Public debt and optimal taxes without commitment

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Abstract

Benhabib and Rustichini [Optimal taxes without commitment, J. Econ. Theory 77 (1997) 231–259] study the properties of optimal capital taxes in economies without commitment and no government debt. They find that capital taxes may be different from zero at steady state. This note shows that, once governments have the possibility of issuing debt and smoothing taxes over time, optimal steady state capital taxes turn out to be zero.

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1. Introduction

The properties of optimal taxation of capital income are well known in environments with full-commitment. Chamley [4] and Judd [5] showed that, when markets are complete, it is optimal to tax capital heavily in the short run when the distortions on capital accumulation are low and not to tax capital income in the long run when those distortions are large. This result is very robust and has been extended to many different scenarios. However, as Chamley [4] suggests, this argument may break down in environments without commitment. This is so because without commitment future governments are tempted to renege on the announced policy plan and take advantage of a capital levy.

This problem is examined by Benhabib and Rustichini [3]. They consider a model with competitive agents and a benevolent government that must tax capital and labor income to finance...
an exogenous stream of government spending in an environment without commitment. This government selects the optimal time-consistent taxes by choosing the policy that maximizes the individual’s welfare subject to the standard feasibility and implementability constraints and an incentive compatibility constraint that embeds the future governments’ incentives. This constraint says that, for all future governments, the welfare value of continuing with the policy that the current government announces must be at least as large as the welfare value of deviating from that policy. Any policy that satisfies this constraint is clearly time-consistent.

Benhabib and Rustichini [3] characterize the optimal time-consistent capital taxes. To do so, they first consider an economy with government debt. However, for simplification purposes, they restrict attention to economies where governments cannot issue debt. In this context, they obtain that optimal capital taxes may be different from zero at steady state. In particular, they show that capital may be subsidized. The intuition for this result is that a subsidy to capital accumulation leads to a very high stock of capital. This stock of private capital is so high that the fear to lose its productive gains deters future governments from deviating from its policy announcement and taxing its rents.

This note reexamines the problem of capital taxation without commitment in an economy with government bonds. We obtain that once governments have the possibility of issuing debt and smoothing taxes over time, optimal steady state capital taxes turn out to be zero. The explanation for this result lies in the large distorting effects of a tax/subsidy on capital income in the long run. As Atkinson et al. [2] observe, a positive (negative) tax on capital at steady state is equivalent to an ever increasing tax (subsidy) on consumption. That policy is so distortionary that it is not optimal if the government has an instrument to smooth the cost of distortionary taxation over time.

Our results have also an intimate connection with those of [3], we show that the issues of government debt during the transition help build a sufficiently high level of capital so that future governments have no incentive to deviate from the long-run zero capital taxes. Thus, in our economy, the subsidies to capital are substituted by government debt as the central commitment device among governments.

The rest of the note is organized as follows. Section 2 presents the economy at steady state. Section 3 characterizes the optimal time-consistent taxes. Section 4 concludes.

2. The economy at steady state

We consider the economy in Benhabib and Rustichini [3]. We focus on the equilibrium at the steady state and make the distinction between incentive-constrained steady states (where the incentive compatibility constraints (2.5) bind) and incentive-unconstrained steady states (where those constraints do not bind). The properties of the optimal taxes without commitment depend crucially on this distinction.

The government’s problem is defined by the Lagrangian (2.6), which yields the first order conditions (2.7)–(2.13). Since we focus on the role of government debt, it is worth examining the necessary condition for debt:

\[-\xi_{t-1} + \beta r \xi_t = \beta \gamma t \beta^{-1} V_b^D (k_t, b_t) .\]

where \(\xi_t\) and \(\gamma t \beta^{-1}\) are, respectively, the Lagrange multipliers on the budget constraint (2.3) and the incentive compatibility constraint (2.5). The first multiplier can be interpreted as the cost of

\(^2\) We do not specify the equations of the model and, in consequence, we refer to those in [3] throughout the note.

\(^3\) Eq. (2.9) in [3] has a small typo; the RHS of this equation should be multiplied by the discount factor \(\beta\).
distortionary taxation and the second as the incentives to deviate from the announced policy. At steady state, we have $\beta r = 1$, and

$$\bar{\xi}_t = \bar{\xi}_{t-1} + \beta \gamma_t \beta^{-t} V^D_k (k, b) .$$

Therefore, if the steady state is incentive unconstrained, i.e. $\gamma_t = 0$, Eq. (1) says that government debt is issued to ensure a complete smoothing of the distortionary taxation over time. Otherwise, if the steady state is incentive constrained, then government debt is issued to smooth taxes over time taking into account the effect of the level of debt on the deviation value.

