exits a finite \( \Omega'' \subset \Omega' \), such that for all set of maturities \( \Omega''' \) so that \( \Omega'' \subset \Omega''' \), the debt structure is not longer irrelevant.

**Proof.** See Appendix A. □

Proposition 3 shows that the irrelevance of the debt structure will never hold for economies with a sufficiently rich maturity structure of debt. The intuition for this result is as follows. We know that the welfare value of continuing with the full-commitment is independent of the set of maturities. Moreover, we demonstrate that the maximum deviation value is increasing in the available maturity and that will always exceed the continuation value of the full-commitment for a sufficiently large set of maturities. Therefore, the best sustainable equilibrium is still the full-commitment, however the debt structure matters. Figure 1 and Table 1 illustrate this result. This figure shows how increases in the set of maturities shift the maximum deviation value above the continuation value.

For economies with no initial debt, we show that there are no incentives to deviate from the full-commitment:

16 We denote by \( V_{FC}^{\Omega} \) the value of continuing with the full-commitment.
Table 1
Changes in parameter and initial values

<table>
<thead>
<tr>
<th>Parameter change</th>
<th>( V_{FC} - V_{D}^{\text{max}} )</th>
<th>( \hat{N} )</th>
<th>( g_{ss} )</th>
<th>( 1 - x_{ss} )</th>
<th>( \beta^{2} V_{D}(t_{bs}^{t+N-1}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( - )</td>
<td>0.0015</td>
<td>13</td>
<td>0.750</td>
<td>0.323</td>
<td>-0.020( \beta^{s} )</td>
</tr>
<tr>
<td>( \beta = 0.975 )</td>
<td>-0.0004</td>
<td>27</td>
<td>0.750</td>
<td>0.328</td>
<td>-0.019( \beta^{s} )</td>
</tr>
<tr>
<td>( \beta = 0.925 )</td>
<td>-0.0017</td>
<td>8</td>
<td>0.750</td>
<td>0.318</td>
<td>-0.021( \beta^{s} )</td>
</tr>
<tr>
<td>( \theta = 0.35 )</td>
<td>-0.0002</td>
<td>14</td>
<td>0.740</td>
<td>0.340</td>
<td>-0.017( \beta^{s} )</td>
</tr>
<tr>
<td>( \theta = 0.31 )</td>
<td>-0.0001</td>
<td>10</td>
<td>0.763</td>
<td>0.300</td>
<td>-0.026( \beta^{s} )</td>
</tr>
<tr>
<td>( \gamma = 2 )</td>
<td>-0.1002</td>
<td>2</td>
<td>0.857</td>
<td>0.322</td>
<td>-0.009( \beta^{s} )</td>
</tr>
<tr>
<td>( \gamma = 0.10 )</td>
<td>-2.6e-7</td>
<td>92</td>
<td>0.217</td>
<td>0.341</td>
<td>-0.001( \beta^{s} )</td>
</tr>
<tr>
<td>( d = 105 )</td>
<td>-0.0013</td>
<td>13</td>
<td>0.750</td>
<td>0.324</td>
<td>-0.021( \beta^{s} )</td>
</tr>
<tr>
<td>( d = 95 )</td>
<td>-3.7e-6</td>
<td>55</td>
<td>0.428</td>
<td>0.326</td>
<td>-0.018( \beta^{s} )</td>
</tr>
<tr>
<td>( \phi b_{0} = 25 )</td>
<td>-0.0056</td>
<td>12</td>
<td>0.750</td>
<td>0.315</td>
<td>-0.022( \beta^{s} )</td>
</tr>
<tr>
<td>( \phi b_{0} = 5 )</td>
<td>-0.0002</td>
<td>13</td>
<td>0.750</td>
<td>0.327</td>
<td>-0.019( \beta^{s} )</td>
</tr>
</tbody>
</table>

\( \hat{N} \) stands for the critical maximum maturity above which \( V_{FC} < V_{D}^{\text{max}} \).
Benchmark parameter values: \( \beta = 0.95, \theta = 1/3, \gamma = 1, d = 100, \phi b_{0} = 10 \).

**Corollary 1.** If \( \phi b_{s} = 0 \) for all \( s \geq 0 \), then the best sustainable equilibrium is the full-commitment independently of the debt structure.

