Incumbency and entry in license auctions: The Anglo–Dutch auction meets another simple alternative

Helmuts Ązacis\textsuperscript{a}, Roberto Burguet\textsuperscript{b,}\textsuperscript{*}

\textsuperscript{a} Cardiff Business School, United Kingdom
\textsuperscript{b} Institute for Economic Analysis (CSIC), Spain

Received 28 February 2006; received in revised form 16 May 2007; accepted 30 May 2007
Available online 6 June 2007

Abstract

The existence of ex-ante strong incumbents may constitute a barrier to entry in auctions for goods such as licenses. Favoring the allocation to entrants is a way to induce entry and thus create competition. Designs such as the Anglo–Dutch auction have been proposed with this goal in mind. We first show that the Anglo–Dutch auction does indeed foster entry and increases the revenues of the seller. However, we argue that a more effective way could be to stage the allocation of the good so that each stage reveals information about the participants. We show that a sequence of English auctions, with high reserve prices in early rounds, fulfils this property and is more efficient than any one-stage entry auction. Moreover, it also outperforms the Anglo–Dutch auction in terms of seller’s revenues.
© 2007 Elsevier B.V. All rights reserved.

JEL classification: D44

Keywords: Anglo–Dutch auction; Entry; License

1. Introduction

One of the salient features of the recent wave of auctions of spectrum licenses has been the disparity of resulting prices. As an example, the Netherlands auction of 3G licenses fetched less than a third of what the British or German auctions raised, in per capita terms.\textsuperscript{1}

One of the explanations proposed for such disparity has been the differences between the corresponding ratios of incumbents to licenses, and the effects of this ratio on entry and competition for licenses (see, for instance, Klemperer, 2002a,b; Milgrom, 2004).

Insufficient entry may limit bidding competition, leading to a low price. Bidders decide whether to enter or not by comparing the costs and expected benefits of participation. For a potential bidder the costs may include resources needed to assess the value of the license, interest foregone on deposits, etc. These are sunk costs that a firm incurs before it knows whether it wins a license. Therefore, the firm will decide to participate only if it believes that its odds of winning the auction are sufficiently high.

The ratio of incumbents to licenses is important in this regard. Incumbents are strong competitors. Indeed, they
may have a client base and lower expected roll-out costs. That is, their expected valuation of a license is higher. Also, their cost of assessing the market may be significantly lower. If there are at least as many incumbents as licenses, then entrants should expect that all licenses will end up in the hands of incumbents with a high probability, and then entry is not likely to occur. That need not be either inefficient or problematic in terms of expected revenue if there are more incumbents than licenses. However, if the number of incumbents and licenses coincide, the lack of entry will destroy all sources of competition, and that will have an enormous effect on revenue.

In this paper we consider the problem of allocating licenses when their number and the number of (ex-ante stronger) incumbents coincide. In order to create competition, the allocation mechanism has to create conditions that favor entry. One way to attain that goal is to favor entrants over incumbents. For instance, Dutch (or first-price) auctions tilt the allocation in favor of ex-ante weaker bidders, and thus weaker bidders prefer Dutch auctions to efficient, English auctions (see Maskin and Riley, 2000). Based on this fundamental insight, Paul Klemperer (see Klemperer, 2002a,b) and others have proposed the use of the so-called Anglo–Dutch auction to allocate \( q \) identical objects (licenses) when also \( q \) ex-ante stronger incumbents are potential buyers.

An Anglo–Dutch auction begins with an ‘English’ phase during which the price is increased until all but \( q + 1 \) bidders drop out. At this moment (and price), the auction switches to a second ‘Dutch’ phase. In this stage, only the remaining \( q + 1 \) bidders can submit (simultaneous, sealed) bids and only bids above the price at which the English phase stopped are allowed.

The first goal of this paper is to show through a very simple model how this auction indeed improves the expected revenues of the seller. Entry exceeds the efficient level, but this results in higher auction revenues. Our second goal is to investigate other simple alternatives that may improve, both in terms of efficiency and revenues, upon the Anglo–Dutch auction. We propose what we could term Anglo–Anglo auction: a two-stage, English auction. The design is inspired by Burguet and Sákovics (1996), and consists of two English rounds, the first one run with a (relatively high) reserve price. Instead of using (induced) allocation preference as the tool to foster entry, the two-stage English auction uses the information conveyed by the (absence of) bidding in the first round. Indeed, if some of the participants in the first rounds (incumbents included) are unwilling to bid above the reserve price, they will be perceived as “weaker than expected” bidders. Thus, potential entrants that did not venture to enter in the first round may consider doing so for the second round. That future competition, on the other hand, may lead the incumbents to accept the reserve price of the first round.

This is the main insight of this paper. Introducing dynamic elements into the design allows the seller to tilt the information, not the allocation, in favor of potential entrants, and may be a better way to overcome the barrier to entry that incumbency represents. Entry in auctions has been studied since the seminal papers by Samuelson (1985) and Levin and Smith (1994). Also, the literature on dynamic auctions is extensive. One of the issues that this literature has dealt with is the consequences for auction design of the lack of commitment by the seller not to resell in the future. There are two recent contributions in this line, Caillaud and Mezzetti (2004) and Skreta (2006). The link between future sale and entry is at the heart of the design that we propose in this paper.

We show that a two-stage, English auction is more efficient than both the English and the Anglo–Dutch auctions. By allowing entry conditional on some private information (entry conditional on bidding behavior), the two-stage entry auction improves upon the most efficient one-stage entry auction, namely, the English auction. Moreover, by choosing the right reserve price in the first round, the gain in efficiency benefits the seller as well. Indeed, the revenues for the seller are higher in the two-stage English auction with appropriate reserve price than in the Anglo–Dutch auction.

The analysis is carried out in an extremely simple model, presented in Section 2, where entrants have to incur a cost before learning their valuations. Moreover, valuations can take only two different values, although incumbents have a higher probability of high valuation. We simplify further by assuming only one unit for sale and one incumbent. The analysis of this model and the results are presented in Section 3.

Our results are valid for the unique separating equilibrium of the two-stage English auction in this simple model. We can guarantee that this is in fact the only equilibrium (in undominated strategies) when the reserve price induces positive entry in the first round under such equilibrium. Unfortunately, the seller’s revenues may be maximized with reserve prices that induce no entry in the first round when bidders play the separating equilibrium. For these reserve prices, the two-

\footnote{We consider the unit-demand case. Notice that if incumbents are willing (and allowed) to bid for more than one license, then entry is not as crucial for competition.}

\footnote{Paul Klemperer has also emphasized the virtues of Anglo–Dutch auctions when dealing with other issues as well, such as collusion, risk aversion, etc.}
stage English auction may have other pooling or partially pooling equilibria, and we cannot guarantee that by choosing the right reserve price the revenues are higher in all equilibria of the two-stage English auction. In Section 4 we discuss this issue, and generalizations of our simple model. In particular, we extend the model in two main directions: multiple units and continuous valuations.

The extension to continuous types is relevant. Indeed, in the two-type model, the Anglo–Dutch is allocationally efficient, in the sense that the object is never assigned to a bidder that competes against a bidder with higher valuation. That is, the only inefficiency may come from inappropriate (excessive) entry. The allocation is still tilted in favor of the ex-ante weaker bidder: when two bidders have the same type, an event with positive probability, the ex-ante weaker bidder will win the auction (will bid higher) with higher probability. Yet one can suspect that by not considering the allocation inefficiency (a tool to reduce informational rents of ex-ante stronger bidders) of Dutch auctions, we underestimate their revenue generation potential. The analysis of the continuous valuations case necessarily relies on numerical computations, since there is no analytical solution to asymmetric Dutch auctions (the second stage of an Anglo–Dutch auction) and the bidding behavior (when to bid) in a two-stage English auction has no simple closed form. Using numerical methods for a family of simple continuous distributions, we obtain the same results as in the base model.

Then we consider multiple units (and the same number of incumbents). Here entry decisions in the second stage of the two-stage English auction depend on the number of units that are sold in the first stage. The larger the number of units left unsold the larger the number of entrants in the second stage. In fact, under very extreme parameter values, the effect that the number of units available has on entry is also very extreme. Unless this is the case, our results for the one unit case hold in the multiple unit case.

2. Rules of the auctions

There are \( q \) identical units available for sale and the same number of incumbents. Also, there is a number of potential entrants. We will only consider pure strategy equilibria in entry. Thus, the exact number of potential entrants is unimportant as long as it is sufficiently large.\(^4\)

In order to learn his valuation and prepare his bid, an entrant has to incur a cost \( c \). To simplify the analysis, we assume that incumbents already know their valuations and incur no further participation costs. Each bidder has demand only for one unit. Valuations are private and independently distributed. The valuations of incumbents and entrants are drawn from distribution functions \( F_1(v) \) and \( F_2(v) \), respectively. Finally, incumbents and entrants recognize themselves as such. We will use perfect-Bayesian as our equilibrium concept. Also, we will consider only equilibria in undominated strategies.