Once we have clarified the role of government debt, we provide three lemmas that will help us determine the properties of the optimal capital taxes at steady state. The first lemma considers steady states $x$ that are incentive unconstrained:

**Lemma 1.** If $x$ is incentive unconstrained, then $\tau^k = 0$.

**Proof.** See the Appendix.

As Lemma 1 states, the optimal capital tax rate is zero at incentive-unconstrained steady states. If the steady state is incentive constrained, the properties of the steady state taxes may be quite different. In order to characterize the steady state, we find, as [3] do, an optimality condition for being at incentive constrained steady states. Under this condition, the Lagrange multipliers $\{\gamma_t\}_t$ associated with the incentive compatibility constraints (2.5) are positive, summable, and converging to zero, which are necessary conditions for optimality.

**Lemma 2.** If $x$ is incentive constrained, then there exists a real number $1 + a \in (0, 1/\beta)$, where $\gamma_{t+1} \beta^{-(t+1)} = (1 + a) \gamma_t \beta^{-t}$.

**Proof.** See the Appendix.

Lemma 2 shows some optimality conditions for incentive-constrained steady states. From Lemma 1, we know that capital taxes are optimally zero at incentive-unconstrained steady states. Are capital taxes different from zero at incentive-constrained steady states? As the next lemma shows, some incentive constrained steady states have the property of an optimal zero capital tax:

**Lemma 3.** If $x$ is incentive constrained and $1 + a \in (0, 1)$, then $\tau^k = 0$.

**Proof.** See the Appendix.

From Lemma 3, we can conclude that being at an incentive-constrained steady state is not a sufficient condition for optimal capital taxes different from zero. In fact, some incentive-constrained steady states satisfy the Chamley–Judd result.

To understand these results better, let us have a look at the first order condition for capital at steady state:

$$ \left( \bar{\xi}_t + \eta_t \right) (r - f_k (k, L)) = \gamma_t \beta^{-t} V^D_k (k, b) - \frac{1}{\beta} (-\eta_t + \eta_{t-1}) .$$

This equation implies an optimal zero capital tax whenever the RHS of (2) equals zero. Notice that, if the steady state is incentive unconstrained or incentive constrained with $1 + a \in (0, 1)$,
the term $\gamma t_1 \beta^{-t}$, which measures the incentives to deviate, is zero or becomes negligible over time. This combined with the first order condition for debt (1) implies that the marginal value of the tax burden is or becomes constant and, in turn, through the first order condition for consumption (5.52), so does the marginal utility of the assets, i.e. $\xi_t = \xi_{t-1}$ and $\eta_t = \eta_{t-1}$, respectively. As in Chamley [4], we then have that the marginal excess burden measured in units of private consumption is constant over time. Consequently, and as can be drawn by simple observation of condition (2), we recover Chamley’s result of optimal zero capital taxes at steady state. In the next section we explore the role of government debt in the optimality of a constant excess burden of taxation and, thus, a zero capital tax at steady state.

3. Optimal capital taxes at steady state

The solution to the government’s problem (2.6) depends on the properties of the deviation. In particular, the relationship between the level of debt and the deviation plays a crucial role in the properties of the optimal capital taxes. How is that relationship? In an economy with some positive level of government debt, the government is always tempted to deviate by defaulting on debt so as to lower the distortionary taxes. When that default is complete, we obtain the following result:

**Proposition 1.** If the deviation is characterized by a complete default on debt, then the steady state $x$ is incentive unconstrained and, therefore, $\tau^k = 0$.

**Proof.** See the Appendix.

The intuition for this result is that whenever a deviating government defaults completely on the outstanding debt, the equilibrium after the deviation is independent of the level of debt. There the first order condition (1) implies that the only role for government debt is to smooth completely the tax burden over time. This requires a zero capital tax at steady state.

Since, in Benhabib and Rustichini [3], capital and debt are completely substitutable and share the same tax rate, a complete default on debt also means a 100% capital levy, which is captured in their model through the assumption of a zero lower bound on the after-tax return on both assets, $r \geq r_{\text{min}} = 0$. Notice, however, that when capital and debt have a different tax rate, a default on debt still makes smoothing the tax burden the only objective of an optimal management of debt and Proposition 1 remains true.

If the default is partial, then the deviation value would depend on the level of debt. This is so because, after the deviation, the government would still need to raise taxes to pay back the remaining debt obligations. To study the properties of the optimal taxes, we examine a generalization of an example in Benhabib and Rustichini [3]. We consider an economy with positive levels of capital and government debt. We assume that the after-tax return on both capital and debt is close to zero but strictly positive $r_{\text{min}} > 0$, therefore, the default on debt is not complete. We specify the consequences of a deviation as such that individuals recognize the incentives to tax heavily capital and to partially default on debt and individuals decide not to save either in the form of

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4 In this context, a complete default on debt occurs even when debt is negative as long as the total assets, the sum of debt and capital, is positive in the economy.