**Proof.** See Appendix A. \( \square \)
We have shown that, under some conditions, the debt structure is not irrelevant. Then, the next question is which is the optimal debt structure. In the following proposition, we identify when the government should issue short or long-term debt:

**Proposition 4.** Consider an initial debt $b_s > 0$ for some $s \geq 0$, and the maturities $i \in \Omega_s \subseteq \{1, 2, \ldots, N\}$ and $j \in \Omega_L \subseteq \{N + 1, \ldots, \infty\}$. We obtain the following results:

(a) If $\beta_i u'(c_i) \leq \beta_j u'(c_j)$ for some $i$ for all $j$ under full-commitment, then selling more debt at maturity $i$ may make the full-commitment sustainable.

(b) If $u'(c_i) \geq u'(c_j)$ for all $i$ for some $j$ under full-commitment, then selling more debt at maturity $j$ may make the full-commitment sustainable.

**Proof.** See Appendix A. □

Proposition 4 provides conditions under which it is optimal to issue more short or long-term debt. What do these conditions mean? Lemma 3 shows that the minimum deviation value is characterized by concentrating all debt in only one maturity. The conditions in (a) and (b) indicate the maturity that yields the minimum deviation value and, thus, the lowest incentives to deviate. Since the minimum deviation value may be higher than the one required to make the full-commitment sustainable, there may exist multiple optimal debt structures that sustain the full-commitment. In particular, it may be possible to eliminate the time-inconsistency problem by selling more of the “right” maturity or decreasing the amount sold at the other maturities rather than eliminating these maturities altogether.

Let us investigate what determines the conditions in (a) and (b). These conditions depend on how the marginal utility of consumption evolves over time and, thus, on the transitional dynamics of the economy. We obtain the following:

**Lemma 4.** If $b_i \geq b_j$, then $u'(c_i) \leq u'(c_j)$ under full-commitment.

**Proof.** See Appendix A. □

Lemma 4 shows that the initial maturity schedule is the determinant of the optimal management of debt. This lemma, in combination with Proposition 4, suggests that if the initial debt matures largely at date $i$, then selling more bonds at the maturity $i$ may sustain the full-commitment.

Figure 2 and Table 2 illustrate these results. Figure 2 depicts the value of deviating from the full-commitment at date 1 for all possible debt structures of one and two-period bonds for four economies that differ only in their initial debt. We graph the deviation and continuation values as a function of one-period bonds maturing at date 1 having the remaining debt covered with two-period bonds. Economies A and B have a larger proportion of debt maturing at date 2 and satisfy the condition in (b). There we see that having only two-period bonds makes the full-commitment sustainable. Economies C and D have an initial maturity schedule of debt concentrated in date 1 and satisfy the condition in (a). For these economies, we find that the minimum deviation value is obtained by issuing only one-period bonds. Moreover, as economies A and D show, concentrating all debt in only
The parameters are the following: $\beta = 0.95$, $\theta = 1/3$, $\gamma = 2$, and $d = 100$.

The table 2 shows the maximum and minimum deviation values for all possible debt structures with one- and two-period bonds.

<table>
<thead>
<tr>
<th>Initial debt</th>
<th>$V_C^{FC}$</th>
<th>$V_{min}^{D}$</th>
<th>$V_{max}^{D(1b_2T)}$</th>
<th>$V^{D(1b_1T)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) $0b_0 = 8$, $0b_1 = 0.4$, $0b_2 = 1$</td>
<td>198.5197</td>
<td>198.3555</td>
<td><strong>198.3555</strong></td>
<td>198.9733</td>
</tr>
<tr>
<td>(b) $0b_0 = 8$, $0b_1 = 0$, $0b_2 = 0.10$</td>
<td>199.2565</td>
<td>199.2561</td>
<td><strong>199.2561</strong></td>
<td>199.4118</td>
</tr>
<tr>
<td>(c) $0b_0 = 8$, $0b_1 = 0$, $0b_2 = 0$</td>
<td>199.3549</td>
<td>199.3566</td>
<td>199.3566</td>
<td>199.4388</td>
</tr>
<tr>
<td>(d) $0b_0 = 8$, $0b_1 = 0.065$, $0b_2 = 0$</td>
<td>199.3368</td>
<td>199.2754</td>
<td>199.3369</td>
<td>199.3956</td>
</tr>
</tbody>
</table>

The table 2 shows the maximum and minimum deviation values for all possible debt structures with one- and two-period bonds.