2.1. Rules of the Anglo–Dutch auction

Before bidding starts entrants decide whether or not to incur cost \( c \) and to learn their valuations. The auction starts as an English auction where bidders continuously raise their bids. We use the clock modelling, so that once a bidder drops out he cannot reenter the auction. When there are exactly \( q+1 \) bidders left, the auction switches to the Dutch auction, or more precisely, to the discriminatory auction, which is a generalization of first-price auction for multiple units. In this stage, surviving bidders bid simultaneously, and the \( q \) bidders with the highest bids each win a license. Winners pay their bids. In this Dutch stage, the price at which the last bidder dropped out in the English stage is set as a reserve price or minimum acceptable bid. To avoid uninteresting equilibria later on, we assume that the \( q+1 \) surviving bidders have to bid in the Dutch stage (offer at least the reserve price). Equivalently, we may realistically assume that there is some positive (perhaps very small) penalty for these bidders if they do not bid.

In the discrete valuation case, several bidders may drop out of the English auction at a given price, leaving less than \( q+1 \) bidders active. In that case, we will assume that some of these simultaneously dropping out bidders are randomly selected to participate in the Dutch stage so that \( q+1 \) bidders are still present.

2.2. Rules of the two-stage English auction

In the first stage the seller sets a reserve price \( r \), common to all units, and then entrants decide whether to enter or not. Then the oral ascending auction starts. Units are awarded to, at most, the \( q \) last bidders to drop out. The price is the maximum of \( r \) and the price at which only \( q \) bidders stay. If less than \( q \) bidders offer the reserve price, so that some units fail to sell in the first stage, these remaining units are auctioned in the second stage with the reserve price now set equal to zero. Before this second stage starts, potential entrants who did not enter in the first stage have a fresh chance to enter and

\(^4\) We may assume that there is an infinite number of potential entrants.
join the competition for the remaining units. Again, if \( q' \) units were left unsold after the first stage, they are awarded to the last \( q' \) active bidders at the price at which only that many bidders are left.

3. Discrete valuations, one unit case

Each bidder’s valuation may be \( \bar{v} \) or \( v \) with \( \bar{v} > v \). Also, \( q=1 \). The probability that bidder 1, the incumbent, has a high valuation \( \bar{v} \) is \( \mu_1 \). For all other bidders (entrants) this probability is \( \mu_2 \). We assume that \( \mu_1 > \mu_2 \).\(^5\) Also, we assume that

\[
\mu_2(\bar{v} - v) > c. \tag{1}
\]

If this condition is not satisfied, then even if the incumbent was known to have a low valuation trying a new entrant would never increase the expected surplus. This would not be an interesting scenario: entry would be zero in any (non-subsidized) auction.

3.1. Anglo–Dutch auction

Consider the subtree of the game that follows entry of \( n \geq 1 \) entrants.\(^6\) Apart from this entry decision, a strategy for a bidder is a complex object that determines, among other things, the dropping out price as a function of when and who has dropped out before. However, a bidder that drops out in the English stage does not take part in the Dutch stage, and therefore dropping out is his last move. Dropping out means a payoff of zero, whereas staying may lead to a positive payoff when the price is below the valuation and to a negative payoff when it is above.\(^7\) That is, there are other bidders’ strategies that result in such positive or negative payoffs. Negative payoffs result from our assumption that bidders selected to play the Dutch stage have to bid at least the reserve price. Positive payoffs result from the possible dropping out of others and then being in a position to bid below the valuation and above the reserve price in the Dutch stage. Therefore, just as in a standard English auction, if we restrict attention to equilibria in undominated strategies, we need only consider strategies where bidders stay in the English stage until the price reaches exactly their valuations independently of the previous history of the auction. Thus, if more than two of the \( n+1 \) bidders have valuation \( \bar{v} \), the English stage stops at price \( \bar{v} \) in any equilibrium in undominated strategies, and then two high valuation bidders are randomly selected to bid in the Dutch stage. On the other hand, if two or fewer bidders have valuation \( \bar{v} \), the English stage stops at price \( v \). In this case, bidders with high valuation, if any, stay for the Dutch stage. If there are fewer than two bidders with high valuation, then bidders among dropping out (low valuation) bidders are randomly selected so that two bidders still take part in the Dutch stage. We are assuming that bidders recognize each other, so that the identity of bidder 1 is known. However, when the English stage stops at price \( v \), the participants in the Dutch stage cannot be sure whether their opponent has been randomly selected among the dropping bidders (i.e., has a low valuation) or not (i.e., has a high valuation).

3.1.1. Bidding in the Dutch stage

As mentioned before, in any equilibrium in undominated strategies the clock stops either at \( v \) or at \( \bar{v} \).\(^8\) In fact, we only need to analyze subtrees in which the clock stops at price \( v \) in the English stage. Indeed, the clock stops at \( \bar{v} \) only when selected bidders have valuation \( \bar{v} \) and therefore bidders bid exactly that value in the Dutch stage independently of their identity.

To avoid open-set problems, we assume that if two bidders with different valuations tie in the Dutch auction then the winner is the one with the higher valuation. We will comment on this tie-breaking rule later. In case of ties of bidders with the same valuation, the winner is selected at random.

Now, assume the clock has indeed stopped at \( v \) in the English stage, and assume that bidder 1 and an entrant are the remaining (or selected) bidders. Conditioning on this event, an entrant’s type is \( \bar{v} \) with probability \( \gamma_2 = \mu_2 / (\mu_2 + (\bar{v} - v) C_0/C_1) \). Indeed, given the stopping price (i.e., conditioning on no more than 1 rival bidder having high valuation), the entrant would be one of the two participants in the Dutch stage with probability one if his valuation is high, and with probability \( \frac{1}{n} \) if his valuation is (as everybody else’s) low. Similarly, the posterior probability that the incumbent has a high valuation is \( \gamma_1 = \mu_1 / (\mu_1 + (1 - \mu_1) C_0/C_1) \).

\(^5\) Note that in case \( \mu_1 = \mu_2 \), the Anglo–Dutch auction is equivalent to a one stage English auction. This case is not interesting, although the results of this paper (except Lemma 2, of course) still hold.

\(^6\) If \( n=0 \), then the auction stops at zero price and the only bidder allowed to bid in the Dutch stage is the incumbent. Thus, the outcome would be a sale to bidder 1 at zero price.

\(^7\) As we mentioned before, with discrete valuations there is a positive probability that two bidders will drop out at the same time, so that one of them is chosen randomly to participate in the Dutch auction. However, this can only happen in equilibrium if the bidder is indifferent as to whether selected for the Dutch auction or not. That is, to all effects the bidder can consider that his continuation payoff is zero once he drops out.

\(^8\) If \( n=1 \), then the clock would stop at 0. Everything else would coincide with the case in which the clock stops at \( v \), so we will not make further reference to this case.
Consider the game that starts at that point with such (common knowledge) beliefs. It is straightforward to see that such a game could not have a pure strategy equilibrium in the bidding game. Also, if an equilibrium in undominated strategies exists bidders bid $v$, the reserve price set by the English stage, if that is their valuation. Finally, in such equilibrium both bidders should play a mixed strategy that puts positive probability (density) on a common interval $[v, b]$, without mass points except perhaps a mass point at $v$ for one of the bidders.\footnote{If a bidder’s strategy puts zero probability in an open interval, the other bidder’s best response also puts zero probability in the interior of that interval. Thus, “holes” are common. This precludes “holes” in the equilibrium strategy of players: given a pair of strategies with a common hole, one bidder benefits from deviating and bidding at the infimum of this “hole”. Thus, “holes” in the equilibrium strategy, including those between $v$ and the infimum of pure strategies played with positive probability density by high valuation bidders, cannot occur. Mass points other than at $v$ could be just as easily ruled out.} In the Proof of Lemma 1 we prove existence (and uniqueness) of such equilibrium by construction. For the moment, let us assume that bidder 1, the incumbent, and some entrant play the Dutch stage. Since $b$ is common and bidders bid $b$ with zero probability (no atoms at that value) then the expected profits of either an entrant or the incumbent that has valuation $\bar{v}$ equals $[\bar{v} - b]$ in equilibrium. Indeed, by bidding $b$ either bidder expects to win with probability 1. Thus, since expected profits should be independent of the pure strategy played in a mixed strategy equilibrium, we will conclude that the entrant and the incumbent will expect the same equilibrium profit upon entry if they have the same valuation.

This is in fact the way the Anglo–Dutch auction is expected to foster entry. Indeed, notice that in a standard English auction bidders have high valuation expect lower profits than incumbents with the same valuation. The Dutch stage tilts competition in favor of entrants, and in the case of discrete valuations, this is enough to perfectly level the field. The incumbent bids less aggressively than entrants in that he puts lower probability on high bids. In fact, the strategy of the incumbent puts positive probability, $\frac{v_1 - v_2}{v_1}$, on a bid $b = v$ that loses with probability 1 to a high valuation entrant’s bid. We summarize all this in the following

**Lemma 1.** If selected to play the Dutch stage when the English stage stops at price $v$, 

i) entrants expect zero profits if their valuation is $v$ and profits $(1 - \gamma_2)(\bar{v} - v)$ if their valuation is $\bar{v}$ independently of the identity of the rival,

ii) entrants bid independently of the identity of the rival, but more aggressively than the incumbent.

**Proof.** See Appendix. $\square$

We should comment now on the role of our tie-breaking assumption. We assume that in case two bidders bid $v$, which occurs with positive probability when the incumbent has valuation $v$ and the entrant has valuation $\bar{v}$, the bidder with high valuation wins. This allows the incumbent to bid $v$ and still expect to win with probability $\gamma_2$. That is, this allows the incumbent to identify the lowest bid that still allows him to defeat the bid of a rival with type $v$. Therefore, the tie-breaking assumption comes to play the role of a smallest unit of money. Notice, however, that we do not need to assume that the incumbent has a preference when bidding against an entrant with type $\bar{v}$.

