The law of conservation of persistence

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Abstract

Understanding the way economic variables respond to shocks is crucial in macroeconomics. A recent strand of literature has emphasized that, in the dynamic case, when the individuals are heterogeneous, estimates of shock persistence based on aggregate data are significantly higher than those derived from disaggregate data, either because the estimates are upwardly biased or because the aggregation process increases overall persistence.

This paper formally analyzes this issue and demonstrates that, in a context where individuals have heterogeneous linear dynamics, the response to an aggregate shock over time is the same, irrespective of the level of aggregation at which it is measured. More specifically, it shows that the aggregate Impulse Response Function (IRF) is simply the expected value of the individual responses. Thus, the aggregation of heterogeneous units does not amplify the response over time to economic shocks: the aggregate process is persistent if its components are (on average) persistent, but not because they are heterogeneous.

We also show that other popular persistence measures, such as the sum of the autoregressive coefficients, the largest autoregressive root, or the first autocorrelation are not suitable for establishing persistence comparisons across aggregation levels and can be highly misleading in this context. Finally, to illustrate the theoretical results, an empirical application using U.S. inflation data has been considered.

Key words: Heterogeneous dynamics, aggregation, persistence, impulse response function, sum of the autoregressive coefficients, U.S. inflation persistence.

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1. INTRODUCTION

A major topic of empirical macroeconomics is the analysis and measurement of shock response. Typically, aggregate data and a variety of tools, such as the Impulse Response Function (hereinafter, IRF) and other related scalar measures, are used to perform the analysis.

A recent strand of literature has emphasized that this approach can be problematic when the representative-agent hypothesis is violated. According to this view, in the dynamic case when the coefficients differ across individuals, shock persistence estimates based on aggregate data are significantly higher than those derived from disaggregate data. Two explanations have been provided to account for this phenomenon. Firstly, building on the results of Pesaran and Smith (1995), it has been argued that estimates computed with aggregate data are biased and that this bias translates into an overestimation of persistence measures (see Imbs et al., 2005). Secondly, other authors state that the aggregation of heterogeneous dynamic processes is not an innocuous operation and that it may tend to increase overall shock response (see Altissimo et al., 2006a, 2006b, among others).\(^1\) Several empirical studies corroborate these arguments by finding estimates of persistence that vary considerably across aggregation levels and are, in general, higher, the higher the level of aggregation (in addition to the above mentioned articles, see Altissimo et al., 2007, Crucini and Shintani, 2006, Clark, 2006, Lünnemann and Mathä, 2004, Abraham and White, 2006, etc). The conclusions of these articles are usually drawn by comparing the average across individuals of some scalar measures of persistence, typically the sum of the autoregressive coefficients, the largest autoregressive root or the first autocorrelation, with the values of the same measures computed with aggregate data.

Dynamic heterogeneity has been found to be relevant in a wide variety of contexts, such as in the speed of reversion to PPP (Imbs et al., 2005 and Crucini and Shintani, 2006), in the speed of reversion of income shocks, (Hu and Ng, 2004), in the dynamics of saving behavior (Haque et al., 2000), in inflation dynamics (Altissimo et al., 2006b, Angeloni et al., 2005), in labor demand across firms (Zhang and Small, 2006), etc. In fact, one could argue that the existence of some degree of heterogeneity across individuals is likely to be the rule rather than the exception in most contexts.

\(^1\)For instance, Altissimo et al (2006b), when summarizing the conclusions of the Inflation Persistence Network, state that “[...] it is apparent that the persistence of aggregated inflation series is typically higher than the average of the persistence of its subcomponents. These patterns illustrate how the time series properties of inflation can be subject to an aggregation effect.”
If the criticisms above are confirmed, they could be potentially very harmful because they imply that the standard approach of assessing the response of the economy to shocks using aggregate data would yield, on many occasions, estimates that would tend to systematically amplify the impact of shocks on the economy.

This paper formally analyzes this issue and shows that the problem is less severe than has been suggested. More specifically, it is shown that in a context where individuals have heterogeneous linear dynamics, the response over time of the aggregate variable to a common shock, as measured by the IRF, is simply the expected value of the individual responses to that shock. Therefore, the (average) response of the economy to an aggregate shock is the same, regardless of the level of aggregation at which it is considered.

The result above does not imply that the micro and the aggregate variables share the same stochastic properties. As is well known, it might be the case that all micro-processes are (short-memory) stationary but that the corresponding aggregate variable presents long-memory or is even non-stationary, (see Robinson, 1978, Granger, 1980 and Zaffaroni, 2004 for a complete characterization of this phenomenon). However, we show that even in these situations, the average response to aggregate shocks is preserved across aggregation levels. This result is particularly interesting because it establishes a direct link between shock responses at different aggregation levels that is invariant to the stochastic properties of the relevant variables. Reversing this same argument, different stochastic properties at the aggregate and the micro level do not imply different (average) shock response behavior.

Thus, if the relations above hold, why has (average) shock response been repeatedly found to be higher the higher the aggregation level? To answer this question, we have explored how other tools that are used to establish shock persistence comparisons perform in this context. The most popular device is probably the sum of the autoregressive coefficients (SAC). This measure was introduced as a summary of the aggregate IRF because it has a direct relation with the cumulative impulse response (CIR), namely, \( \text{CIR} = 1/(1-\text{SAC}) \). Thus, the values of the SAC can be easily interpreted since they exactly correspond to different values of the CIR. Typically, when applied to disaggregate data, the SAC is computed for each of the micro units and its average is used as a measure of persistence. Then, this value is usually compared to the SAC obtained with aggregate data. Notice, however, that, given the non-linear character of the above-described relation between the CIR and the SAC, it is no longer true that the average CIR equals

\[ \text{SAC} \]

\[ \frac{1}{1-\text{SAC}} \]

An stationary process is short (long)-memory if its autocorrelation function is (is not) summable.
the inverse of 1 minus the average SAC and, therefore, the latter value has not a meaningful interpretation. We show that, while the (average) CIR remains constant, the (average) SAC increases systematically with the aggregation level. Thus, by relying on the latter measure, one would erroneously conclude that shock response increases with the level of aggregation when, in fact, is not the case. Similar problems appear when other tools such as the largest autoregressive root (LAR) or the first autocorrelation are employed for these purposes.