5 Benhabib and Rustichini’s [3] example is for an economy that allows a 100% taxation of capital income, i.e. $r_{\text{min}} = 0$, but does not allow for government debt. We relax both assumptions.
private capital or government debt. To allow this equilibrium to be sustainable, we need a $r_{\min}$ sufficiently low and a production function for which capital is not essential, for example the one considered by Benhabib and Rustichini [3] 6:

\[ f(k, L) = A(\varepsilon)k + BL + \varepsilon k^2 l^{1-z} \quad \text{with} \quad A(\varepsilon) = \frac{1}{\beta} - z\varepsilon^p > 0, \]  

(3)

where $z \in (0, 1)$, $B > 0$, $\varepsilon \geq 0$, $z > 0$ and $p > 0$. We consider the utility function

\[ u(c_t) = c_t^{1-\sigma} \quad \text{and} \quad v(L_t) = \frac{L_t^{1+e}}{1+e}, \]  

(4)

with $\sigma \geq 0$ and $e > 0$. For this example, we obtain the following result:

**Proposition 2.** There exists $\bar{r} > 0$ such that if $r_{\min} \leq \bar{r}$, then $\tau^k = 0$.

**Proof.** See the Appendix.

The intuition for this result is that when a deviation is characterized by a partial default on debt, the issues of government debt have two effects: smoothing taxes and affecting the incentives to deviate. If that default is partial but sufficiently large, i.e. $r_{\min}$ sufficiently low, then the first effect dominates and the optimal capital taxes are zero at steady state. 7

We now illustrate our results with some numerical examples. We first consider some steady states studied in Benhabib and Rustichini [3]. As Table 1 shows, for those parameter values, we obtain that negative levels of debt support optimal zero capital taxes at steady state. Moreover, the stock of private capital has increased. Next, in Table 2, we consider economies with positive levels of debt at steady state. 8 There we see that a lower positive level of debt can support a higher level of capital and a zero capital tax at steady state. 9 To sum up, for a given level of debt, the government reduces the amount of debt to such a point that it is no longer worthwhile for the government to deviate from a zero capital tax. This is so because a lower level of debt reduces the benefits of a default and, moreover, increases the level of capital making more attractive continuing with the announced policy.

6 If the production function were Cobb–Douglas, then the worst sustainable equilibrium would be non-Markov. That equilibrium could be characterized using the APS method, developed by Abreu et al. [1] and extended to dynamic games by Phelan and Stacchetti [6] for economies with private capital. Due to computational difficulties, this method has not been applied yet to economies with both capital and debt. However, if the tax rates on debt and capital were different and a complete default on debt were allowed, then the worst sustainable equilibrium would be characterized by a complete default on government debt, Proposition 1 would hold and capital taxes would be zero.

7 If a deviation is not characterized by a large or complete default on debt, then the two roles of government debt would be both present and may be strong. There the discount factor could be important to determine which effect dominates and whether the optimality of zero capital taxes at steady state holds. All other results are independent of the discount factor.

8 A positive level of government debt may occur at steady state under different circumstances, for example, when the initial government is not allowed to default on the initial debt or when the stream of government spending is such that involves higher spending in the transition than at steady state.

9 The assumption of a common tax on debt and capital affects critically the results in Table 1, but does not affect the results in Table 2. If different tax rates were considered, a government would never default on negative debt. We would need then to specify other consequences after a deviation. Since, for that deviation, government debt would still have the objective of smoothing the tax burden over time, the result of optimal zero capital taxes could hold.
Table 1
Steady states from Benhabib and Rustichini [3]a,b

<table>
<thead>
<tr>
<th>ε</th>
<th>(k, b)</th>
<th>τL</th>
<th>τk</th>
<th>Time-consistent</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0001</td>
<td>(0.0367, 0.0000)</td>
<td>0.3997</td>
<td>0</td>
<td>No</td>
</tr>
<tr>
<td></td>
<td>(0.0368, −0.034)</td>
<td>0.3989</td>
<td>0</td>
<td>Yes</td>
</tr>
<tr>
<td>0.001</td>
<td>(0.0879, 0.0000)</td>
<td>0.3996</td>
<td>0</td>
<td>No</td>
</tr>
<tr>
<td></td>
<td>(0.0881, −0.061)</td>
<td>0.3981</td>
<td>0</td>
<td>Yes</td>
</tr>
<tr>
<td>0.002</td>
<td>(0.1003, 0.0000)</td>
<td>0.3994</td>
<td>0</td>
<td>No</td>
</tr>
<tr>
<td></td>
<td>(0.1005, −0.056)</td>
<td>0.3980</td>
<td>0</td>
<td>Yes</td>
</tr>
<tr>
<td>0.0025</td>
<td>(0.1047, 0.0000)</td>
<td>0.3993</td>
<td>0</td>
<td>No</td>
</tr>
<tr>
<td></td>
<td>(0.1049, −0.052)</td>
<td>0.3981</td>
<td>0</td>
<td>Yes</td>
</tr>
</tbody>
</table>