We now turn to the analysis of entry and the English stage.

3.1.2. Entry in the Anglo–Dutch auction

Note that an entrant expects positive profits only when his type is $\bar{v}$ and no more than one other bidder has this type. In this event, his profits are independent of the pure strategy (among those inside the support of his equilibrium mixed strategy) that he chooses to play. One of these strategies is to bid $b = v$, in which case he wins $(\bar{v} - v)$ when all rivals have $v$ or when the rival is the incumbent and bids $v$. This event has probability $\left(1 - \mu_1 + \mu_1 \frac{\bar{v} - v}{v}ight)(1 - \mu_2)^{n-1}$. Thus, the expected profits of an entrant are

$$\Pi^e(n) = \mu_2(1 - \mu_2)^{n-1} \left[\frac{(1 - \mu_1)}{(n - 1)\mu_2 + 1} + (1 - \mu_1)\right](\bar{v} - v) - c.$$  

(2)

This is a decreasing function of $n$. Entry occurs up to the point where the above expression is non-negative, and the same expression is negative for $n+1$. That is, treating $n$ as a continuous variable, the number of entrants in the Anglo–Dutch auction, $n^*$, satisfies $\Pi^e(n^*) = 0$.\footnote{When $\Pi^e(1) < 0$, entry will not take place either in the Anglo–Dutch auction or the English auction.}

Compare this entry decision with the entry decision in a standard English auction. Again, treating $n$ as a continuous variable, the number of entrants in a standard English auction, $n^\prime$, when $n^\prime > 0$, solves

$$\mu_2(1 - \mu_1)(1 - \mu_2)^{n^\prime-1}(\bar{v} - v) - c = 0.$$  

(3)

Since \( \frac{(\mu_1 - \mu_2)}{\sigma^2} > 0 \), and \( \Pi(n) \) is decreasing in \( n \), we conclude that

Lemma 2. The Anglo–Dutch auction promotes entry beyond what is obtained in the standard English auction: \( n^\circ \geq n^r \).

To conclude with the Anglo–Dutch auction, we can compute the profits expected by the incumbent. Again, when bidding \( v \) in the Dutch stage if his type is \( v \) and the clock stops at \( v \) in the English stage, the incumbent’s expected profits are

\[
\mu_1 (1 - \mu_2)^n (v - \bar{v}).
\]

(4)

3.2. Two-stage English auction

The first stage of a two stage English auction is an oral ascending auction with an opening price \( r \). The value of \( r \) is a choice parameter for the seller. Let \( k \) be the number of entrants in the first stage and, \( (k, r) \) for \( r \in (v, \bar{v}) \) the number of entrants in the second stage. We look for a separating equilibrium where high valuation bidders in the first stage prefer to bid (i.e., prefer to stay past the reserve price \( r \)) while low valuation bidders abstain from bidding in the first stage. We will come to the issue of uniqueness in the next section.

Equilibrium bidding in the second stage for all \( k \) participants is simple: dropping out at a price equal to valuation is weakly dominant. Thus, in equilibrium treating \( l \) as a continuous variable, it satisfies

\[
\mu_2 (1 - \mu_2)^{l-1} (\bar{v} - v) = c.
\]

(5)

The left hand side is the expected profits upon entry, and the right hand side is the cost of entry. In order to simplify the analysis, we will assume that the condition holds with equality. All results carry through otherwise assuming that the seller sets a small extra fee in the second stage to keep entrants at their indifference level. Note that this is in the interest of the seller, and can only increase the seller’s revenues in the two-stage English auction.

The value of \( l \) that solves Eq. (5) is independent of \( r \) and \( k \). Thus, we will drop the arguments of the function \( l \). Also, note that given Eq. (1), \( l \geq 1 \) and, if we compare Eq. (5) with Eq. (2) and the result of Lemma 2, we conclude that \( l \geq n^\circ + 1 \geq n^r + 1 \). That is, when a separating equilibrium exists the number of entrants in the second stage is strictly higher than the number of entrants in the Anglo–Dutch and English auctions.

For a bidder that bids in the first stage, i.e., for a bidder with valuation \( v \) that has stayed past the opening price \( r \), dropping out only when the price reaches \( \bar{v} \) is weakly dominant: dropping out earlier (which requires that at least two bidders are still active) implies zero profits. Thus, the only choice for a bidder with valuation \( \bar{v} \) present in the first stage is to wait in the hope that there is a second one. For the incumbent to indeed prefer to bid in this first stage we should have that \( (1 - \mu_2)^{k} (\bar{v} - r) \geq (1 - \mu_2)^{k+1} (\bar{v} - v) \). Indeed, under the belief that all bidders with valuation \( \bar{v} \) and only these bidders bid in the first stage, he will only earn positive profits, \((\bar{v} - \bar{y})\) in the second stage or \((\bar{v} - r)\) in the first, if no other participant has valuation \( v \). Similarly, for a first stage entrant with high type to prefer to bid rather than to wait, we need \((1 - \mu_1)(1 - \mu_2)^{k-1} (\bar{v} - r) \geq (1 - \mu_1) \) \( (1 - \mu_2)^{k+1} (\bar{v} - v) \). In both cases, the restriction can be written as:

\[
r \leq r(0) \equiv \bar{v} - (1 - \mu_2)^{l} (\bar{v} - v) = \bar{v} - \frac{1 - \mu_2}{\mu_2} c.
\]

(6)

The last equality above is obtained by substituting Eq. (5). For values of \( r > r(0) \) no separating equilibrium can exist. In fact, it is easy to show that the only equilibrium outcome for such \( r \) involves no entry and no bidding in the first stage, and then the two-stage English auction is nothing but a delayed English auction. Thus, in the sequel we will restrict attention to the interval \( (\bar{v}, r(0)) \). Note that given (1), \( r(0) > \bar{v} \).

We now turn to the entry decision in the first stage. The zero profit condition for \( k > 0 \) entrants in the first stage is

\[
\mu_2 (1 - \mu_1) (1 - \mu_2)^{k-1} (\bar{v} - r) = c.
\]

Thus, we define \( k(r) \) for \( k=1, 2 \ldots \) as the solution in \( r \) of this equation. That is,

\[
r(k) = \bar{v} - c \frac{1}{\mu_2 (1 - \mu_1) (1 - \mu_2)^{k-1}}.
\]

(7)

Note that \( r(k) \) is decreasing in \( k > 0 \). Also, note that \( r(0) > r(1) \). Thus, for \( r \in (\max \{ \bar{v}, r(1) \}, r(k) \) for \( k \geq 0 \), \( k \) entrants would enter in the first stage of a separating equilibrium. In particular, \( k > \max \{ \bar{v}, r(1) \} \) only the incumbent participates in the first stage of the auction in a separating equilibrium. For \( v = \bar{v} \), Eq. (7) is the entry condition in the English auction Eq. (3) when \( n^r \geq 1 \). It follows that \( k \leq n^r \) in any separating equilibrium.

3.3. Comparing total surplus

In this section, we assume that bidders play the separating equilibrium obtained in the previous section. In the next section we will discuss when other equilibria are possible, and the implication for revenue and surplus comparisons.
One feature of both the Anglo–Dutch auction and the two-stage English auction in this setting is that the license is assigned to the user that values it most among those present at the round in which it is assigned. (In a more general, continuous type model this is true for the two-stage English auction, but not for the Anglo–Dutch auction.) Thus, in our setting the expected total surplus depends only on entry.

The standard English auction maximizes the gains from trade among the mechanisms at which entry occurs only at one point in time. Indeed, given $n$ entrants, a new entrant adds surplus only if the $n-1$ previous entrants and the incumbent all have valuation $v$ and the new entrant has valuation $\bar{v}$. For $n \geq 1$, this event has probability $\mu_2(1-\mu_1)(1-\mu_2)^{n-1}$ and the increase in surplus is $(\bar{v} - v)$ in this case. Trading this increased expected surplus with the cost of entry $c$ results in Eq. (3), the entry condition in an English auction. In this sense, the Anglo–Dutch auction fosters entry beyond the efficient level.

In a two-stage English auction, entry may take place at more than one point in time. If the license is not assigned in the first stage, then new entrants will enter to take part in a final, English auction. Second-stage entry conditional on all first-stage participants having low type $y$ is also (conditionally) efficient. Indeed, the expected surplus, given that there is a second stage, is $v + [1 - (1 - \mu_2)^l](\bar{v} - v) - lc$ where $l$ is given by Eq. (5). Now,

$$[1 - (1 - \mu_2)^l](\bar{v} - v) = (\bar{v} - v) \sum_{m=1}^{l} \mu_2(1 - \mu_2)^{m-1}. $$

(8)

Note that there are $l$ terms on the right hand side of Eq. (8), and they are decreasing in $m$. Thus, the smallest one is $\mu_2(1 - \mu_2)^{l-1}(\bar{v} - v)$. But this term is equal to $c$, if Eq. (5) is satisfied.

Then, it should not come as a surprise that the surplus in a two-stage entry mechanism may be higher than the surplus in even the most efficient one-shot entry mechanism. In fact, this is so for any reserve price choices where the incumbent bids in the first stage.

**Proposition 3.** The expected surplus in any separating equilibrium of the two-stage English auction with $r \in (y, r(0))$, is higher than in a standard English auction and therefore also higher than in an Anglo–Dutch auction.