Another aspect that could bring about differences among persistence estimates is the incorrect specification of the micro and/or the aggregate processes. Under heterogeneity, the dynamics of the aggregate process can become very complex and, hence, the risk of misspecification is higher (in which case, the results in Pesaran and Smith, 1995 would apply). Nevertheless, as opposed to what has been often argued in the literature, the existence of individual heterogeneity does not necessarily lead to misspecification of the aggregate model. We will also discuss how consistent estimation of the relevant quantities can be accomplished at the aggregate level.

Finally, to illustrate the theoretical results, the persistence properties of U.S inflation have been analyzed at different aggregation levels and it is shown that shock response remains fairly constant across them.

The structure of this paper is as follows. Section 2 presents a simple model for the individual data, the Koyck lag or random coefficients model, and defines some tools for measuring the average impact of an aggregate shock on the economy. Section 3 compares the measures of average micro-persistence derived in Section 2 with those obtained from the corresponding aggregate model. Section 4 considers aspects that are relevant for the empirical application of the results derived in Section 3, such as the holding of a Law of Large Numbers that relates the aggregate model (obtained as the expected value of the micro-relations) with real data (defined as a weighted sum of individual variables), as well as some estimation issues. Section 5 extends the results obtained in previous sections in several directions. Section 6 illustrates the theoretical results by analyzing the persistence properties of U.S. inflation at different aggregation levels and Section 7 concludes. The Appendix reports the results of some Monte Carlo experiments that compare the response to shocks over time across different aggregation levels computed with standard time series estimation techniques on simulated data.
2. HETEROGENEITY AND PERSISTENCE AT THE MICRO-LEVEL

The concept of persistence has been used in different contexts and with different meanings in the economic literature. In this paper we are interested in assessing the persistence of economic shocks, that is, the speed and pattern of adjustment of the process (or processes) of interest to shocks of different natures.

Suppose that, in an economy where agents have heterogeneous dynamics, one wants to evaluate the average impact of an aggregate shock affecting all individuals. This section presents a standard disaggregate model where this issue can be analyzed and reviews measures of shock persistence that can be employed, or have been employed in the past, for analyzing this issue. Section 3 applies the same measures to the aggregate model and compares the two sets of results.

A simple but commonly postulated model for microeconomic behavior that allows for heterogeneous dynamics is the dynamic random coefficients model. This model describes an economy in which each agent $i$ satisfies a linear regression with a Koyck lag given by:

$$y_{it} = a_i y_{i,t-1} + b_i' x_{it} + \nu_{it}, \quad i = 1, \ldots, N, \quad t = 1, \ldots, T, \quad (1)$$

$$\nu_{it} = \rho_i u_t + \varepsilon_{it}, \quad (2)$$

where $t$ denotes time, $y_{it}$ and $x_{it}$ are observable variables and $a_i = \bar{a} + \eta^a_i$, $b_i = \bar{b} + \eta^b_i$, $\rho_i = 1 + \eta^\rho_i$ are unknown coefficients, where $\eta^k$, for $k \in \{a, b, \rho\}$, are mutually independent, zero-mean random variables with variance $\sigma^2_k$. The innovation $\nu_{it}$ is the sum of two orthogonal, zero-mean martingale difference sequences, one common to all agents and one idiosyncratic, with variances $\sigma^2_u > 0$ and $\sigma^2_\varepsilon$, respectively. The distribution of $a$ has bounded support in the interval $[-1,1]$ and it is assumed that $E_I(a^h)$ exists for all $h$, where $E_I(.)$ denotes expectation across the distribution of individuals. The process $x_{it}$ is assumed to be independent of $u_t$ and the expectation $E_I(b' x_t)$ is assumed to exist. If $x_{it}$ is just a constant, then (1) is simply the first-order autoregressive model.

The distribution of agents can be discrete or continuous and the number of agents is assumed to be countably or uncountably infinite.

Suppose now that, at time $t$, a unitary aggregate shock occurs and one is interested in measuring its average impact over time on an economy like (1), populated by heterogeneous individuals. For

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4A sufficient, although not necessary, condition for this assumption to hold for all $h$ is that the support of $a$ is strictly contained in the interval $[-1,1]$. 

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each agent $i$, the impact of this shock can be evaluated through the Impulse Response Function, defined as the difference between two forecasts (see Koops et al., 1996 and Jorda, 2005)

$$IRF^i(t, h) = E(y_{it+h}|u_t = 1; z_{it-1}) - E(y_{it+h}|u_t = 0; z_{it-1})$$  

(3)

where the operator $E(\cdot|\cdot)$ denotes the best mean squared error predictor and $z_{it-1} = (y_{it-1}, y_{it-2}, \ldots x_{it-1}, x_{it-2}, \ldots)'$. Application of this definition to (1) yields

$$IRF^i(t, h) = \rho_i a^h_i, \text{ for } h \geq 0.$$  

(4)

Then, a natural measure of average individual response to a unitary common shock can be obtained by averaging (4) over the distribution of agents, that is,

$$IRF_{dis}(t, h) = E_I(IRF(t, h)) = E_I(a^h), \text{ for } h \geq 0,$$  

(5)

since $E(\rho)$ has been normalized to 1. The disaggregate IRF (denoted as $IRF_{dis}$), defined as the expected value of the individual micro responses, is simply given by the $h$-th moment of the distribution of $a$ for the simple DGP considered in this section.