a For ε = 0.00001, we obtain a steady state capital of 0.03675. For a capital stock k = 0.00001, we find τk = 0.0089.
bParameter values: σ = 0, ϵ = 1.15, β = 0.95, G = 2, α = 0.34, B = 3, z = 1, p = 7/8 and rmin = 0.

Table 2
Steady state with an optimal zero capital tax and positive debta

<table>
<thead>
<tr>
<th>ε</th>
<th>(k, b)</th>
<th>τL</th>
<th>τk</th>
<th>Time-consistent</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0075</td>
<td>(0.1280, 0.2283)</td>
<td>0.4043</td>
<td>0</td>
<td>No</td>
</tr>
<tr>
<td></td>
<td>(0.1290, 0.0082)</td>
<td>0.3987</td>
<td>0</td>
<td>Yes</td>
</tr>
<tr>
<td>0.01</td>
<td>(0.1360, 0.0857)</td>
<td>0.4002</td>
<td>0</td>
<td>No</td>
</tr>
<tr>
<td></td>
<td>(0.1362, 0.0429)</td>
<td>0.3991</td>
<td>0</td>
<td>Yes</td>
</tr>
<tr>
<td>0.02</td>
<td>(0.1540, 0.3706)</td>
<td>0.4056</td>
<td>0</td>
<td>No</td>
</tr>
<tr>
<td></td>
<td>(0.1550, 0.1930)</td>
<td>0.4011</td>
<td>0</td>
<td>Yes</td>
</tr>
<tr>
<td>0.05</td>
<td>(0.1830, 0.7027)</td>
<td>0.4082</td>
<td>0</td>
<td>No</td>
</tr>
<tr>
<td></td>
<td>(0.1832, 0.6783)</td>
<td>0.4076</td>
<td>0</td>
<td>Yes</td>
</tr>
</tbody>
</table>

aParameter values: σ = 0, ϵ = 1.15, β = 0.95, G = 2, α = 0.34, B = 3, z = 1, p = 7/8 and rmin = 0.

4. Conclusions

Benhabib and Rustichini [3] showed that the optimal time-consistent capital taxes may be different from zero at steady state. This notes demonstrates that their result comes from the lack of both a commitment technology and an instrument to smooth taxes over time. Once governments have the possibility of issuing debt and smoothing distortionary taxes over time, steady state capital taxes are zero.

This note is related to Phelan and Stacchetti [6]. They study the properties of capital taxation without commitment and no government debt. In particular, they compute the whole set of sustainable equilibria and show that if the incentive compatibility constraints are binding, then capital taxes may be different from zero at steady state. This note shows that, when we allow for government debt, those incentive constraints may not bind at steady state.
We believe our results are quite robust and could be extended to more complex frameworks. As commented earlier, if the level of government debt is positive, then the optimal punishment after a deviation, namely, the worst sustainable equilibrium, would be generally characterized by a complete default on government debt. If that is the case, then the sole objective of the optimal management of government debt at steady state is to smooth completely the tax burden over time and, as we have shown, that requires a zero tax rate on capital income.

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Appendix

The steady state is characterized by the first order conditions (2.17) and (3.14)–(3.18) and the constraints (3.19)–(3.23). This system can be simplified by plugging (3.16), (3.17) and (3.20) into the remaining equations, yielding the following set of equations\(^{10}\):

\[
w u'(c) = v'(L),
\]

\[
\left(\frac{1}{\beta} - 1\right) (k + b) + wL = c,
\]

\[
k + c + G = f(k, L),
\]

\[
\left(\frac{1}{1 - \beta}\right) (u(c) - v(L)) \geq V^D(k, b),
\]

\[
\eta_t = -\left(1 + \beta^{-t} (\gamma * \beta)_t\right) u'(c) - \sigma_c \zeta_t + \left(\frac{k + b}{c}\right) \sigma_c \beta \gamma_{t+1} \beta^{-(t+1)} V^D_b(k, b),
\]