**Proof.** See Appendix. □

According to the above proposition, any reserve price, including $r = y(1+\epsilon)$, which maximizes entry in the first round and still allows a positive probability of new entry in the second round, induces more efficient entry than the most efficient one-shot entry auction. But what is the efficient level of first-stage entry in the two-stage English auction?

To answer this question, first observe that in (the separating equilibrium of) the two-stage English auction, since $l$ is independent of $r$ and $k$, the total surplus depends only on the level of entry in the first period, $k$. Thus, let us denote by $k^*$ the level of entry that maximizes this total surplus. When there is a second opportunity to experiment, i.e., to obtain valuation draws, assigning the license in the first round has an opportunity cost above $y$, so that efficient entry in the first stage needs not be the highest compatible with screening low valuation types. Indeed, one more entrant in the first round, if of high type, adds $(\bar{v} - v)$ surplus only with probability $(1 - \mu_2)^l$, the probability that the future $l$ entrants all have low valuation. This additional entrant, again if of high type, also saves $lc$ entry costs. Thus, taking into account this higher opportunity cost of assigning the license in the first round, we have:

**Lemma 4.** Maximizing surplus in a separating equilibrium of the two-stage English auction requires limiting first-stage entry (as compared to the maximum, non-subsidized entry). In particular, if $k^* \geq 1$, it satisfies

$$r(k^*) = \bar{v} - [(1 - \mu_2)^l + l \mu_2(1 - \mu_2)^{l-1}](\bar{v} - v). $$

(9)

**Proof.** See Appendix. □

**Corollary 5.** Efficient entry in the first stage of the two-stage English auction is strictly positive if and only if $\mu_2 \leq \mu_2(l-1)(1-\mu_2)$. That is, if and only if $\frac{\mu_1}{1-\mu_1} \leq \frac{\mu_2}{\Phi}$, where $\Phi = \log(1-\mu_2)/\log(\mu_2) - \log(\bar{v} - v)$.

The corollary results from Eqs. (5), (9), and the definition of $r(k)$.

### 3.4. Comparing revenues

Revenues and efficiency are intimately related. Indeed, disregarding the integer problem, entrants expect zero profits (net of entry cost) both in a standard English, an Anglo–Dutch, and a two-stage English auction. Therefore, when comparing the seller’s revenues in both auctions we need only to look at total surplus (net of entry costs) and the incumbent’s profits. The incumbent’s profits are $\mu_1(1 - \mu_2)^l(\bar{v} - v)$ both in the standard English
and in the Anglo–Dutch auctions, except that \( n \) may differ in both. Thus, the revenues for the seller in each case are

\[
R(n^i) = \bar{v} - (1 - \mu_2)^{n^i} (\bar{v} - \mu_1) - n^i c, \tag{10}
\]

for \( i = e, a \). Using Eq. (10), it is straightforward to check that \( R(n) - R(n - 1) \) is decreasing in \( n \). Also, from Eq. (5), this first difference is increasing for \( n < l \), and decreasing for \( n > l \). Thus, \( R(n) \) attains a maximum at \( n = l \) (again, treating \( n \) as a continuous variable). Compare this with entry decisions in Anglo–Dutch and English auctions, obtained respectively from Eqs. (2) and (3) above. Since \( l - 1 \geq n^a \geq n^e \), we have the following

**Proposition 6.** The revenues of the seller are weakly higher in the Anglo–Dutch auction than in the standard English auction. They are strictly higher if the Anglo–Dutch auction induces additional entry.

As conjectured, the Anglo–Dutch auction increases the revenues of the seller by fostering entry.

Now, consider the two-stage English auction. When maximizing revenues, the seller needs only consider the maximum of the reserve prices compatible with any amount of entry, \( k \); that is, \( r(k) \). Indeed, on the one hand any two values for the reserve price that induce the same first period entry (and also the same second period entry) also induce the same total (gross) surplus and the same cost of entry. On the other hand, given \( k \), the profits of both the first period entrants and the incumbent are lower the higher the reserve price. In fact, the expected profit of the incumbent for different values of \( r(k) \) for \( k \geq 1 \) are \( \mu_1 (1 - \mu_2)^{k} (\bar{v} - r(k)) \). Substituting the definition of \( r(k) \), we can write these expected profits for \( k \geq 1 \) as \( \frac{\mu_1 - \mu_2}{\mu_1} \mu_2/\mu_1 \). Thus,

**Lemma 7.** The expected profits of the incumbent evaluated at \( r(k) \), are independent of \( k \), for \( k \geq 1 \).

The implication is that, from the point of view of the seller’s revenues, the reduction in \( r \) that is necessary to attract one more entrant in the first round exactly compensates for the increase of competition thus obtained. Also, the entrants’ profits are zero at \( r(k) \), for \( k \geq 1 \). But even when \( k^* \geq 1 \), the ranking of the seller’s revenues when \( r = r(k^*) \) and when \( r = r(0) \) is ambiguous. Thus,

**Corollary 8.** From the point of view of the seller, the optimal reserve price is either \( r = r(k^*) \) or \( r(0) \).

We are now ready to compare the seller’s revenues in the Anglo–Dutch auction and in the two-stage English auction.

**Proposition 9.** The separating equilibrium of a two-stage English auction with either a reserve price \( r(k^*) \) or a reserve price \( r(0) \) results in higher revenues for the seller than the Anglo–Dutch auction.

**Proof.** See Appendix. \( \square \)

In an Anglo–Dutch auction, the seller sacrifices surplus to foster entry and obtain higher revenues. A two-stage English auction increases the revenues of the seller by improving the efficiency of entry decisions. As a result, both the revenues of the seller and the efficiency of the allocation are higher than in an Anglo–Dutch auction.

4. Equilibrium uniqueness and generalizations

In this section we analyze both the possible existence of other equilibria in the two-stage English auction and how our results generalize to more complex models.

4.1. Equilibrium uniqueness

The results in the previous section hold as long as bidders play the unique separating equilibrium. There is still a question about uniqueness, since the two-stage English auction may well have other, non-separating equilibria. In this section we analyze this question. First, notice that the optimal two-stage English auction may imply a reserve price \( r(k^*) \) for \( k^* \geq 1 \), or a reserve price \( r(0) \). In the first case, uniqueness is not an issue.

**Lemma 10.** Assume \( r(1) > \bar{y} \). Then for \( r \in (\bar{y}, r(1)) \), the equilibrium in a two-stage English auction is unique.

**Proof.** See Appendix. \( \square \)

When the optimal two-stage English auction requires setting \( r(k) \) for \( k \geq 1 \), not only is the separating equilibrium unique, but there is no other equilibrium, either pooling or in mixed strategies. Entering in the first round when planning not to bid even if the type is high amounts to committing to entering in the second round whatever information is learnt from the first round. In a sense this possibility does not reduce effective competition in the second round. Thus, if bidding in the first round is profitable under the assumption that all high type bidders that are present will bid in the first round (that is, when competition is as fierce as possible in that round), it will also be profitable when some bidders present in the first round may wait with positive probability.
We should consider a case that cannot satisfy the conditions of the lemma, and may be an important one: the case where \( r(1) < \bar{y} \). As we argued before, this is the case when an English auction does not induce entry. Also, in a separating equilibrium of a two-stage English auction, no matter what the \( r \) is, first-round entry is zero. Thus, the optimal reserve price is \( r(0) \). Yet, the Anglo–Dutch auction may induce some positive entry.

As we have seen, both surplus and revenues for the seller are higher in the separating equilibrium of the two-stage English auction with reserve price \( r(0) \). However, in the interval \((\max\{\bar{y}, r(1)\}, r(0)]\) the equilibrium is not necessarily unique. There may be other equilibria where the entrants (if there are any) or the incumbent may abstain from bidding in the first stage with positive probability even when their valuation is \( \bar{y} \).

In our simple model, eliminating such equilibrium would be easy for the seller: for instance, the incumbent (or the incumbent and the potential entrants in the first round) could face some bidding cap in the second round.\(^{11}\) It suffices that the incumbent cannot raise his bid above \( \bar{y} \) if there is a second round.

This may not be a satisfactory solution in some applications. Then we need to worry about the multiplicity problem. For instance, at \( r=r(0) \) there is a pooling equilibrium, besides the separating one. Indeed, if the incumbent is not expected to bid in the first round independently of his valuation, then entry in the first round is still zero, and we have an ordinary English auction conducted in the second round. We know that, if \( n^* > n^s \), then the revenues in this new equilibrium of the two-stage English auction are lower than in the Anglo–Dutch auction. In such cases, we do not have results guaranteeing that a reserve price \( r \) exists such that all equilibria of the two-stage English auction result in revenues higher than those of the Anglo–Dutch auction.

### 4.2. Continuous distributions

In the previous sections we have analyzed a very stylized model of competition, where buyers’ valuations could take one of only two values. This was enough to illustrate the main insights behind the proposal to use Anglo–Dutch auctions to foster entry in the presence of a strong incumbent. Indeed, the incumbent bids less aggressively in the Dutch part, so that the chance that an entrant will obtain the license at a profit is enhanced. This fosters entry and increases the revenue for the seller at an efficiency cost: excessive entry. The stylized model also shows that a two-stage English auction may be more appropriate to attain the goal of high revenues with no cost to (and even enhancing) efficiency.