Since the IRF is an infinite vector of numbers, it is a rather unwieldy measure of persistence. For this reason, scalar measures are frequently preferred. In most applications where economies with heterogeneous agents are considered, results are derived from the SAC and, to a lesser extend, from the LAR and the first autocorrelation. Typically, they are computed for each individual time series and, then, averages (or distribution quantiles) are reported as measures of average micro persistence. For some applications, see Altissimo et al. (2006a, 2007), Bilke (2005), Clark (2006), Lünnemann and Mathä (2004), Abraham and White (2006), etc.

Other popular scalar tools that summarize the information of the IRF are the Cumulated Impulse Response (CIR), which measures the total cumulative effect of a shock over time, and the half life (HL), defined as the number of periods it takes until half the effect of a shock dissipates. Using expression (5), it is straightforward to define the corresponding versions for measuring persistence at the micro level. The disaggregate CIR ($CIR_{dis}$) can be computed as

$$CIR_{dis} = \sum_{h=0}^{\infty} IRF_{dis}(t, h),$$  

(6)

which, for the simple model considered in this section, yields $CIR_{dis} = E_I\left(\frac{1}{1-a}\right)$. The disaggregate
HL (HL\textsubscript{dis}) can be defined as the value of \( h \) that verifies\(^6\)

\[
IRF_{\text{dis}}(t,h)\big|_{h=\text{HL}_{\text{dis}}} = 0.5.
\] (7)

Finally, other authors have employed an alternative definition of ‘disaggregate’ IRF to the one presented in this section to draw their conclusions about persistence. This is the case of Imbs et al. (2005) and Crucini and Shintani (2006), who first construct an ‘artificial’ representative agent model, \( y^*_t \), that has the same autoregressive structure as the units \( y_{it} \) but whose AR coefficients are given by the average of the individual-specific coefficients. Then, they compute the ‘disaggregate’ IRF by applying the standard definition of the IRF in (3) on \( y^*_t \).

In the following section we assess the performance of the above-defined measures of persistence when used to compare persistence across aggregation levels.

3. INDIVIDUAL HETEROGENEITY AND PERSISTENCE AT THE MACRO LEVEL

In most cases, the task of evaluating the impact of an aggregate shock on the economy is performed using only aggregate data. However, there is a widespread belief that in an economy with heterogeneous agents like (1), the magnitude and the persistence of the response is amplified by the aggregation process, making the shock response estimates higher, the higher the aggregation level. These claims can be found in many empirical papers (to mention only two, see Imbs et al., 2005 and Altissimo et al., 2006b).

This section compares the response to the same aggregate shock over time when it is computed from the micro model (1) or from its corresponding aggregate process. To establish the relation, we first focus on the IRF and, then, the behavior of other popular measures of shock persistence is also examined. For the sake of clarity, this section deals with a very simple model. Section 5 will extend the present analysis by relaxing some of the assumptions that will be imposed below.

How is the IRF computed from aggregate data related to the average of the individual impulse responses, defined in Section 2? In a representative agent economy, the answer would be trivial. This is because the individual and the aggregate models would share the same dynamics so the IRFs derived from each model would also be the same. Nevertheless, it is well-known that when the individuals are heterogeneous, the dynamics of the aggregate process become, in general,

\(^6\)Notice that, since the HL is a nonlinear function of the IRF and, thus, the average of the individual HLs does not coincide in general with the HL associated with IRF\textsubscript{dis}, defined in (7).

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much more complex. Thus, in order to compute the aggregate IRF, we first need to consider the aggregation of model (1). This problem has been addressed by Lewbel (1994), who followed the “stochastic” approach to aggregation introduced by Kelejian (1980) and formalized by Stoker (1984). The latter author defines an aggregate function as the expected value over the distribution of agents of the micro relations (see Stoker, 1984, Definition 2)

\[ Y_t = E_I (y_t) = \int y_t P(y_t|\theta_t) \, dy_t, \]  

where \( \theta (t) \) is a parameter vector that could vary over time but not across individuals and \( E_I (.) \) gives the time path of the dependent variable mean.

To derive the aggregate model, we follow the exposition in Lewbel (1994). In addition to the assumptions in Section 2, we assume that \( B = E_I (b) \), \( X_t = E_I (x_t) \), and \( E_I (\nu_t) = u_t \) exist. For simplicity, we also consider that the variables \( b \) and \( x \) are uncorrelated for all \( t \), and that \( a \) is independent of the distribution of \( (b'x + \nu) \). Some of these conditions can be easily relaxed without any change in the main result, as will be shown in Section 5. So,

\[ Y_t = E_I (ay_{t-1}) + B'X + u_t, \]  

where \( Y_t = E_I (y_t) \). Lewbel (1994) showed that, under the above-mentioned assumptions, expression (9) can be written as

\[ Y_t = \sum_{s=1}^{\infty} A_s Y_{t-s} + B'X_t + u_t, \]  

for constants \( A_1, A_2, ..., \) defined as \( A_s = E (\alpha_s) \), where \( \alpha_1 = a \) and \( \alpha_s = (\alpha_{s-1} - A_{s-1}) a \) for \( s > 1 \). These constants can be easily shown to satisfy the equation

\[ A_s = m_s - \sum_{r=1}^{s-1} m_{s-r} A_r. \]  

Thus, model (10) can be employed to evaluate how the aggregate variable \( Y_t \) responds to changes in the aggregate shock, \( u_t \).