The parameters are the following: $\beta = 0.95$, $\theta = 1/3$, $\gamma = 2$, and $d = 100$.

$1b_2T$ stands for having only two-period bonds, thus, $1b_1 = 0$. Similarly for $1b_1T$.

bonds at the “right” maturity is not a requirement; the full-commitment can be sustained by simply selling less bonds at the other maturity.

We should discuss further the cases of default and of restrictions on the level debt. If the initial debt position entails a large rollover, then a balanced budget without default does not exist and the worst sustainable equilibrium is a balanced budget with default. There the debt structure is still relevant because it would be still optimal to concentrate debt in few maturities in order to maintain the default equilibrium as the worst sustainable in the
future. However, this reasoning would break down if, as in Cole and Kehoe (2000), these debt positions also introduced the possibility of a liquidity crisis in equilibrium. There the government would prefer to concentrate debt in few maturities but not too much in order to reduce the incentives to deviate but also avoid the crisis zone.

As noted earlier, the existence of restrictions on government debt may allow for other punishments after deviation. In particular, for large rollovers, the government may prefer to sell as much debt as possible after a deviation instead of defaulting on debt payments. If the upper limit on debt is sufficiently large, this equilibrium after a deviation may be quite severe and allow sustaining better outcomes. Therefore, in the presence of debt restrictions, the government would also have an incentive to concentrate debt in few maturities.

5. Conclusions

This paper has studied the time-inconsistency problem of optimal fiscal policy in an economy with a limited debt structure and a reputation mechanism. We have explored the role of the debt structure in relation to the time-inconsistency problem. We have obtained that the structure of government debt will be always relevant in economies with a large set of maturities of debt. We have shown that when there is an optimal debt structure that eliminates the time-inconsistency problem, this debt structure is generally simple, concentrated in few maturities rather than spread over many maturities. Moreover, we have identified to which maturities should the government debt be narrowed. We have found that if the government faces a maturity schedule with more debt obligations in the short run than in future periods, issuing more short term debt can enforce commitment.

These findings contrast with earlier results. We have found that reputation mechanisms and the backward induction method lead to time-consistent policies with very different properties. Using backward induction, Rogers (1989) showed that more simple debt structures (with a smaller set of maturities) increase the costs of the time-inconsistency problem. However, in contrast to Rogers (1989), we have found that simple debt structures may be the optimal response to time-inconsistency problems.

It would be interesting to consider different extensions. The natural extension of this model to a monetary economy would be that of Alvarez et al. (2004). They show that if the Friedman rule is optimal, then the economy becomes equivalent to a real economy and a maturity structure of debt indexed to consumption, like the one that Lucas and Stokey (1983) proposed, would make the Ramsey policy time-consistent. In this economy, we could introduce a reputation by threat of reversion to the worst sustainable equilibrium. It is clear that the worst sustainable equilibrium would still be autarky. Following the same

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17 Here we have considered a single default rate on all outstanding debt independently of the maturity. If the default rate had varied with the maturity and the default cost were a cost per default, there would be further incentives to concentrate debt in few maturities even when the default equilibrium is achieved.

18 This result raises some questions that are worth exploring. As Stockman (2004) shows, the presence of debt restrictions reduce the gains from issuing debt and make a default equilibrium most likely. However, if there is a direct cost of default, then these debt restrictions may also change the equilibrium after a deviation and default does not need to occur.
argument as above, if we consider economies where the Friedman rule is optimal, then this economy would become equivalent to the one we analyze in this paper, except for that the government has now also access to nominal debt. It is then clear that the debt structures we propose in this paper would still allow the government to carry out the Ramsey plan. Furthermore, given the convex costs of distortionary taxation, we would also obtain that the government could reduce the incentives to deviate by concentrating the debt (nominal and real) in few maturities.

An extension, where our results may not hold in general, would be to consider an economy with uncertainty. As Angeletos (2002) shows, in the absence of contingent debt, there would be a role for increasing the number of maturities since the new maturities could insure the economy against uncertain events.