Yet, there is one aspect of Anglo–Dutch auctions that the discrete case does not reflect: the Dutch stage may introduce inefficiency that goes beyond excessive entry, i.e., allocation inefficiency. Indeed, in asymmetric settings, a Dutch auction may assign the good or license to a buyer other than the one most willing to pay for it. This inefficiency is in general to the advantage of both the entrant and the seller (see Maskin and Riley, 2000). Thus, we may consider continuous distributions of types, where this allocation inefficiency appears. Assume that \( v_i \), a buyer’s valuation, is a (independent) realization of a continuous random variable. The incumbent draws his type from a distribution \( F_1(v) \), whereas all entrants draw their types from a distribution \( F_2(v) \). We further assume that \( F_1 \) stochastically dominates \( F_2 \).

In the Anglo–Dutch auction, it is still weakly dominant for bidders to stay in the auction up to the moment when the price reaches their respective valuations. If the clock stops at price \( \rho \), with two entrants as the remaining bidders, then these bidders participate in a symmetric Dutch auction with reserve price \( \rho \), the one with highest valuation wins, and the revenue for the seller equals the expected value of the second largest valuation. When one of the two remaining bidders is the incumbent, however, bidders participate in an asymmetric Dutch auction. In general, bidding strategies in this case, and expected revenues for the seller, can only be obtained through numerical methods.

With respect to the two-stage English auction, and for any given reserve price set by the seller, \( r \), both the incumbent’s and entrants’ optimal behavior is to drop out at a price equal to their willingness to pay, if they do participate in any of the stages. Thus, we only need to analyze participation decisions. We can conjecture that these will be characterized by two cut-off values, \( w_1 \), \( w_2 \), such that the incumbent decides to participate in the first period if \( v_1 \geq w_1 \), and any first period entrant \( i \) participates if \( v_i \geq w_2 \). Both of these values will depend on the number of entrants in the first period and how many are expected in the second. Treating entry as a continuous variable, the four conditions (equations) that characterize \((w_1, w_2, k, l)\) are: zero profits for entrants in both stages; and indifference as to whether to participate or to wait both for type \( w_2 \) entrants and for the type \( w_1 \) incumbent.

Using numerical computations, we have solved for the continuous model with asymmetric, uniform distributions, \( F_i(v) = \frac{v_i}{\bar{v}_i}, \ i = 1, 2, \) with \( 1 = \bar{v}_2 \leq \bar{v}_1 \). The details are

---

\(^{11}\) Alternatively, since the problem may only appear when the seller targets a no-entry in the first round equilibrium, it suffices to run this round as a take it or leave it offer to the incumbent at a price \( r(0) \). This solution however may not be good outside the two-type model.
shown in the Appendix. In all cases analyzed, the total surplus is higher under a two-stage English auction than under an Anglo-Dutch auction. Revenues also follow this ranking except for one case: when \( v_1 = 1.2 \), and \( c = 0.072 \). This cost was chosen (as in all examples) so that entrants in the Anglo-Dutch auction break even, the most favorable case from the point of view of the seller. In this particular case, 2 entrants enter the Anglo-Dutch auction. For these same values, the optimal reserve price is \( r = \frac{2}{3} \), which generates zero entry in the first stage of the two-stage English auction, and 2 entrants in the second stage. \( (r = \frac{2}{3}) \) is the maximum reserve price compatible with the incumbent “bidding” in the first stage with positive probability, since the expected highest bid from the second period entrants is \( \frac{2}{3} \). Consequently, \( w_1 = 1 \). This obviously results in lower revenues for the seller (0.528, instead of 0.531). Why is entry not higher in the two-stage English auction? The answer has to do with the integer nature of entry. Indeed, with \( l = 2 \), entrants expect positive profits, equal to 0.0116. (Recall that entrants expect zero profits in the Anglo-Dutch auction with these parameter values.) Yet with \( l = 3 \) they would expect negative profits. As we were assuming in the previous section, to improve upon the Anglo-Dutch auction the seller only needs to set an entry fee above 0.0016 (much lower than 0.0116) in the second stage of the two-stage English auction, which entrants would be willing to pay.

4.3. More than one unit

We should also check whether our results extend to the case where the seller has more than one unit for sale, and the equality between the number of units and the number of incumbents still holds. In the Appendix we consider the case where 2 units are to be sold and there are 2 incumbents. All other assumptions are as in the model analyzed in Section 3. The main novelty here is that, in the second stage of a two-stage English auction, there is now a chance that one of the units for sale will be assigned in the first stage, but the other unit is still available in the second stage. Let \( l_1 \) be the number of second-stage entrants when 1 unit is left and \( l_2 \) the number of second-stage entrants when 2 units are still available. These are defined by

\[
\mu_2 (1 - \mu_2)^{(l_1 - 1)}(\bar{v} - y) = c,
\]

(11)

and

\[
\mu_2 [(1 - \mu_2)^{l_2 - 1} + (l_2 - 1)\mu_2 (1 - \mu_2)^{l_2 - 2}](\bar{v} - y) = c.
\]

(12)

Notice that the second stage is independent of the identity of the winner of the first, given our assumptions. In the Appendix we show that the socially efficient entry and allocation in one-stage auctions is achieved using, for instance, an English auction. However, a two-stage English auction always results in higher surplus than any one-stage auction. Thus, our results on efficiency carry over to this multiple-unit case.

With respect to the seller’s revenues a sufficient condition for the ranking to be the same as in the one-unit case is \( l_2 \leq 2(l_1 + 1) \). In fact, this leaves a lot of slack. We have obtained cases where the Anglo-Dutch auction performs better than the two-stage English auction. However, these cases involve both extremely low values of \( \mu_2 \) and \( c \) so that even though no entry would take place in an English auction, a large number of firms would enter in an Anglo-Dutch or a second stage of the two-stage English auction.

5. Concluding remarks

We have offered theoretical support to the claim that an Anglo-Dutch auction results in higher revenues than ascending auctions when there are as many licenses as incumbents. By favoring ex-ante weaker entrants, the Anglo-Dutch auction fosters entry and this results in higher prices for licenses at the cost of efficiency. However, we have also proposed another simple alternative to this Anglo-Dutch auction: a two-stage English auction; what we could term an Anglo-Anglo auction. Instead of relying on inefficient allocations to induce entry, the two-stage English auction relies on information revealed through bidding, information on which entrants can condition their entry decisions. As a result, entry is more efficient and surplus is higher, which works to the advantage of the seller. Thus, this simple design not only increases revenues in general, but also the gains from trade.

There is one aspect in which the two-stage English auction is more complex than the Anglo-Dutch auction: its intrinsic multistage nature. Indeed, the Anglo-Dutch auction requires two stages, but no time-lag is required between them. On the contrary, in a two-stage English auction potential new entrants should be given enough time to “enter” (ascertain their valuation, prepare a bidding strategy) before the second stage takes place. In cases where this waiting is costly, this cost would have to be weighed against the gains that we have discussed. Also, there is another dimension in which the Anglo-Dutch auction is simpler, and then more robust: the seller does not need to “compute” reserve prices. As such, the two-stage English auction is less “distribution-free”.

On
the other hand, it is simpler from the point of view of buyers: under private values, the bidding strategies are straightforward, and the only decision is when to bid. In Anglo–Dutch auctions bidders need to assess the valuation distributions of their opponents. Certainly some information about these distributions is still required when deciding whether to enter a two-stage English auction, but this is also required for entering an Anglo–Dutch auction. In any case, the conclusion of this work is that the possibility for learning afforded by dynamic designs may be a better alternative than inefficiencies when the goal is to foster competition through entry.

Appendix A. Proofs of results

Proof of Lemma 1. Let \( H_2 \) be the cdf of the entrant’s strategy when his valuation is high, and \( H_1 \) that of the incumbent. Given the posterior beliefs \( \gamma_i, i = 1, 2 \) defined before, and since bidders should be indifferent as to which strategy in \((v, b)\) they use, we have

\[
\pi_1(b) = \left(1 - \gamma_2\right) + \gamma_2 H_2(b) \right)(v-b), \tag{13}
\]

for all \( b \) in \((v, \bar{b})\). Similarly, for bidder 2, and for all \( b \) in \((v, \bar{b})\)

\[
\pi_2(b) = \left(1 - \gamma_1\right) + \gamma_1 H_1(b) \right)(v-b). \tag{14}
\]

Thus, given \( \bar{b} \), Eqs. (13) and (14) characterize \( H_i \), for \( i = 1, 2 \). Note that since \( \pi_1(b) = \pi_2(b) \) but \( \gamma_1 > \gamma_2 \), \( H_2(b) < H_1(b) \) for all \( b \). Now we apply the condition that the infimum of the intervals of both mixed strategies should be \( v \). Since only one bidder can have a mass point at that infimum, and \( H_2(b) < H_1(b) \) for all \( b \), we conclude that \( H_2(v) = 0 \). That is, from Eq. (13)

\[
\pi_1 = \left(1 - \gamma_2\right)(v - v) = \pi_2. \tag{15}
\]

Substituting in Eqs. (13) and (14), we obtain

\[
H_2(b) = \frac{1 - \gamma_2}{\gamma_2} \frac{b - v}{v - \bar{b}}, \tag{16}
\]

and

\[
H_1(b) = \frac{1 - \gamma_1}{\gamma_1} \frac{b - v}{v - \bar{b}} + \frac{\gamma_1 - \gamma_2}{\gamma_1} \frac{v - v}{v - \bar{b}}. \tag{17}
\]

so that \( H_1(v) = \frac{1 - \gamma_2}{\gamma_2} \). Also, using \( H_2(\bar{b}) = 1 \) we obtain

\[
\bar{b} = \gamma_2 \bar{v} + (1 - \gamma_2) \frac{v}{\gamma_2}. \tag{18}
\]

This completes the characterization of the equilibrium bidding when the incumbent and an entrant play the Dutch stage.