### 3.1 Comparing Aggregate versus Disaggregate response

As in Section 2, we consider a unitary common shock occurring at time \( t \) and now we are interested in its effect on the aggregate variable \( Y_{t+h} \), for \( h \geq 1 \). The (aggregate) IRF (denoted henceforth as \( IRF_{AG} \)) can be easily computed as

\[ IRF_{AG}(t, h) = E (Y_{t+h}|u_t = 1; Z_{t-1}) - E (Y_{t+h}|u_t = 0; Z_{t-1}), \]
where $Z_{t-1} = (Y_{t-1}, Y_{t-2}, ..., X_{t-1}, X_{t-2}, ...)$. Application of this definition to (10) yields

$$IRF_{AG}(t, 1) = A_1; IRF_{AG}(t, 2) = (A_1^2 + A_2);$$

$$IRF_{AG}(t, 3) = (A_1 (A_1^2 + A_2) + A_2 A_1 + A_3);$$

and, in general,

$$IRF_{AG}(t, h) = \begin{cases}
IRF_{AG}(t, 0) = 1, \\
\sum_{j=1}^{h} A_j IRF_{AG}(h - j), & \text{if } h \geq 1.
\end{cases} \quad (13)$$

Although this expression seems complex, it can, in fact, be notably simplified. Notice that equation (11) can be rewritten as $m_s = \sum_{r=0}^{s-1} m_r A_{s-r}$ and, iterating the latter expression, it is straightforward to show that

$$IRF_{AG}(t, 1) = m_1 = E(a) = IRF_{dis}(t, 1)$$

$$IRF_{AG}(t, 2) = m_2 = E(a^2) = IRF_{dis}(t, 2)$$

$$\ldots$$

$$IRF_{AG}(t, h) = m_h = E(a^h) = IRF_{dis}(t, h). \quad (14)$$

It is also easy to check that expression (13) provides the $h$-th coefficient of the polynomial $A(L)^{-1}$ and, then, from (14), it follows that $A(z)^{-1} = M(z)$, where

$$M(z) = \sum_{j=0}^{\infty} E(a^j) z^j. \quad (15)$$

Thus, there is a direct link between the micro and the aggregate response to a common shock: the IRF computed in the aggregate model is just the expected value of the individual IRFs and, as will be shown in the next section, this is also true under less stringent assumptions than the ones imposed above. This result is very interesting because it implies that the aggregation of heterogeneous processes does not amplify the response to a given shock. It also implies that if the aggregate process is highly persistent, it is not so because the micro-units are heterogeneous but because, on average, they are highly persistent too.

The result above does not imply that other key properties of the micro processes, often related to the concept of persistence, are invariant to aggregation. As is well known, the aggregation of

\footnote{A related and independently developed result has recently been put forward by Caballero and Engel (2007). They consider a model of infrequent price adjustment at the disaggregate level and show that the response to a monetary shock is the same at the micro and at the macro level.}
short-memory stationary processes may produce an aggregate process that is long-memory or even non-stationary (see Section 4 for details). Nevertheless, even in this case, the relation described above between the behavior of the IRF across different aggregation levels would apply.

To illustrate this argument, consider again model (1) under the assumptions of Sections 2 and 3, with \( a \) having support in the interval \((0, 1)\). Clearly, since \( a_i < 1 \) for all \( i \), all individual processes are (short-memory) stationary. Thus, they admit a Wold representation (for simplicity, we assume that \( b_i = 0 \) for all \( i \))

\[
y_{it} = \nu_{it} + a_i \nu_{it-1} + a_i^2 \nu_{it-2} + \ldots, \quad i = 1, \ldots, N, \quad t = 1, \ldots, T. \tag{16}
\]

The aggregate process \( Y_t = E_I (y_{it}) \) could be computed by taking expectations in (16), provided the expected value of the right-hand side yields a well-defined MA representation. This will happen provided the resulting MA coefficients are square-summable, that is, if \( \sum_{j=0}^{\infty} E_I (a^j)^2 < \infty \). As shown by Zaffaroni (2004), the holding of this condition crucially depends on the behavior of the distribution of \( a \) in a neighborhood of 1, as will be explained in more detail in Section 4. Consider, for instance, that \( a \) follows a Beta\((p, q)\) distribution on the interval \((0, 1)\), with density

\[
f (a) = \frac{2}{B(p, q)} a^{2p-1} (1 - a^2)^{q-1}, \quad 0 < a < 1,
\]

where \( p, q > 0 \) are the parameters of the distribution and \( B(p, q) \) is the beta function. The moments of \( a \) are given by

\[
m_j = E (a^j) = \frac{B(p + j/2, q)}{B(p, q)},
\]

and for large \( j \),

\[
m_j \approx (p + j/2)^{-q}. \tag{17}
\]

Clearly, for values of \( q \) smaller than 0.5, the sequence \( \{m_j\}_{j=0}^{\infty} \) is not square-summable, implying that the MA representation of \( Y_t \) is not mean-square convergent and, then, \( Y_t \) is not covariance-stationary. Notice, however, that the AR expansion described in (9) exists and is well defined, since \( A(L) Y_t \) is a mean-square convergent random variable. If \( 0.5 < q < 1 \), \( Y_t \) is stationary and the k-th autocorrelation can be approximated by \( \rho_k \approx ck^{1-2q} \). Then, if \( q \in (0.5, 1) \) the correlations are not summable and \( Y_t \) is a long-memory process.\(^8\)

Notice however that the aggregate IRF derived from \( Y_t \) would equal the average of the micro IRFs derived from the stationary processes, since all the conditions needed for obtaining (14) are fulfilled.