Acknowledgments

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Appendix A. Proofs

Proof of Lemma 1. The first-order conditions for government bonds (16) yield $\lambda_t = \lambda_s$ for all dates $t$ and all maturities $s \geq t + 1$. This amounts to the same number of conditions as periods and as many variables to solve for as maturities in each period, $b_{k,t,s}$ for all $s \geq t + 1$. Therefore, there are infinite possible combinations of the debt structure satisfying the first-order conditions (16).

Proof of Proposition 1. We first show that a balanced budget is a sustainable equilibrium. Firstly, we prove the sustainability of closing down the market for government debt. If the government specifies prices $\{q^{B,t,s}\}_{s=t+1}^{\infty} \{q^{S,t,s}\}_{s=t+1}^{\infty}$ that are sufficiently high (low), then the individuals find optimal not to purchase (sell) any bonds. Next, if the individuals’ choice is not to trade any bonds, then the government is indifferent about the prices $\{q^{B,t,s}, q^{S,t,s}\}_{s=t+1}^{\infty}$. Secondly, if the initial debt $\{b_{i,t}\}_{i=t}^{\infty}$ allows, then a balanced budget without default is a sustainable equilibrium. Otherwise, a balanced budget with default is sustainable.

We next show that a balanced budget is the worst sustainable equilibrium. Suppose that there exists a sustainable equilibrium $(\sigma', f')$ that provides less welfare than a balanced budget. Given that the government can always achieve a balanced budget (either without or
with default) unilaterally, the government would optimally deviate to the balanced budget. Thus, the pair \((\sigma', f')\) fails to satisfy condition (ii) to be sustainable.

**Proof of Proposition 2.** Suppose first that a given policy plan and allocation rule are sustainable. Then, they must form a competitive equilibrium at date 0. Moreover, a reversion to the worst sustainable equilibrium is always feasible. Hence, the welfare value of this pair must be at least as large as the welfare value after deviation.

Next, suppose that a policy plan and allocation rule satisfy conditions (i) and (ii) of Proposition 2. We must show that they are sustainable. First, this pair is optimal since it is a competitive equilibrium at date 0 by condition (i). Second, no deviation will be desirable since, by condition (ii), they provide a greater welfare than any deviation. Thus, they form a sustainable equilibrium.

**Proof of Lemma 2.** We analyze the outcome after a deviation, which we mark with a superscript \(D\) to distinguish it from a sustainable outcome. The economy after a deviation is characterized by Eqs. (1), (18), (19) and (20) and yields a deviation value \(V^D(\{t^D_{bs}\}_{s=t}^\infty)\).

The deviation value is differentiable with respect to the initial debt \(t^D_{bs}\) for all maturities \(s\) because the welfare (2) and Eqs. (1), (18), (19) and (20) are continuously differentiable. The derivative of the deviation value with respect to \(t^D_{bs}\) is

\[
\frac{\partial V^D(\{t^D_{bs}\}_{s=t}^\infty)}{\partial t^D_{bs}} = \beta^{t-t'} \left[ u'(c_x) \frac{\partial c^D_x}{\partial t^D_{bs}} + v'(x_x) \frac{\partial x^D_x}{\partial t^D_{bs}} + z'(g_x) \frac{\partial g^D_x}{\partial t^D_{bs}} \right].
\]  
(23)

By implicit derivation of Eqs. (1), (18), (19) and (20), we obtain

\[
\left( \frac{\partial c^D_x}{\partial t^D_{bs}} \right) + \left( \frac{\partial x^D_x}{\partial t^D_{bs}} \right) + \left( \frac{\partial g^D_x}{\partial t^D_{bs}} \right) = 0,
\]  
(24)

\[
\left( \frac{\partial c^D_x}{\partial t^D_{bs}} \right) \left[ u' + u'(c^D_x - b_x) \right] + \frac{\partial x^D_x}{\partial t^D_{bs}} \left[ v' - v''(d - x^D_x) \right] = u',
\]  
(25)

\[
\left[ u'' + \lambda_x^D \left( 2u'' + u'''(c^D_x - b_x) \right) \right] \left( \frac{\partial c^D_x}{\partial t^D_{bs}} \right) + \left( \frac{\partial \lambda_x^D}{\partial t^D_{bs}} \right) \left( u' + u''(c^D_x - b_x) \right) - \frac{z''}{\partial t^D_{bs}} \right] = 0.
\]  
(26)