\[
\xi_t ((1 + \sigma_L) w - f_L(k, L)) - \left(1 + \beta^{-t} (\gamma * \beta)_t\right) v'(L) = \eta_t f_L(k, L),
\]

\[
(\xi_t + \eta_t) \left(\frac{1}{\beta} - f_k(k, L)\right) = \gamma_t \beta^{-t} V^D_k(k, b) - \frac{1}{\beta} (-\eta_t + \eta_{t-1}),
\]

\[
\xi_t = \xi_{t-1} + \beta \gamma_t \beta^{-t} V^D_b(k, b),
\]

\[\boxed{\text{Appendix}}\]

\[\boxed{\text{We obtain (ss.5) as opposed to (5.52), because Eq. (2.9) entails } \gamma_{t+1} \beta^{-(t+1)} \text{ instead of } \gamma_t \beta^{-t}.}\]
with \( \sigma_c \equiv -\frac{u''(c)c}{u'(c)} \) and \( \sigma_L \equiv \frac{u''(L)L}{v'(L)} \). Next, combining (ss.5) and (ss.6), we obtain

\[
\xi_t = (1 + \beta^{-t} (y * \beta)_t) D + \gamma_{t+1} \beta^{-(t+1)} F, \quad \text{with}
\]

\[
D \equiv \frac{u'(c) (w - f_L(k, L))}{((1 + \sigma_L) w - (1 - \sigma_c) f_L(k, L))} < 0 \quad \text{and}
\]

\[
F \equiv \frac{f_L(k, L) \sigma_c \left(\frac{k+b}{c}\right) \beta V_b^D(k, b)}{((1 + \sigma_L) w - (1 - \sigma_c) f_L(k, L))} \leq 0,
\]

which is our corrected version of Eq. (5.63). \(^{11}\)

**Proof of Lemma 1.** If the incentive compatibility constraint (ss.4) is non-binding, then the Kuhn–Tucker conditions imply \( \gamma_t = 0 \). The first order condition for capital (ss.7) can be written as

\[
\text{Kuhn–Tucker conditions imply}
\]

\[
\text{Proof of Lemma 1.}
\]

If the incentive compatibility constraint (ss.4) is non-binding, then the Lagrange multiplier \( \xi_t \) is constant and, through (ss.5), the Lagrange multiplier \( \eta_t \) is also constant. Therefore, the first order condition for capital (ss.7) can be written as \( (\xi_t + \eta_t) \left( \frac{1}{\beta} - f_k(k, L) \right) = 0 \), which yields \( t^k = 0 \). \( \square \)

**Proof of Lemma 2.** We next show that there exists a real number \( a \) such that \( \gamma_{t} \beta^{-t} = (1 + a) \gamma_{t-1} \beta^{-(t-1)} \), or equivalently, \( \gamma_{t} \beta^{-t} = (1 + a)t \gamma_{0} \). We manipulate Eq. (5) to obtain

\[
\xi_t = \xi_{t-1} + \gamma_{t+1} \beta^{-(t+1)} F + \gamma_t \beta^{-t} (D - F),
\]

which combined with (ss.8) yields

\[
\gamma_{t+1} \beta^{-(t+1)} F = \gamma_t \beta^{-t} \left( \beta V_b^D(k, b) - D + F \right).
\]

The term \( F \) equals zero whenever \( V_b^D(k, b) = 0 \) or \( \sigma_c = 0 \). If \( V_b^D(k, b) = 0 \), then \( \gamma_t = 0 \), as shown later. Since we focus on incentive constrained steady states, let \( V_b^D(k, b) \neq 0 \). If \( \sigma_c > 0 \), we obtain

\[
\gamma_{t+1} \beta^{-(t+1)} = \gamma_t \beta^{-t} \left( \frac{\beta V_b^D(k, b) - D + F}{F} \right) \quad \text{and} \quad 1 + a = \frac{\beta V_b^D(k, b) - D + F}{F}.
\]

If \( \sigma_c = 0 \), then \( \gamma_t \beta^{-t} (\beta V_b^D(k, b) - D) = 0 \), which implies either \( \gamma_t = 0 \) or \( V_b^D(k, b) = \frac{1}{\beta} D \). Suppose that \( V_b^D(k, b) = \frac{1}{\beta} D \), then plugging (ss.5) and (5), Eq. (ss.7) becomes

\[
(1 + \beta^{-t} (y * \beta)_t) (D - 1) \left( \frac{1}{\beta} - f_k(k, L) \right) = \gamma_t \beta^{-t} \left( V_k^D(k, b) - \frac{1}{\beta} \right),
\]

which yields

\[
\gamma_{t+1} \beta^{-(t+1)} \left( V_k^D(k, b) - \frac{1}{\beta} \right) = \gamma_t \beta^{-t} \left( V_k^D(k, b) - \frac{1}{\beta} \right).
\]