\(^8\)More specifically, \( Y_t \) could be approximated by a fractionally-integrated process with order of integration \( d = (1 - q) \). See Granger (1980) for details.
Finally, the relation between the micro and the aggregate IRFs can also be exploited in other ways. If disaggregate data is available, it is possible to completely determine the dynamics of the aggregate process. If the latter is stationary, the disaggregate IRF defined in (5) equals the coefficients of the Wold representation of the aggregate process. Then, the macro dynamics can be identified in a very simple way from the (average) of the micro impulse responses (and vice versa). If the aggregate process is not stationary, its Wold representation does not exist. However, the coefficients of the polynomial \( A(z) = M(z)^{-1} \), where \( M(z) = \sum_{j=0}^{\infty} m_j z^j \) are the coefficients of the disaggregate IRF, are those corresponding to the AR(\( \infty \)) representation of the aggregate process. These relationships could potentially be used to improve the accuracy of the estimation combining micro and macro data, in the spirit of Imbens and Lancaster (1994).

Moreover, it would be possible to recover micro information when only aggregate data is available. Since the aggregate IRF identifies all the moments of the distribution of the autoregressive parameter, \( a \), it is possible to infer some interesting information on the distribution of this parameter only from the aggregate data. For instance, probability bounds for \( a \) can be easily computed from the aggregate data by applying Markov and Chebichev inequalities. Furthermore, under certain conditions, the whole distribution of the autoregressive parameter \( a \) can be recovered using only aggregate data. This is the well known “moment problem”, that consists of inverting the mapping that takes a probability measure to the sequences of moments. This problem has already a long tradition, started by Stieljes (1894) and developed by subsequent authors (see, for instance, Lin, 1997 for a description of the conditions needed to recover the density from the moments).

### 3.2 Other measures of persistence

We now examine how the remaining measures of persistence mentioned in Section 2 perform in this framework. The aggregate versions of the CIR and the HL are defined as

\[
CIR_{AG} = \sum_{h=0}^{\infty} IRF_{AG}(t, h),
\]

and

\[
IRF_{AG}(t, h)|_{h = HL_{AG}} = 0.5,
\]

respectively. Clearly, the equality between the IRF \( AG \) and the IRF \( dis \), established in (14), implies that the CIR and the HL also present the same population values across aggregation levels.

However, most of the empirical papers focus on the sum of the autoregressive coefficients for comparing persistence across aggregation levels. In these articles, the SAC computed with aggre-
gate data (denoted henceforth as SAC\(_{AG}\)) is compared with the average of the individual SACs (SAC\(_{dis}\)), see for instance Altissimo et al. (2006a) and the references therein.

The SAC was originally introduced by Andrews and Chen (1994) because it has a direct relation with the CIR through the relation

\[ SAC = 1 - CIR^{-1}, \tag{18} \]

and so, “different values of the SAC can be interpreted easily in terms of persistence because they correspond straightforwardly to different values of the CIR” (Andrews and Chen, 1994).

However, although this relation holds when a single series is employed, it is clear that the relation (18) no longer holds when the SAC and the CIR are replaced by the corresponding averages. In fact, although the CIR remains constant across aggregation levels, the SAC increases systematically with the level of aggregation. To see this, notice that for the simple model described in (1), CIR\(_{dis}\) is given by

\[ CIR_{dis} = 1 + E_I(a) + E_I(a^2) + \ldots = E_I \left( \frac{1}{1-a} \right) \]

and recall that CIR\(_{dis}\) = CIR\(_{AG}\). Since \( \frac{1}{1-a} \) is a convex function, Jensen’s inequality implies that

\[ SAC_{AG} = 1 - \left( E_I \left( \frac{1}{1-a} \right) \right)^{-1} > 1 - \left( \frac{1}{1 - E_I(a)} \right)^{-1} = E_I(a) = SAC_{dis}, \tag{19} \]

so the aggregate SAC is bigger than the average of the individual SACs unless there is no heterogeneity, in which case both measures are equal. It follows that SAC\(_{dis}\) is a poor summary of the CIR\(_{dis}\) and then, it is not a suitable tool for establishing persistence comparisons.

The inequality in (19) can be illustrated with a simple example. Consider, again, a collection of individuals behaving as in (1) (with \( b_i = 0 \), for all \( i \) for simplicity) with a heterogenous autoregressive parameter, \( a \), that follows a U(0,1) distribution. Thus, the (population) value of the average of the individual SACs, SAC\(_{dis}\), is equal to 0.5. With respect to CIR\(_{dis}\), the moments of the uniform distribution, given by \( E(a^h) = (h+1)^{-1} \), are not summable and so CIR\(_{dis}\) = M(1) = \( \sum_{h=0}^{\infty} E(a^h) = \infty \). As for the corresponding aggregate values, notice that, in this case, \( Y_t \) can be obtained by taking expectations in (16) since the aggregate MA(\( \infty \)) representation is well defined.\(^9\) Recalling that \( A(z) = M(z)^{-1} \) for all \( z \), it follows that if \( z = 1 \) then \( A(1) = M(1)^{-1} = 0 \). Thus, SAC\(_{AG}\) = 1 − A(1) = 1. With respect to the CIR, it can easily checked that CIR\(_{AG}\) = M(1) = \( \infty \). Thus, although the CIR remains constant across aggregation levels, the SAC increases considerably (from 0.5 to 1), which could clearly lead to the wrong conclusion that shocks are more persistent at the aggregate than at the micro level.