Solving for \(\frac{\partial c^D_x}{\partial t^D_{bs}}\) and \(\frac{\partial g^D_x}{\partial t^D_{bs}}\) as a function of \(\frac{\partial x^D_x}{\partial t^D_{bs}}\) in Eqs. (24) and (25) and then using the optimality conditions (19) and (20), the derivative (23) becomes

\[
\frac{\partial V^D(\{t^D_{bs}\}_{s=t}^\infty)}{\partial t^D_{bs}} = \beta^{t-t'} \left[ \frac{u'(u' - z')}{u' + u''(c^D_x - b_x)} \right] + \left( \frac{\partial x^D_x}{\partial t^D_{bs}} \right) \left[ \frac{(v''(d - x^D_x) - v')(u' - z')}{u' + u''(c^D_x - b_x)} + v' - z' \right].
\]
\[ = -\beta_s \lambda_s^D u'(c_s) < 0. \]

Second, since for the set of maturities \( \Omega' \supseteq \Omega \), the government could still issue only bonds with maturities \( \Omega \), it is obvious that \( V_{\text{max}}^D(\Omega') \geq V_{\text{max}}^D(\Omega) \) and \( V_{\text{min}}^D(\Omega') \leq V_{\text{min}}^D(\Omega) \).

**Proof of Example 1.** From the proof of Lemma 2, it follows that

\[
\frac{\partial^2 V_D}{\partial tbs \partial tbs}(\{tbs\}_{t=1}^{\infty}) = -\beta^s u'(c_s) \left( \frac{\partial \lambda_s^D}{\partial tbs} \right) + u''(c_s) \left( \frac{\partial c_s}{\partial tbs} \right),
\]

Suppose first that \( \partial c_s / \partial tbs \leq 0 \), then Eq. (25) implies \( \partial c_s / \partial tbs > 0 \). Next, subtracting Eq. (27) from (26) and using conditions (19) and (20), we get

\[
\left[ u'' + \lambda_s^D (2u'' + u''' (c_s - b_s)) \right] \left( \frac{\partial \lambda_s^D}{\partial tbs} \right) - \left[ u'' + \lambda_s^D (2v'' - v''' (d - x_s^D)) \right] \left( \frac{\partial x_s^D}{\partial tbs} \right)
\]

This allows us to write

\[
\frac{\partial^2 V_D}{\partial tbs \partial tbs}(\{tbs\}_{t=1}^{\infty}) = -\beta^s \left( A u' + (u'' - Bu') \left( \frac{\partial c_s}{\partial tbs} \right) \right).
\]

For the utility function (22), it is easy to see that \( A \) is positive whenever \( c \geq g \). From (19), we obtain \( c \geq g \) for all \( \theta \geq \gamma \). The term \( u'' - Bu' \) is positive because \( u'' - (\lambda_s^D u'/ (z' - u')) u'' > 0 \). Therefore, the condition \( \theta \geq \gamma \) is sufficient to guarantee that \( \partial^2 V_D(\{tbs\}_{t=1}^{\infty}) / (\partial tbs \partial tbs) < 0 \).

**Proof of Lemma 3.** To determine the debt structure that yields \( V_{\text{max}}^D(\Omega) \), we consider the full-commitment solution at date \( t \) and find the issues of debt \( \{b_{k,t}^h\}_{h \in \Omega} \) that maximize the deviation value (2) subject to constraints (1), (18), the implementability constraint at period \( t \):

\[
-K_t + \sum_{k=5} b_{s,t} \sum_{h \in \Omega} \beta^{s-t} u'(c_s) b_{k,t}^h = 0,
\]

(28)
where $K_t = -u'(c_t)(c_t - b_t) + v(x_t)(d - x_t) > 0$, and the non-negativity constraints. We obtain the following necessary and sufficient, since $\partial V^D((1,b_s)_{s=1}^\infty)/(\partial b_s \partial b_s) < 0$, conditions:

$$\Pi_t u'(c_t) = \lambda^D_t u'(c^D_s) \quad \text{for all } s \in \Omega,$$

where $\Pi_t$ is the Lagrange multiplier on constraint (28). The condition (29) implies the equalization of the marginal gain and cost from issuing one unit more of a bond with maturity $s$. The LHS of (29) is the marginal gain from relaxing the budget constraint (28) at date $t$; this gain is independent of the amount of debt maturing at date $s$. The RHS is the marginal cost from paying the debt at date $s$; this cost increases with the amount of debt maturing at date $s$. The condition (29) becomes $\Pi_t = \lambda^D_t u'(c^D_s)$ for all $s$, at steady state, which clearly implies $t+1_b = b > 0$ for all maturities $s \in \Omega$.