\(^{11}\) The signs of \( D \) and \( F \) come from the optimality of \( t^l \in \left( 0, \min \left[ \frac{\sigma_c + \sigma_L}{1 + \sigma_L}, 1 \right] \right) \) and that \( V_b^D(k, b) < 0 \).
Thus, if $V^D_k = \frac{1}{\beta}$, then $\tau^k = 0$. Otherwise, $1 + a \equiv \frac{V^D_k(k,b) - \frac{1}{\beta}}{V^D_k(k,b) - \frac{1}{\beta} - (1 + D)(f_k(k,l) - \frac{1}{\beta})}$. Therefore, incentive constrained steady states satisfy

$$1 + a \equiv \begin{cases} \frac{\beta V^D_b(k, b) - D + F}{F} & \text{for } \sigma_c > 0, \\ V^D_k(k, b) - \frac{1}{\beta} - (1 + D)(f_k(k, l) - \frac{1}{\beta}) & \text{for } \sigma_c = 0. \end{cases}$$

**Proof of Lemma 3.** The proof of Lemma 2 shows that a real number $a$ exists such that $\gamma_t \beta^{-t} = (1 + a)^t \gamma_0$. If $1 + a \in (0, 1)$, then $\gamma_t \beta^{-t}$ approaches zero and, in turn, $1 + \beta^{-t} (\gamma_\star \beta)_t$ converges to a positive constant. From conditions (ss.8) and (ss.5), we obtain that both $\xi_t$ and $\eta_t$ become constant. Therefore, by simple observation of the first order condition for capital (ss.7), $(\eta_t + \xi_t) \left( \frac{1}{\beta} - f_k(k, l) \right)$ approaches zero and, thus, $\tau^k = 0$ at the steady state. □

**Proof of Proposition 1.** If a deviation is characterized by a complete default on government debt, then the deviation value is independent of the level of debt, i.e. $V^D_b(k, b) = 0$. Then, the first order condition for debt (ss.8) can be written as $\tau^k = \tau^k = 0$ at steady state. Next, Eq. (5) implies that $1 + \beta^{-t} (\gamma_\star \beta)_t$ must be also constant, which, in turn, means that $\gamma_t \beta^{-t}$ must be zero for all $t$ at steady state. Therefore, the steady state is incentive unconstrained and $\tau^k = 0$. □

**Proof of Proposition 2.** We prove this proposition by showing that there exists $\tilde{r} > 0$ such that if $r_{\min} \leq \tilde{r}$, then $V^D_b(k_t, b_t) > \frac{1}{\beta} D$, which implies $1 + a = 0$ for $\sigma_c = 0$, and $1 + a < 1$ for $\sigma_c > 0$; and, in turn, $\tau^k = 0$. Let us denote the variables after a deviation with a superscript $d$ and compute $V^D_b(k_t, b_t)$. The economy after a deviation is characterized by

$$r_{\min}(k_t + b_t) + u^d_t L^d_t = c^d_t, \quad (6)$$

$$u^d_t u'(c^d_t) = v'(L^d_t), \quad (7)$$

$$c^d_t + G = f(k_t, L^d_t), \quad (8)$$

at date $t$, and

$$w^d_{t+s} L^d_{t+s} = c^d_{t+s},$$

$$u^d_{t+s} u'(c^d_{t+s}) = v'(L^d_{t+s}),$$

$$c^d_{t+s} + G = f(0, L^d_{t+s}),$$

at date $t + s$ for all $s > 0$. Thus, the welfare after a deviation at date $t$ is

$$V^D(k_t, b_t) = (c^d_t)^{1-\sigma} \frac{1}{1-\sigma} - \frac{(L^d_t)^{1+e}}{1+e} + \left[ \frac{\beta}{1-\beta} \right] \left( \frac{(c^d_{t+s})^{1-\sigma}}{1-\sigma} - \frac{(L^d_{t+s})^{1+e}}{1+e} \right).$$
which allows us to write
\[
V^D_b (k_t, b_t) = \frac{\partial V^D (k_t, b_t)}{\partial b_t} = \left( c^d_t \right)^{-\sigma} \frac{\partial c^d_t}{\partial b_t} - \left( L^d_t \right) \frac{e}{\alpha} \frac{\partial L^d_t}{\partial b_t}.
\] (9)