\(^9\)\( Y_t \) is stationary with variance \( \sigma_u^2 \sum_{j=0}^{\infty} (j+1)^{-2} < \infty \).
The example above also illustrates another potential pitfall of the SAC. This measure is usually interpreted in the following way: the closer it is to 1, the higher the persistence of the process. A value of the SAC equal to one is usually taken as evidence of permanent shock behavior. However, the usual interpretation of the SAC can be highly misleading since \( SAC = 1 \) does not imply permanent shock behavior. In fact, it only implies that the spectral density has a pole at frequency zero. The set of processes verifying this condition is very large, containing \( I(1) \) processes (for which the above-mentioned interpretation of permanent shock behavior is valid) but also stationary and non-stationary ones with non-permanent shock behavior that are much less persistent than \( I(1) \) processes. As is well known, processes verifying \( A(1) = 0 \) are very likely to arise in practice when the aggregation of heterogeneous units is considered.\(^{10}\) Therefore, it is important to bear in mind these considerations when applying these measures to macroeconomic data.

As for the LAR, it suffers from similar problems. For instance, for the example above, it is easy to see that \( \text{LAR}_{\text{dis}} = \text{SAR}_{\text{dis}} = 0.5 \) and \( \text{LAR}_{\text{AG}} = \text{SAR}_{\text{AG}} = 1 \), where \( \text{LAR}_{\text{dis}} \) and \( \text{LAR}_{\text{AG}} \) are the average of the individual LARs and the LAR associated to \( Y_t \), respectively. Thus, the LAR should not be employed either to compare persistence across aggregation levels.

Comparing the averages of the first autocorrelation of the individual models with the first autocorrelation of the aggregate data is also a problematic strategy since, in general, the average value of the first autocorrelation of the individual processes (\( \text{FAC}_{\text{dis}} \)) does not equal the first autocorrelation of the aggregate process, in spite of sharing the same IRF. Firstly, notice that the aggregate population autocorrelation function may not even exist, since the aggregate process can be nonstationary. If the aggregate process is stationary, Pesaran and Smith (1995) have shown, in a framework similar to the one considered in this paper, that the first autocorrelation of the aggregate process is not a consistent estimator of the \( \text{FAC}_{\text{dis}} \). Again, this should not be interpreted as different (average) shock response at the micro and the macro level. Instead, this result suggests that the autocorrelation function is not a suitable instrument for comparing shock responses.

Finally, in some applications, an alternative definition of disaggregate IRF to the one presented in this paper has been employed (see Imbs et al., 2005, and Crucini and Shintani, 2006). As mentioned in Section 2, the strategy consists of fitting autoregressive models to each of the

\(^{10}\)See Zaffaroni (2004) for a semiparametric characterization of the family of distributions that generate this behavior.
heterogeneous processes and to compute the average of these AR coefficient estimates. Next, an “averaged” model is constructed, having the same AR structure as the individuals but whose coefficients are given by these averaged estimates. Then, the IRF of this artificially generated process is computed as in a model with homogeneous coefficients. Thus, estimates of the function

\[ \overline{IRF}(t, h) = E_I (a)^h, \text{ for } h \geq 0, \]  

are provided as measures of micro-persistence. But clearly, since the IRF is not a linear function, (20) does not capture the average response of the micro units. Furthermore, it can be easily seen that, in most empirically relevant cases, (20) systematically underestimates the true average response. Whenever the support of \( a \) is positive, which is a very realistic assumption in many applications, \( a^h \) is strictly convex and the application of Jensen’s inequality yields

\[ E_I (IRF(t, h)) = E_I \left( a^h \right) > \overline{IRF}(t, h) = E_I (a)^h, \text{ for all } h > 1. \]  

Thus, if estimates of (20) are compared with estimates of IRF_{AG}, important differences are likely to arise but mainly due to the underestimation of shock responses at the micro level rather than an overestimation of the response when aggregate data is employed.\(^{11}\)

The discussion above helps to explain why so many papers have reported evidence of different persistence levels when they are measured using disaggregate or aggregate data: the comparison is usually carried out using tools that are not appropriate for this purpose. The LAR, the SAC and the first autocorrelation present, in general, different values across aggregation levels even when the (average) response to a common shock is identical.

4. AGGREGATION, THE LAW OF LARGE NUMBERS AND OTHER ISSUES

The discussion contained in Section 3 is based on results that are true for the population. But, in order to be of interest to practitioners, it must be possible to obtain good approximations to the population values when the corresponding sample counterparts are employed, at least when \( N \) and \( T \) are sufficiently large. Thus, two issues deserve consideration here. Firstly, the population aggregate model is defined as the expected value of the individual processes. However,\(^{11}\)

\(^{11}\)In fact, Gadea and Mayoral (2007) have shown that the reduction in real exchange rate persistence found by these authors using sectoral real exchange rates is due to their definition of average micro persistence. When the disaggregate IRF defined in (5) is employed, standard estimates of persistence, highly compatible with those obtained from aggregate data, are recovered.
aggregate data, denoted as \( \bar{Y}_{Nt} \) henceforth, is usually constructed as (a possibly weighted) average of the individual data. Hence, \( \bar{Y}_{Nt} \) would approximately follow the aggregate model in (10) if a LLN relating \( Y_t \) and \( \bar{Y}_{Nt} \) holds. Secondly, the aggregate model contains an infinite number of coefficients. Thus, one could think that the estimation of this model with a finite sample could be problematic so that it could be the case that individual and aggregate divergences arise at this step. In this section we briefly examine these two issues.