The concavity of $V^D((1,b_s)_{s=1}^\infty)$ guarantees that the minimum for a non-negative debt structure $b_s \geq 0$ will be at a corner solution, where all financial needs are covered by only one maturity. Therefore, the debt structure $\{b^k\}_{k=1}^\infty$ that attains $V^D_{\min}(\Omega)$ satisfies $t+1_b = K_t/(\beta^{s-t} u'(c_t)) > 0$, for a particular maturity $s$, and zero otherwise. □

Proof of Proposition 3. The proof of (i) is obvious. If $V^C_{FC} \geq V^D_{\max}(\Omega)$ holds for all dates $t$, then the full-commitment satisfies conditions (i) and (ii) of Proposition 2 to be sustainable for all possible debt structures. We know that $V^D_{\max}(\Omega)$ is not decreasing in $\Omega$ and $V^C_{FC}$ is independent of $\Omega$. Hence, for all $\Omega' \subset \Omega$, $V^C_{FC} \geq V^D_{\max}(\Omega')$ holds and the full-commitment is sustainable independently of the debt structure.

The proof of (ii) lies on the value of deviating without reputational costs, $V^D_{NRC}$. Given the time-inconsistency of the full-commitment, we know that $V^C_{FC} < V^D_{NRC}$. Moreover, we now show that for $\Omega_\infty = 1, \ldots, \infty$, there is a debt structure for which the deviation value equals that without reputational costs and, thus, exceeds the value of continuing with the full commitment, $V^C_{FC} < V^D((1,b_s)_{s=1}^\infty) = V^D_{NRC}$. Let us characterize $V^D_{NRC}$ for the debt structure $b_s = b$ for all $s \geq 1$. If the reputation costs were absent, then the optimal choices at date 1 would solve the resource constraint (1) and the following necessary conditions:

$$u'(c_1) + \lambda_1(u'(c_1) + u''(c_1)(c_1 - b)) = z'(g_1),$$
$$v'(x_1) + \lambda_1(v'(x_1) - v''(x_1)(d - x_1)) = z'(g_1),$$

for all $t$, where $\lambda_1$ is determined by the implementability condition:

$$u'(c_1)(c_1 - b) - v'(x_1)(d - x_1) + \sum_{s=2}^{\infty} \beta^{s-1}[u'(c_s)(c_s - b) - v'(x_s)(d - x_s)] = 0.$$

It is obvious that all variables are constant over time and that a balanced budget is satisfied. Hence, $V^C_{FC} < V^D((1,b_s)_{s=1}^\infty) = V^D_{NRC} \leq V^D_{\max}(\Omega_\infty)$ for this debt structure. In sum, from Lemma 4, it is easy to see that $V^D_{\max}(\Omega)$ is strictly increasing in $\Omega$. Moreover, we have now shown that, for $\Omega_\infty$, the maximum deviation value exceeds the value of continuing with the full-commitment. Then, there exists a finite maximum maturity $\Omega''$, $\Omega \subset \Omega''$, such that for all set of maturities $\Omega'' \subset \Omega''$, we obtain $V^C_{FC} < V^D_{\max}(\Omega'')$. Therefore, for all $\Omega''$, the full-commitment is still sustainable, however the debt structure is not longer irrelevant. □
Proof of Corollary 1. If \(q_b = 0\) for all \(s \geq 0\), then the full-commitment economy is always at steady state and the government runs a balanced budget. From Lemma 3, the debt structure maximizing the deviation value is \(b_d^i = \bar{b} = 0\) for all maturities \(s\), which yields the full-commitment outcome. Therefore, \(V_D^C = V_D^{\max}(\Omega) \geq V_D((b_i)_{i=1}^\infty)\) under all possible debt structures and the full-commitment policy is sustainable independently of the debt structure. \(\square\)