For the utility function (4), the system (6)–(8) can be written as
\[
\begin{align*}
 r_{\min} (k_t + b_t) + \left( c^d_t \right)^{\sigma} \left( L^d_t \right)^{1+\epsilon} - c^d_t &= 0, \\
 c^d_t + G - f \left( k_t, L^d_t \right) &= 0,
\end{align*}
\] (10)

which by implicit differentiation yields
\[
\begin{align*}
 &- \left( 1 - \sigma \left( c^d_t \right)^{-1} \left( L^d_t \right)^{1+\epsilon} \right) \frac{\partial c^d_t}{\partial b_t} + (1 + \epsilon) \left( c^d_t \right)^{\sigma} \left( L^d_t \right) \frac{e}{\alpha} \frac{\partial L^d_t}{\partial b_t} = -r_{\min}, \\
 &\frac{\partial c^d_t}{\partial b_t} - f \left( k_t, L^d_t \right) \frac{\partial L^d_t}{\partial b_t} = 0.
\end{align*}
\] (11)

Solving for \( \frac{\partial c^d_t}{\partial b_t} \) and \( \frac{\partial c^d_t}{\partial b_t} \), the derivative (9) becomes
\[
V^D_b (k_t, b_t) = r_{\min} \frac{u' \left( c^d_t \right) \left( w^d_t - f \left( k_t, L^d_t \right) \right)}{(1 + e) \left( 1 - \sigma \right) f \left( k_t, L^d_t \right) - \sigma r_{\min} \left( \frac{k_t + b_t}{c^d_t} \right) f \left( k_t, L^d_t \right)}.
\] (12)

The inequality \( V^D_b (k_t, b_t) > \frac{1}{\beta} \) can be written as
\[
\begin{align*}
 r_{\min} \frac{u' \left( c^d_t \right) \left( w^d_t - f \left( k_t, L^d_t \right) \right)}{(1 + e) \left( 1 - \sigma \right) f \left( k_t, L^d_t \right) - \sigma r_{\min} \left( \frac{k_t + b_t}{c^d_t} \right) f \left( k_t, L^d_t \right)} \\
> \frac{1}{\beta} \frac{u' \left( c \right) \left( w - f \left( k, L \right) \right)}{(1 + e) \left( 1 - \sigma \right) f \left( k, L \right)},
\end{align*}
\] or equivalently as
\[
Y * \left( \frac{\tau^L d}{(1 + e) \left( 1 - \tau^L d \right)} - (1 - \sigma) \right) < \left( \frac{\tau^L}{(1 + e) \left( 1 - \tau^L \right)} - (1 - \sigma) \right),
\] (13)

with
\[
Y = \left( \frac{r_{\min} u' \left( c^d_t \right)}{\beta u' \left( c \right)} \right) \left( \frac{(1 + e) \left( 1 - \tau^L d \right) - (1 - \sigma)}{(1 + e) \left( 1 - \tau^L \right) - (1 - \sigma) - \sigma r_{\min} \left( \frac{k_t + b_t}{c^d_t} \right)} \right).
\] (14)

Obviously, this inequality holds whenever \( \tau^L d < \tau^L \) and \( Y < 1 \).

We next show that \( \tau^L d < \tau^L \) for all \( r_{\min} \leq \frac{1}{\beta} - 1 \). From (ss.3) and (11), we get
\[
c^d_t - f \left( k, L^d_t \right) > c - f \left( k, L \right).
\] (15)
Given $r_{\text{min}} \leq \frac{1}{\beta} - 1$, Eqs. (ss.1)–(ss.2) and (10) yield

\[ c_t^d - \left( c_t^d \right)^{\sigma} \left( L_t^d \right)^{1+e} \leq c_t - c^{\sigma} L^{1+e}. \]  

(16)

By total differentiation of Eqs. (15) and (16), we obtain

\[ \triangle c - f_L (k, L) \triangle L > 0, \]  

(17)

\[ \left( 1 - \sigma \frac{wL}{c} \right) \triangle c - (1 + e) w \triangle L \leq 0, \]  

(18)

where $\triangle z$ denotes the change in $z$ from the best sustainable to the equilibrium after a deviation; $\triangle z > 0$ means $z < z_t^d$. Moreover, since $\frac{c}{e} = 1 - \frac{c^{\sigma} L^e}{f_L (k, L)}$, we can write

\[ \triangle t_e = - \left( \frac{1}{f_L (k, l) L} \right) \left[ \frac{wL}{c} \triangle c + \left( e - \left( \frac{f_{LL} (k, L) L}{f_L (k, L)} \right) \right) \left( 1 - \sigma \frac{wL}{c} \right) \triangle c \right]. \]  

(19)