For simplicity, we assume that the aggregate data \( \bar{Y}_{Nt} \) is constructed as a simple average of a large number of individual processes,

\[
\bar{Y}_{Nt} = \frac{1}{N} \sum_{i=1}^{N} y_{it},
\]

where \( y_{it} \) is defined as in (1) with \( b_i = 0 \) for all \( i \). \( \bar{Y}_{Nt} \) can be written as the sum of two terms

\[
\bar{Y}_{Nt} = \frac{1}{N} \left( \sum_{i=1}^{N} \frac{\varepsilon_{it}}{1 - a_i L} + \sum_{i=1}^{N} \rho_i u_t (1 - a_i L) \right),
\]

that will be referred to as the idiosyncratic and common components, respectively. Then, \( \bar{Y}_{Nt} \) would be a good approximation of the aggregate model derived in Section 3 provided that a LLN applies, such that \( \bar{Y}_{Nt} \) and \( Y_t = E_t(y_t) \) are close for large \( N \). However, it has been argued that, for the disaggregate model considered in this paper, such a LLN might not hold (see, for instance, Forni and Lippi, 1997, pp. 17). Since the applicability of the results developed in Section 3 relies on this convergence, it is worth considering this issue in more detail.

The holding of a LLN relating \( \bar{Y}_{Nt} \) and \( Y_t \) hinges on whether the limit of \( \bar{Y}_{Nt} \) when \( N \to \infty \) is stationary or not. So, before considering the convergence of \( \bar{Y}_{Nt} \) and \( Y_t \), the asymptotic properties of \( \bar{Y}_{Nt} \) as \( N \) increases should be reviewed.

This issue has been analyzed by Zaffaroni (2004) and we briefly summarize the results that are relevant to the problem considered here. The asymptotic behavior of \( \bar{Y}_{Nt} \) critically depends on the properties of the distribution of \( a \) around 1. As shown by Granger (1980), if the support of \( a \) is given by \( [\delta_1, \delta_2] \) with \( \delta_2 < 1 \), the corresponding aggregate process is I(0), for any shape of the distribution of \( a \).\(^{12}\) On the other hand, if \( \delta_2 = 1 \) and the distribution of \( a \) is such that \( P(a = 1) > 0 \), then \( \bar{Y}_{Nt} \) converges to an I(1) random variable. An interesting intermediate case arises whenever \( \delta_2 = 1 \) and \( a \) belongs to a family of absolutely continuous distributions such that \( P(a = 1) = 0 \). To characterize the convergence in this case, Zaffaroni (2004) considers the

\(^{12}\)See Davidson (2007) for a precise definition of I(0)-ness.
following semiparametric specification of the density of \( a \in (0, 1) \) around unity: \( ^{13} \)

\[
f(a) \sim c_b (1 - a)^{-b}, \quad \text{as } a \to 1, \quad 0 < c_b < \infty, \quad b \in [0, 1)
\]

In this case, \( \bar{Y}_{Nt} \) converges to a stationary random variable provided \( b < 0.5 \) and to a nonstationary one otherwise. Interestingly, if \( 0 \leq b < 0.5 \), the limit of \( \bar{Y}_{Nt} \) is a long-memory process and if \( b > 0 \), the limit process can be characterized as a fractionally integrated process with order of integration \( d = b \).

Under similar assumptions as the ones adopted in this paper, Zaffaroni (2004) shows that provided the limit of \( \bar{Y}_{Nt} \) is stationary, then a strong LLN holds and \( \bar{Y}_{Nt} \overset{L^2}{\to} E_t(y_t) = Y_t \). In this case, the idiosyncratic component converges almost surely to zero while the common component converges in \( L^2 \) to the corresponding expectation: \( ^{14} \)

\[
N^{-1} \sum_{i=1}^{N} \frac{\rho_i u_t}{(1 - a_i L)} \overset{L^2}{\to} E_t \left( \frac{\rho u_t}{1 - a L} \right) = \sum_{j=0}^{\infty} E_t \left( a^j \right) u_{t-j}.
\]

However, whenever the limit of \( \bar{Y}_{Nt} \) is a nonstationary random variable, the convergence above fails: the idiosyncratic component no longer vanishes because the variance of \( N^{-1} \sum_{i=1}^{N} \frac{\varepsilon_{it}}{(1 - a_i L)} \) tends to infinity. On the other hand, the common component does not converge to its expected value because neither the Bochner nor the Pettis integral of this component exist.

In principle, this could be a major drawback for the results established in Section 3. If nonstationary variables are observed, \( \bar{Y}_{Nt} \) is not a good proxy for \( Y_t \). Then, one should not expect that persistence estimates obtained with aggregate data, \( \bar{Y}_{Nt} \), are close to those obtained with the corresponding disaggregate variables. Notice, however, that this problem has an easy solution. Taking first differences to the original aggregate data, \( \bar{Y}_{Nt} \), we obtain

\[
(1 - L) \bar{Y}_{Nt} = \frac{1}{N} \left( \sum_{i=1}^{N} \frac{(1 - L) \varepsilon_{it}}{(1 - a_i L)} + \sum_{i=1}^{N} \frac{(1 - L) \rho_i u_t}{(1 - a_i L)} \right), \tag{23}
\]

and, in this case, the same results as in the case where the limit of \( \bar{Y}_{Nt} \) is stationary are recovered, that is, the idiosyncratic component in (23) converges to zero while the common one converges.

\(^{13}\)This condition is semiparametric because the behavior of the density for any given interval \( [0, \gamma_2] \) with \( \gamma_2 < 1 \) is unspecified. Standard distributions, such as the Uniform or the Beta, are contained in this specification by setting \( d = 0 \) and \( d \geq 0 \), respectively.