Proof of Proposition 4. Since the minimum deviation value is at a corner, it suffices to compare the maturity \(i \in \Omega_S \subseteq \{1, 2, \ldots, N\}\) that yields \(V_{\text{min}}^D(\Omega_S)\) and the maturity \(j \in \Omega_L \subseteq \{N + 1, \ldots, \infty\}\) that yields \(V_{\text{min}}^D(\Omega_L)\). The deviation value with only the two maturities \(i\) and \(j\) is

\[V_D((b_i)_{i=1}^{\infty}, j) = \beta^{i-1}W(b_i) + \beta^{j-1}W(b_j) + \left[\frac{1}{1 - \beta} - \beta^{i-1} - \beta^{j-1}\right]W(0),\]

where \(W(b_i)\) represents the per period utility in date \(s\). We study non-negative debt structures, thus \(i_b^s = q_b + b_k^s \geq 0\) for \(s = i, j\). The budget constraint at period 0

\[-K_0 + \beta^j u'(c_i)b_i^k + \beta^j u'(c_j)b_j^k = 0,\]

can be written as

\[i_b + (\frac{\beta^j u'(c_j)}{\beta^j u'(c_i)})b_j = q_b + \left(\frac{K_0}{\beta^j u'(c_i)}\right)b_j + \left(\frac{\beta^j u'(c_j)}{\beta^j u'(c_i)}\right)b_j.\]  

(30)

Hence, the maximum values that \(i_b\) and \(i_b\) can take are

\[i_bT = q_b + \left(\frac{K_0}{\beta^j u'(c_i)}\right)b_j \quad \text{and} \quad i_b = \left(\frac{\beta^i u'(c_i)}{\beta^i u'(c_j)}\right)b_b.\]

The deviation values at the two possible corners are

\[V_D(i_b) = \beta^{i-1}W(i_bT) + \left[\frac{1}{1 - \beta} - \beta^{i-1}\right]W(0),\]

\[V_D(i_b) = \beta^{j-1}W(i_b) + \left[\frac{1}{1 - \beta} - \beta^{j-1}\right]W(0).\]

If \(\beta^j u'(c_i) \leq \beta^j u'(c_j)\), then \(i_bT \leq i_b\), \(W(i_bT) \geq W(i_b)\) and

\[V_D(i_bT) - V_D(i_b) = \beta^{i-1}W(i_bT) - \beta^{j-1}W(i_b) - \left[\beta^{i-1} - \beta^{j-1}\right]W(0)\]

\[\leq \left[\beta^{i-1} - \beta^{j-1}\right]W(i_bT) - W(0) < 0.\]

Hence, if \(\beta^j u'(c_i) \leq \beta^j u'(c_j)\) for some \(i \in \Omega_S\) for all \(j \in \Omega_L\), then \(V_D(i_b) < V_D(i_bT)\), and thus, \(V_D^{min}(\Omega_S) < V_D^{min}(\Omega_L)\) holds. Therefore, the full-commitment may be sustained by selling more debt at maturity \(i \in \Omega_S \subseteq \{1, 2, \ldots, N\}\).

Next we consider \(u'(c_i) \geq u'(c_j)\). From Eq. (30), we get \(i_b = \frac{[\beta^j u'(c_i)]/[\beta^j u'(c_j)]}{(i_bT) - i_b}\), which allows us to write the deviation value as a function of only \(i_b\):

\[V_D((b_i)_{i=1}^{\infty}, j) = \beta^{i-1}W(i_b) + \beta^{j-1}W\left(\left[\frac{\beta^j u'(c_i)}{\beta^j u'(c_j)}\right](i_bT) - i_b\right)\]

\[+ \left[\frac{1}{1 - \beta} - \beta^{i-1} - \beta^{j-1}\right]W(0).\]
This has an inverted-U shape with a maximum at $b_i^*$, which satisfies

$$\left(1 + \frac{\beta_i' u'(c_i)}{\beta_j' u'(c_j)}\right) b_i^* = b_i T.$$ 

Plugging the value $b_i T$ into the deviation value, we get

$$V^D(\{1 b_i\}_{i=1}^n) = \beta^{i-1} W(1 b_i) + \beta^{j-1} W\left(\left(\frac{\beta_i' u'(c_i)}{\beta_j' u'(c_j)} + 1\right) b_i^* - \left(\frac{\beta_i' u'(c_i)}{\beta_j' u'(c_j)}\right) 1 b_i\right) + \left[\frac{1}{1 - \beta} - \beta^{i-1} - \beta^{j-1}\right] W(0). \quad (31)$$