Characterizing the equilibrium when two entrants play the Dutch auction is even simpler. Here both bidders are symmetric and update to \( \gamma_2 \) the probability that the rival has high type when they themselves do. Then, their expected profits are given by Eq. (13), and using \( H_2(v) = 0 \) we obtain that these profits equal Eq. (15). This implies that Eq. (16) is now the common equilibrium bidding strategy for both rivals.

Note that \( H_i \) for \( i = 1, 2 \) are unique. Thus, they uniquely define equilibrium bidding in the Dutch stage when the English stage stops at \( v \). □

Proof of Proposition 3. The total net surplus from a two-stage English auction is

\[
\bar{v} - (1 - \mu_1)(1 - \mu_2)^k[(1 - \mu_2)(\bar{v} - v) + l_c] - kc \tag{18}
\]

We have already established that the second stage entry \( l \) is conditionally efficient, i.e., that \( l \) minimizes \( (1 - \mu_2)(\bar{v} - v) + l_c \), and that \( l \) is independent of the reserve price, and therefore independent of first stage entry \( k \). Also, since \( l \geq n^2 + 1 \) we have that \( l > n^2 - k \). Therefore, total net surplus in Eq. (18) is strictly greater than

\[
\bar{v} - (1 - \mu_1)(1 - \mu_2)^k[(1 - \mu_2)(\bar{v} - v) + (n^2 - k)c]
\]

\[
-kc = \bar{v} - (1 - \mu_1)(1 - \mu_2)^k(\bar{v} - v) - (1 - \mu_1)
\]

\[
	imes (1 - \mu_2)^k(n^2 - k)c - kc \geq \bar{v} - (1 - \mu_1)(1 - \mu_2)^k
\]

\[
	imes (\bar{v} - v) - n^2 c,
\]

where the last inequality follows from \( n^2 \geq k \) and \( (1 - \mu_1)(1 - \mu_2)^k < 1 \). The last line is the expected surplus in the standard English auction and the result follows. □

Proof of Lemma 4. The marginal contribution of a new entrant in the first stage of the two stage English auction, for \( k \geq 1 \) is

\[
\mu_2(1 - \mu_1)(1 - \mu_2)^{k-1}[(1 - \mu_2)(\bar{v} - v) + l_c] - c.
\]

Indeed, \( \mu_2(1 - \mu_1)(1 - \mu_2)^{k-1} \) is the probability that the entrant has a high valuation \( \bar{v} \) and the remaining entrants have low valuation \( v \). In this event, the entrant in the first stage adds (gross) surplus \( (\bar{v} - v) \) if future entrants were to have low type as well, an event with probability \( (1 - \mu_2)^k \). Also in this event, the first stage entrant saves second stage entry costs \( l_c \). Now, treating \( k \) and \( l \) as continuous variables and substituting Eq. (5), entry in the first stage should take place until the point where

\[
\mu_2(1 - \mu_1)(1 - \mu_2)^{k-1}[(1 - \mu_2)(\bar{v} - v) + l_c] + l \mu_2(1 - \mu_2)^{k-1}(\bar{v} - v) = c.
\]

Comparing above expression with Eq. (7), we obtain Eq. (9). □
Proof of Proposition 9. The revenues for the seller in an Anglo–Dutch auction are

$$R^{AD} = R(n^2) = \bar{v} - (1 - \mu_2)^{n^2}(\bar{v} - y) - n^2 c,$$

whereas the revenues of the seller in the separating equilibrium of a two-stage English auction evaluated at $$r = r(k)$$, can be written as

$$R^{2S}(r(k)) = \bar{v} - (1 - \mu_1)(1 - \mu_2)^k((1 - \mu_2)^{(\bar{v} - y)} + lc) - kc - \mu_1(1 - \mu_2)^k(\bar{v} - r(k)),$$

where the last term represents the profits of the incumbent.

First, consider the case when $$k = k^* \geq 1$$. Using the definition of $$r(k^*)$$, we have

$$R^{2S}(r(k^*)) = \bar{v} - (1 - \mu_2)^k((1 - \mu_2)^{(\bar{v} - y)} + lc) - k^* c.$$

Again we use the fact that $$l$$ minimizes $$(1 - \mu_2)^{(\bar{v} - y)} + lc$$, and that $$l \geq n^2 - 1$$ and $$n^2 \geq n^2 \geq k$$ in any separating equilibrium, so that $$l > n^2 - k^*$$. Therefore, the total net surplus in Eq. (18) is strictly larger than

$$R^{2S}(r(k^*)) > \bar{v} - (1 - \mu_2)^k((1 - \mu_2)^{(\bar{v} - y)} + lc) - k^* c - (n^2 - k^*)c = \bar{v} - (1 - \mu_2)^n(\bar{v} - y) - \mu_1 lc + (1 - \mu_2)^n(\bar{v} - y) - n^2 c - \mu_1 lc = R^{AD}.$$

Now, consider $$r(0)$$. We have

$$R^{2S}(r(0)) = \bar{v} - (1 - \mu_2)^y(\bar{v} - y) - (1 - \mu_1)lc = \bar{v} - (\bar{v} - y)(1 - \mu_2)^y - \mu_1 lc + (1 - \mu_2)^n(\bar{v} - y) - n^2 c - \mu_1 lc = R^{AD}.$$

Waiting means that bidder $$i$$ expects a payoff at least as high in the second stage than when bidding in the first stage:

$$\theta_{-i} \times (\bar{v} - r) \leq \theta_{-i} \times (1 - \mu_2)^y(\bar{v} - y),$$

where $$\theta_{-i}$$ is the probability that none of the $$k'$$ other potential bidders in the first stage bids, and $$\lambda_{-i}$$ is the probability that conditional on this, the other bidders active in the first round all have valuation $$\gamma$$. There are $$l'$$ entrants in the second stage means that the $$l' + 1$$st entrant expects (strictly) negative net profits from entry:

$$\lambda_{-i} \left[ \frac{1 - \mu_1}{p_i \mu_i + (1 - \mu_i)} \right] \mu_2 (1 - \mu_2)^y(\bar{v} - y) < c.$$

Substituting the second equation in the first and rearranging, we have

$$r > \bar{v} - \left[ \frac{p_i \mu_i + (1 - \mu_i)}{1 - \mu_i} \right] c \geq \bar{v} - \frac{c}{\mu_2 (1 - \mu_1)} = r(1),$$

where the second inequality is obtained by substituting $$p_i = 1$$ and $$\mu_i = \mu_1$$ since the expression is decreasing both in $$p_i$$ and $$\mu_i$$. We obtain a contradiction since $$r \in (\bar{v}, r(1)].$$

Appendix B. Continuous valuations

Consider the Anglo–Dutch auction. There are initially $$n$$ entrants plus the incumbent participating in the English stage. A weakly dominant strategy for bidders is to stay in the auction up to the moment when the price reaches their respective valuations and then to drop out. Suppose that the bidder with the third highest valuation drops out at the valuation $$\rho$$. Then the remaining two bidders participate in the Dutch stage with reserve price $$\rho$$. Suppose that two bidders with the highest valuations are entrants. Then the expected revenue for seller from the Dutch stage is

$$\int_{\rho}^{\bar{v}_2} v(1 - F_2(v))f_2(v)dv,$$

which is the expected value of the second highest valuation given that it exceeds $$\rho$$ times the probability of this event. The density (on $$[0, \bar{v}_2]$$) of the highest valuation $$\rho$$ (reserve price) among the remaining $$n - 1$$ bidders is

$$f_{n-1:n-1}(\rho) = F_2(\rho)^{n-2}f_1(\rho) + (n - 2)F_2(\rho)^{n-3}F_1(\rho)f_2(\rho).$$
There are \( n(n - 1) \) permutations where the two bidders with the highest valuations are entrants. Thus, the revenue for seller (times the probability of the event) accruing when two entrants play the Dutch part is

\[
R_{w,w} = n(n-1) \int_0^{\bar{v}_2} \left( \int_0^{\bar{v}_2} v(1 - F_2(v)) f_2(v) dv \right) \\
\times dF_2(\rho)^{n-2} F_1(\rho) = n(n-1) \\
\times \int_0^{\bar{v}_2} v(1 - F_2(v)) F_2(v)^{n-2} F_1(v) f_2(v) dv.
\]

Suppose now that one of the two bidders with the highest valuations is the incumbent. Define the truncated distributions

\[
G_2(v) = \frac{F_2(v) - F_2(\rho)}{1 - F_2(\rho)} \\
G_1(v) = \frac{F_1(v) - F_1(\rho)}{1 - F_1(\rho)}
\]