For $1 - \sigma \frac{wL}{c} \geq 0$, the inequalities (17)–(18) imply $L \leq L^d$ and $c < c^d$, and, in turn, $\tau^L > \tau^L$. For $1 - \sigma \frac{wL}{c} < 0$, we find $c < c^d$. If $L \leq L^d$, then it is clear that $\tau^L > \tau^L$. Consider $L > L^d$, that is, $\triangle L < 0$. Using Eq. (18), we find

\[ \triangle t_e \leq - \left( \frac{1}{f_L (k, l) L} \right) \left[ \frac{wL}{c} \triangle c + \left( e - \left( \frac{f_{LL} (k, L) L}{f_L (k, L)} \right) \right) \left( 1 - \sigma \frac{wL}{c} \right) \triangle c \right], \]

which is negative because $- \frac{f_{LL} (k, L) L}{f_L (k, L)} < 1$. Hence, if $r_{\text{min}} \leq \frac{1}{\beta} - 1$, then $\tau^L > \tau^L$.

For $\sigma_c = 0$, it is clear that $\tau^L < 1$, since $r_{\text{min}} < \frac{1}{\beta}$. For $\sigma_c > 0$, we next show that

\[ \left[ \frac{1}{2 - \beta} \right] \left[ \frac{1 - \beta}{\beta} \right] \geq r_{\text{min}} \text{ implies that } \tau^L < 1. \]

We can write condition $Y < 1$ as

\[ \left[ \frac{1}{\beta} u^\prime (c) - r_{\text{min}} u^\prime (c_t^d) \right] \left[ (1 + e) \left( 1 - \tau^L \right) - (1 - \sigma) \right] > \sigma r_{\text{min}} \left( \frac{k_t + b_t}{c_t^d} \right). \]  

(20)

Next, the condition $1 + a < \frac{1}{\beta}$ can be written as

\[ V_b^D (k, b) > \frac{1}{\beta} \frac{u^\prime (c) (w - f_L (k, L))}{(1 + e) w - (1 - \sigma) f_L (k, L) - \left( \frac{1}{\beta} - 1 \right) f_L (k, L) \sigma \left( \frac{k + b}{c} \right)}, \]

which, as $\tau^L > \tau^L$, implies

\[ (1 + e) \left( 1 - \tau^L \right) - (1 - \sigma) > \sigma \left( \frac{1}{\beta} - 1 \right) \left( \frac{k_t + b_t}{c_t^d} \right). \]  

(21)
Combining Eqs. (20) and (21), we obtain

\[ r_{\text{min}} < \left[ \frac{1 - \beta}{\beta} \right] \left[ \left( \frac{c}{c^d} \right) + (1 - \beta) \left( \frac{u'(c^d)}{u'(c)} \right) \right]^{-1}. \]

Since \( \left( \frac{c}{c^d} \right) < 1 \), it results that \( r_{\text{min}} \leq \left( \frac{1}{\frac{1}{2} - \beta} \right) \left( \frac{1 - \beta}{\beta} \right) \) is a sufficient condition for \( Y < 1 \).

We have obtained the following results. For \( \sigma_c = 0 \), we have \( Y < 1 \) for all \( r_{\text{min}} \) and \( \tau^d \leq \tau^l \) for all \( r_{\text{min}} \leq \frac{1}{\beta} - 1 \). Let then \( \bar{\tau} \equiv \frac{1}{\beta} - 1 \). If \( r_{\text{min}} \leq \bar{\tau} \), then the necessary condition \( V^D_b (k_t, b_t) = \frac{1}{\beta} D \) for being at incentive constrained steady states does not hold. Hence, the steady state must be incentive unconstrained and, thus, by Lemma 1, we find \( \tau^k = 0 \).

For \( \sigma_c > 0 \), we have \( Y < 1 \) for all \( r_{\text{min}} \leq \left( \frac{1}{\frac{1}{2} - \beta} \right) \left( \frac{1 - \beta}{\beta} \right) \) and \( \tau^d \leq \tau^l \) for all \( r_{\text{min}} \leq \frac{1}{\beta} - 1 \). Let then \( \bar{\tau} \equiv \left( \frac{1}{\frac{1}{2} - \beta} \right) \left( \frac{1 - \beta}{\beta} \right) \). If \( r_{\text{min}} \leq \bar{\tau} \), then the inequality \( V^D_b (k_t, b_t) > \frac{1}{\beta} D \) holds, which implies \( 1 + a < 1 \). Hence, the steady state is either incentive unconstrained or incentive constrained with \( (1 + a) \in (0, 1) \), then by Lemmas 1 and 3, we obtain \( \tau^k = 0 \). \( \Box \)

References