We now show that the slope of the deviation value (31) is flatter on the RHS of the maximum $b_i^*$ and, given that $b_i^*$ is closer to $1 b_i T$ than to 0, then $V^D(1 b_i T) > V^D(1 b_j T)$ holds.

On the LHS of $b_i^*$, at $1 b_i L = b_i^* - z$, $1 b_j L = b_i^* + \left(\frac{\beta_i' u'(c_i)}{\beta_j' u'(c_j)}\right) z$,

we have

$$V(\{1 b_i\}_{i=1}^n) = \beta^{i-1} W(b_i^* - z) + \beta^{j-1} W\left(b_i^* + \frac{\beta_i' u'(c_i)}{\beta_j' u'(c_j)} z\right) + \left[\frac{1}{1 - \beta} - \beta^{i-1} - \beta^{j-1}\right] W(0), \quad (32)$$

On the RHS, at $1 b_i R = b_i^* + z$, $1 b_j R = b_j^* - \left(\frac{\beta_i' u'(c_i)}{\beta_j' u'(c_j)}\right) z$,

we have

$$V(\{1 b_i\}_{i=1}^n) = \beta^{i-1} W(b_i^* + z) + \beta^{j-1} W\left(b_i^* - \frac{\beta_i' u'(c_i)}{\beta_j' u'(c_j)} z\right) + \left[\frac{1}{1 - \beta} - \beta^{i-1} - \beta^{j-1}\right] W(0), \quad (33)$$

If $u'(c_i) \leq u'(c_j)$, then $1 b_i L > 1 b_i R > 1 b_j L > 1 b_j R$ and similarly for $\lambda_i^D u'(c_i)$. Moreover, since $1 b_i L - 1 b_i R = 1 b_i L - 1 b_i R$, we get

$$\left(\lambda_i^D u'(c_i^L) - \lambda_i^D u'(c_i^R)\right) > \left(\lambda_j^D u'(c_j^L) - \lambda_j^D u'(c_j^R)\right),$$

which implies that the sum of both derivatives, (32) and (33), is positive. Thus, the slope is flatter on the RHS and then $V^D(1 b_i T) > V^D(1 b_j T)$. Hence, if $u'(c_i) \leq u'(c_j)$ for all $i \in S$. 


for some \( j \in \Omega_L \), then \( V^D_{\min}(\Omega_L) < V^D_{\min}(\Omega_S) \) holds. Therefore, the full-commitment may become sustainable by selling more debt at maturity \( j \in \Omega_L \subseteq \{N + 1, \ldots, \infty\} \). □

**Proof of Lemma 4.** If the full-commitment is sustainable, then the economy is described by constraints (1), (4), (11), and (12), and the conditions (13)–(15), which can be written as

\[
\begin{align*}
  z'(g_t) &= u'(c_t) + \lambda_0 \left[ u'(c_t) + u''(c_t)(c_t - qb_t) \right], \\
  z'(g_t) &= v'(x_t) + \lambda_0 \left[ v'(x_t) - v''(x_t)(d - x_t) \right],
\end{align*}
\]

(34)

(35)

for all dates \( t \). Let us compare the economy at date \( i \) and \( j \) when \( qb_i \geq qb_j \). Observe that the LHS of (35) is decreasing in \( g_t \) and the RHS is decreasing in \( x_t \). Thus, if \( g_i \leq (\geq) g_j \), then \( x_i \leq (\geq) x_j \) and, using the resource constraint (1), \( c_i \geq (\leq) c_j \). Moreover, the LHS of (34) is decreasing in \( g_t \) and the RHS is decreasing in \( c_t \). Hence, if \( qb_i \geq qb_j \), then \( g_i \leq g_j \) and \( c_i \geq c_j \), and therefore, \( u'(c_t) \leq u'(c_j) \). □

**References**