Let \( v = \phi_2(b) \) and \( v = \phi_1(b) \) be inverse bid functions, respectively, for entrant and incumbent, defined on \([\rho, b^*]\), such that \( \phi_2(\rho) = \rho \) and \( \phi_1(b^*) = \bar{v}_i \) for \( i = 1, 2 \). When an entrant bids \( b \) he wins with probability \( G_1(\phi_1(b)) \) and pays his bid. Similarly, the distribution of an entrant’s bid is \( G_2(\phi_2(b)) \). Therefore the expected revenue for the seller when accruing from an entrant-winner is \( R_2 = \int_0^{b^*} b G_1(\phi_1(b)) dG_2(\phi_2(b)) \), and the expected revenue accruing from an incumbent-winner is \( R_1 = \int_0^{b^*} b G_2(\phi_2(b)) dG_1(\phi_1(b)) \). The distribution of \( \rho \) is \( f_{n-1,n-1}(\rho) = (n-1) F_2(\rho)^{n-2} f_2(\rho) \). Then, the expected revenue accruing to the seller from an entrant when the incumbent is one of the two bidders with the highest valuations is

\[
R_{w,s} = n \int_0^{\bar{v}_2} R_2(1 - F_2(\rho))(1 - F_1(\rho)) \\
\times (n-1)F_2(\rho)^{n-2} f_2(\rho) d\rho,
\]

and the expected revenue accruing to the seller from the incumbent is

\[
R_{s,w} = n \int_0^{\bar{v}_2} R_1(1 - F_2(\rho))(1 - F_1(\rho)) \\
\times (n-1)F_2(\rho)^{n-2} f_2(\rho) d\rho.
\]

Thus, the total revenues of the seller are \( R_{w,w} + R_{w,s} + R_{s,w} \). Notice that \( R_1 \) and \( R_2 \) depend on the (inverse) bidding functions \( \phi_2(b) \). These have to be computed using numerical methods.

Now we turn to a two-stage English auction. Again assume that, besides the incumbent, \( k \) entrants enter in the first stage and, if nobody bids, some additional \( l \) bidders enter in the second stage. Since first stage entrants must bid at least the reserve price \( r \), they will decide to participate in the first stage bidding if their valuations exceed a cut-off value \( w_2 \). Similarly, the incumbent will decide to participate in the bidding if his valuation is above a cut-off value \( w_1 \).

Define truncated distributions for \( i = 1, 2 \) as

\[
H_i(v) = \frac{F_i(v)}{F_i(w_i)}
\]

We can distinguish three cases: (1) \( \bar{v}_2 \geq w_2 \geq w_1 \geq 0 \), (2) \( \bar{v}_2 \geq w_1 \geq w_2 \geq 0 \), (3) \( \bar{v}_1 \geq w_1 \geq \bar{v}_2 \geq w_2 \geq 0 \). (And two more separate cases when \( k = 0 \) since then \( w_2 \) does not exist; \( \bar{v}_2 \geq 0 \) and \( \bar{v}_1 \geq w_1 \geq \bar{v}_2 \).) Here we present only derivations of cut-off points for the first case. Note that once bidders decide to bid (both in the first and second stages) it is weakly dominant for them to bid their true valuations. The cut-off point \( w_1 \) for the incumbent is found when he is indifferent as to whether to obtain the object in stage 1 at reserve price \( r \) or waiting till stage 2 and obtaining it at the highest valuation among \( k+l \) entrants. Thus,

\[
(w_1 - r)F_2(w_2)^k = \int_0^{w_1} (w_1 - v) dF_2(v)/F_2(v)^k \\
= \int_0^{w_1} \frac{F_2(v)^k}{F_2(v)^k} dv,
\]

and the cut-off point \( w_2 \) for each of the \( k \) entrants satisfies

\[
(w_2 - r)F_2(w_2)^k F_1(w_1) + \int_{w_1}^{w_2} (w_2 - v) F_2(w_2)^k F_2(v) F_1(v) dv \\
= \int_{w_1}^{w_2} (w_2 - v) F_1(w_1) dF_2(v)/F_2(v)^k F_1(v) \\
+ \int_0^{w_1} (w_2 - v) dF_1(v) F_2(v) F_2(v) F_1(v) dv.
\]

Here we have an extra term since an entrant with valuation \( w_2 \) will win against an incumbent whose valuation takes value in \((w_1, w_2]\). Rearranging,

\[
(w_1 - r)F_2(w_2)^k F_1(w_1) + \int_{w_1}^{w_2} \frac{F_2(w_2)^k F_1(v) dv}{F_2(w_2)^k F_1(v)} \\
= \int_0^{w_1} \frac{F_2(v)^k}{F_2(v)^k} F_1(v) dv \\
+ \int_{w_1}^{w_2} F_2(v) F_2(v)^k F_1(v) dv.
\]

We can express both conditions using truncated distributions

\[
\int_0^{w_1} F_2(v)^k H_2(v)^k dv \\
\int_{w_1}^{w_2} F_2(v)^k H_2(v)^k F_1(v) dv \\
+ \int_{w_1}^{w_2} F_2(v)^k H_2(v)^k F_1(v) dv.
\]
Combining both equations gives
\[
\int_0^{w_1} F_2(v) H_2(v)^k dv + F_1(w_1)^{-1} \int_{w_1}^{w_2} F_1(v) dv \\
= \int_0^{w_1} F_2(v) H_2(v)^{k-1} H_1(v) dv \\
+ \int_{w_1}^{w_2} F_2(v) H_2(v)^{k-1} dv.
\]

(21)

Let us define the revenue of the seller from the incumbent as \( R_s \), from each of the first-stage entrants as \( R_k \), and from each of the second-stage entrants as \( R_l \). The total revenue to the seller then is \( R_s + kR_k + lR_l F_2(w_2)^k F_1(w_1) \). Also define the expected profit of each of the first-stage entrants as \( P_k \), and of each of the second-stage entrants as \( P_l \).

For fixed entry \((k, l)\) we solve the following maximization problem:
\[
\max_{w_1, w_2} R_s(w_1, w_2) + kR_k(w_1, w_2) + lR_l(w_1, w_2) F_1(w_1) F_2(w_2)^k
\]
subject to constraint (21), and inequalities \( w_2 \geq w_1 \geq 0 \), \( P_k(w_2, w_1) \geq c \) and \( P_l(w_2, w_1) \geq c \). After solving for this, we can find the reserve price \( r \) from either Eq. (19) or (20). Next we present expressions for revenues \( R_s, R_k, R_l \), and profits \( P_k, P_l \).

The expected revenue from the incumbent is
\[
R_s = \int_{v_1}^{v_2} J_1(v) f_1(v) dv + \int_{v_2}^{v_2} J_1(v) F_2(v)^k f_1(v) dv \\
+ \int_{w_1}^{w_2} J_1(v) F_2(w_2)^k f_1(v) dv \\
+ \int_0^{v_2} J_1(v) F_2(v)^{k-1} f_1(v) dv,
\]

where, as usual, \( J_i(v) = v - \frac{1 - F_i(v)}{f_i(v)} \), for \( i = 1, 2 \). The expected revenue from each of the \( k \) first stage entrants is
\[
R_k = \int_{v_1}^{v_2} J_2(v) F_1(v) F_2(v)^{k-1} f_2(v) dv \\
+ \int_{v_1}^{v_2} J_2(v) F_1(w_1) F_2(v)^{k-l} f_2(v) dv \\
+ \int_0^{v_2} J_2(v) F_1(v) F_2(v)^{k-1} f_2(v) dv,
\]

and the expected revenue from each of the \( l \) second stage entrants is
\[
R_l = \int_{v_1}^{v_2} J_2(v) F_2(v)^{l-1} f_2(v) dv \\
+ \int_{v_1}^{v_2} J_2(v) H_2(v)^k f_2(v)^{l-1} f_2(v) dv \\
+ \int_0^{v_2} J_2(v) H_1(v) H_2(v)^k F_2(v)^{l-1} f_2(v) dv.
\]

The expected profit of each of the \( k \) first stage entrants is
\[
P_k = \int_{v_1}^{v_2} (1 - F_2(v)) F_1(v) F_2(v)^{k-1} f_2(v) dv \\
+ \int_{v_1}^{v_2} (1 - F_2(v)) F_1(w_1) F_2(v)^{k-l} f_2(v) dv \\
+ \int_0^{v_2} (1 - F_2(v)) F_1(v) F_2(v)^{k-1} f_2(v) dv \geq c,
\]

and the expected profit of each of the \( l \) second stage entrants is
\[
P_l = \int_{v_1}^{v_2} (1 - F_2(v)) F_1(v) F_2(v)^{l-1} f_2(v) dv \\
+ \int_{v_1}^{v_2} (1 - F_2(v)) H_2(v)^k F_2(v)^{l-1} f_2(v) dv \\
+ \int_0^{v_2} (1 - F_2(v)) H_1(v) H_2(v)^k F_2(v)^{l-1} f_2(v) dv \geq c.
\]

For numerical simulations we assume that valuations come from the uniform distributions on \([0, \tilde{v}_1]\), with \( F_1(v) = \frac{v}{\tilde{v}_1} \), and \( \tilde{v}_2 \leq \tilde{v}_1 \). The results from numerical simulations are summarized in Table 1. We have fixed \( \tilde{v}_1 = 1 \) for all simulations. Table 1 illustrates results when \( \tilde{v}_1 \) varies. The entry cost \( c \) was chosen to ensure that \( n \) entrants in the Anglo–Dutch auction earn exactly zero net profits. With the uniform distributions, and when first-stage entry is positive, one can show that \( w_1 = w_2 \) satisfies the equations for cut-off points. Among the results presented in Table 1 only in one case are the seller’s revenues lower in the two-stage English auction than in the Anglo–Dutch auction, namely, in the auction that does not induce a strictly larger (overall) number of entrants than the Anglo–Dutch auction: \( \tilde{v}_1 = 1.2, c = 0.072 \), where \( R^{AD} \) is equal to 0.531, and \( R^{SS} = 0.528 \). Yet, this is due to an integer problem. Indeed, two
Entrants expect substantial positive profits in the second stage of the two-stage English auction (0.0116 net of entry cost, in this case), yet a third one would expect negative profits. If we keep the profits of entrants in the second stage to zero, for instance, charging these entrants an entry fee of 0.0116, which would be paid if the incumbent did not bid (an event with probability 1/1.2), then $R^{2S}=0.547$ and then the revenues for the seller would again be greater in the two-stage auction.

**Appendix C. The two unit case**

Again we show that the two-stage English auction is more efficient than both the standard English auction and the Anglo–Dutch auction for all reserve prices $r$ when a separating equilibrium is played. The expected surplus for any one-stage auction is given by

$$S^1 = 2\bar{v} - \{2\mu_1(1 - \mu_1)(1 - \mu_2)^k + (1 - \mu_1)^2[2(1 - \mu_2)^k + n\mu_2(1 - \mu_2)^{n-1}]\} + (\bar{v} - \mu) - nc,$$

which is maximized by $n$ such that

$$\mu_2[(1 - \mu_1)^2\{2(1 - \mu_2)^k + n(1 - \mu_2)^{n-1}\} + 2\mu_1(1 - \mu_1)(1 - \mu_2)^{n-1}]\} + (\bar{v} - \mu) = c,$$

and is achieved using an English auction. Thus, necessarily expected surplus from the Anglo–Dutch auction is at most equal to the expected surplus from the standard English auction. The expected surplus for the two-stage English auction is given by

$$S^2 = 2\bar{v} - \{2\mu_1(1 - \mu_1)(1 - \mu_2)^k + (1 - \mu_1)^2k\mu_2\}$$

$$\times \{(1 - \mu_2)^{k-1}\} + (\bar{v} - \mu) + lc]$$

$$- (1 - \mu_1)^2(1 - \mu_2)^k\{2(1 - \mu_2)^k + l_2\mu_2\}$$

$$\times (\bar{v} - \mu) - lc - kc,$$

$$\{(1 - \mu_2)^{k-1}\} + (\bar{v} - \mu) + l_2\mu_2\}$$

$$\times (\bar{v} - \mu) - lc - kc.$$

First we observe that $k \leq n$ since the highest first-stage entry is achieved by setting the reserve price $r = \bar{v} + \varepsilon$. Define $l = n - k$ and rewrite expression for $S^1$ as follows:

$$S^1 = 2\bar{v} - \{2\mu_1(1 - \mu_1)(1 - \mu_2)^k + (1 - \mu_1)^2k\mu_2\}$$

$$\times \{(1 - \mu_2)^{k-1}\} + (\bar{v} - \mu) + lc]$$

$$- (1 - \mu_1)^2(1 - \mu_2)^k\{2(1 - \mu_2)^k + l_2\mu_2\}$$

$$\times (\bar{v} - \mu) - lc - kc$$

$$- (1 - \{2\mu_1(1 - \mu_1)(1 - \mu_2)^k + (1 - \mu_1)^2[(1 - \mu_2)^k$$

$$+ k\mu_2(1 - \mu_2)^{k-1}]\} + l_2\mu_2\}$$

Since $l_1$ and $l_2$ maximize expected surplus of the second stage when one and two units, respectively, are available, it follows that

$$(1 - \mu_2)^{l_1}\{(\bar{v} - \mu) + lc]\} + l_1c \leq (1 - \mu_2)^{l_2}\{(\bar{v} - \mu) + lc]\},$$

$$\{2(1 - \mu_2)^{l_2} + l_2\mu_2(1 - \mu_2)^{l_2-1}\}$$

$$\leq \{2(1 - \mu_2)^{l_1} + l_1\mu_2(1 - \mu_2)^{l_1-1}\},$$

and then the revenues for the seller would again be greater in the two-stage auction.

Observe that the previous argument easily extends to more than two units.

The revenues of the seller are the difference between net surplus and the profits of incumbents. We have already demonstrated that the surplus is higher in the two-stage English auction than in the Anglo–Dutch auction. For the Anglo–Dutch auction to yield higher revenues the expected social surplus received by incumbents must be smaller in the Anglo–Dutch auction than in the two-stage English auction.

It can be shown, as in the case of one unit, that the revenues in a two-stage auction are maximized either when there is no entry in the first stage and we set the highest reserve price that induces incumbents with high valuations to bid in the first stage, or when the entry level in the first stage is (almost) socially efficient. (It will not be an exactly socially efficient entry level because now incumbent profits are not independent of $k$.) We provide partial results on revenue ranking in both auctions.

**Lemma 11.** The expected surplus in any separating equilibrium of the two-stage English auction is higher than in a standard English auction and an Anglo–Dutch auction.

Observe that the previous argument easily extends to more than two units.
It can be shown that the incumbent’s expected utility in the Anglo–Dutch auction takes the same expression as in a (one-stage) English auction

\[ \mu_1[(1 - \mu_2)^n + (1 - \mu_1)n\mu_2(1 - \mu_2)^{n-1}](\bar{v} - \mu). \]

(25)

Thus the expected revenue of the seller in the Anglo–Dutch auction can be written as

\[
R^1 = 2\bar{v} - 2\mu_1(1 - \mu_1)[(1 - \mu_2)^n(\bar{v} - \mu) + nc]
\]

\[
- (1 - \mu_1)^2\{(2(1 - \mu_2)^n + n\mu_2(1 - \mu_2)^{n-1}\}
\]

\[
\times(\bar{v} - \mu) + nc - 2\mu_1(1 - \mu_2)^n
\]

\[
+ (1 - \mu_1)^2\{(1 - \mu_2)^n + n\mu_2(1 - \mu_2)^{n-1}\}\]

\[
\times(\bar{v} - \mu) - \mu_1nc.
\]

(26)

It can be shown that the number of entrants in the Anglo–Dutch auction is given by

\[
n = \max_{\mu_2} \frac{2\mu_2}{m(m + 1)} \left\{ \mu_2^2(1 - \mu_2)^{m-1} + \frac{1}{m}\mu_1(1 - \mu_1)(1 - \mu_2)^{m-1}
\right. \\
\left. + \frac{m - 1}{m}\frac{\mu_2}{(1 - \mu_2) + \frac{1}{m+1}(1 - \mu_1)(1 - \mu_2) + (1 - \mu_1)\mu_2}
\right\}
\]

\[
\times(1 - \mu_2)^{m-2} + \frac{4(m - 1)}{m(m + 1)}(1 - \mu_2)^2(1 - \mu_2)^{m-1}
\]

\[
+ \frac{2}{m}\(1 - \mu_1\)^2(n - 1)\mu_2(1 - \mu_2)^{m-2}\}
\]

\[
\times\frac{1}{m+1}(1 - \mu_2)^2 + \frac{\mu_2}{m+1}(1 - \mu_2) + \frac{1}{m}\mu_2^2
\]

\[
+ \frac{(m - 1)(m - 2)}{(m + 1)m}(1 - \mu_1)^2(1 - \mu_2)^{m-1}
\]

\[
+ \frac{m - 2}{m}(1 - \mu_1)^2(m - 1)\mu_2(1 - \mu_2)^{m-2}\}(\bar{v} - \mu) \geq c.
\]

(27)

We want to know when the revenue of the seller is higher in the two-stage English auction (Eq. (24)), where entry is given by conditions (11) and (12), than in the Anglo–Dutch auction where entry is given by the condition (27). First, observe that when \(\mu_1 = 0\), \(n\) is given by

\[
\mu_2[(1 - \mu_2)^{n+1} + (n + 1)\mu_2(1 - \mu_2)^n](\bar{v} - \mu) = c
\]

implying \(n = l_2 - 2\). The derivative of the expression in square brackets in Eq. (27) with respect to \(\mu_1\) is negative. The expression in square brackets in Eq. (27) is also declining with respect to \(m\). Therefore we may conclude that higher probability \(\mu_1\) leads to lower entry \(n\), and it is, at most, \(l_2 - 2\). Assuming that \(n = l_2 - 2\) for all \(\mu_1\) holds, when comparing Eqs. (24) and (26), we obtain that \(R^{2S} \geq R^{1S}\) if

\[
2\mu_1(1 - \mu_1)[(1 - \mu_2)^{l_1}(\bar{v} - \mu) + l_1c]
\]

\[
+ 2\mu_2^2(1 - \mu_2)^{l_2}(\bar{v} - \mu) \leq 2\mu_1(1 - \mu_1)
\]

\[
\times[(1 - \mu_2)^{l_2-2}(\bar{v} - \mu) + (l_2 - 2)c]
\]

\[
+ 2\mu_2^2(1 - \mu_2)^{l_2-2}(\bar{v} - \mu) + \mu_1^2(l_2 - 2)c
\]

or

\[
(1 - \mu_2)^{l_1}(\bar{v} - \mu) + l_1c - \mu_1l_1c \leq (1 - \mu_2)^{l_2-2}(\bar{v} - \mu) + (l_2 - 2)c,
\]

where the last inequality holds since \(l_1\) was chosen to minimize \((1 - \mu_2)^{l_1}(\bar{v} - \mu) + l_1c\).

References