\(^{14}\)The expectation of the idiosyncratic component is taken with respect to the Pettis integral (see Uhlig, 1986). This is because the Bochner integral (which extends the definition of the Lebesgue integral to functions taking values in a Banach space) of that component may not exist. This is the well known measurability problem (see Judd, 1985).
to the corresponding expectation. Thus, it holds that

\[(1 - L) \tilde{Y}_{Nt} \overset{L^2}{\to} (1 - L) Y_t, \tag{24}\]

where \((1 - L) Y_t\) is the first differences of \(Y_t = E_i(y_t)\) and is a stationary process. Thus, whenever nonstationarity is detected, the usual procedure of first differentiating the data would be sufficient in order to guarantee the convergence of \((1 - L) \tilde{Y}_{Nt}\) to \((1 - L) Y_t\). The IRF of \(Y_t\) can be estimated by first estimating the IRF associated with \((1 - L) \tilde{Y}_{Nt}\), and, then, cumulating the corresponding values. That is,

\[
IRF_{AG}(t, h) = \sum_{j=1}^{h} IRF_{(1-L)\tilde{Y}_N}(j, t),
\]

where \(IRF_{AG}(t, h)\) and \(IRF_{(1-L)\tilde{Y}}(j, t)\) denote the estimates of the IRFs associated to \(Y_t\) and to \((1 - L) Y_t\), respectively. Fortunately, the stationarity of the aggregate process can be tested using standard techniques without requiring any further knowledge of the micro data.

As regards the estimation of the infinite-order AR aggregate process \(Y_t\), consistent estimates of the model can be obtained using standard techniques. If \(Y_t\) is an \(I(0)\) random variable, the results of estimation of AR(\(\infty\)) processes developed by Berk (1974) can be applied. He showed that consistent estimates are achieved by estimating an AR(\(k\)) process, where \(k\) grows at an appropriate rate with respect to the sample size \(T\).\(^{15}\) Likewise, if the aggregate model is integrated of order 1, similar results can also be applied on the first differences of \(Y_t\). If the aggregate model has a fractional order of integration, \(d\), verifying that \(0 < d < 1\), first differences should also be applied to the original data whenever \(d \geq 0.5\), for the reasons stated above. Then, the vast literature on estimation of fractionally integrated processes can be employed (see, for instance, Robinson, 2003). Alternatively, provided \(d < 0.5\), a long AR(\(k\)) specification would also deliver consistent estimates of the parameters (see Godet, 2007). Although a full analysis of this issue is beyond the scope of this paper, the results of some Monte Carlo experiments illustrating how standard time series techniques allows us to achieve highly compatible persistence estimates across different levels of aggregation are provided in the Appendix.

The above-mentioned characteristics of the aggregate data can be tested with techniques that are already standard in the time series literature. Thus, in sum, the arguments above imply that, when applying the appropriate estimation techniques, micro and macro estimates of persistence

\[^{15}\text{More specifically, consistency is achieved provided } k, T \to \infty \text{ and } k^3/T \to 0.\]
should be very close if the assumptions of the previous sections hold and \( N \) and \( T \) are sufficiently large.

5. EXTENSIONS

A more realistic approach than the one analyzed in Section 3 would entail considering vector autoregressive processes, fewer assumptions of independence and uncorrelatedness between the variables of the model as well as allowing for more general individual dynamics. This section extends the results of Section 3 in these directions.

We now consider vector autoregressive (VAR) systems at the disaggregate level. For this, equation (1) is reinterpreted so that \( y_{it} \) and \( \nu_{it} = \rho_i u_t + \varepsilon_{it} \) are now \( J \) vectors of random variables, \( a_i \) is a \( J \times J \) matrix of random coefficients, \( \rho_i \) is a diagonal \( J \times J \) matrix verifying that \( \mathbb{E}(\rho_i) = I_J \), where \( I_J \) is the identity matrix of order \( J \), and for simplicity, \( b_i \) is set equal to 0 for all \( i \).

Following Jordá (2005) the individual IRF associated with this model can be obtained by

\[
IRF_i(t,h) = \mathbb{E}(y_{it+h}|u_t = \bar{\delta}; z_{it-1}) - \mathbb{E}(y_{it+h}|u_t = \bar{\delta}; z_{it-1})
\]

where \( z_{it-1} = (y_{it-1}, y_{it-2}, \ldots)' \), \( \bar{\delta} \) is a \( J \) vector of zeroes and \( \delta \) are the relevant experimental shocks. Under the assumptions of Section 3, it is easy to check that, in the vector case, the individual IRF is given by

\[
IRF_i(t,h) = a_i^h \rho_i \delta, \quad \text{for} \ h \geq 0.
\]

and, then, the average response can be defined as

\[
IRF_{dis}(t,h) = \mathbb{E}( IRF(t,h) ) = \mathbb{E}( a^h ) \delta, \quad \text{for} \ h \geq 0.
\]

The aggregate VAR can be obtained using a similar strategy as in the scalar case. Taking expectations in (1) and adding and subtracting the matrix \( A_1 = E(a_1) \), we get

\[
Y_t = E_I(y_t) = A_1 Y_{t-1} + E_I((A_1 - E(A_1)) (\alpha_{t-2} + \nu_{t-1})) + u_t.
\]

Defining the \( J \times J \) matrix \( A_s = E(\alpha_s) \), where \( \alpha_s = (\alpha_{s-1} - A_{s-1}) a \) and \( \alpha_1 = a \), one can iterate the procedure above to obtain the aggregate model

\[
Y_t = \sum_{s=1}^{\infty} A_s Y_{t-s} + u_t.
\]

The aggregate IRF can be obtained as,

\[
IRF_{AG}(t,h) = \mathbb{E}(Y_{t+h}|u_t = \delta; Z_{t-1}) - \mathbb{E}(Y_{t+h}|u_t = \bar{0}; Z_{t-1})
\]
where \( Z_{t-1} = (Y_{t-1}, Y_{t-2}, \ldots) \), that yields the expression \( IRF_{AG}(t, 0) = \delta \) and

\[
IRF_{AG}(t, h) = \sum_{j=1}^{h} A_j IRF_{AG}(h - j), \quad \text{for } h \geq 1. \tag{28}
\]

As in the scalar case, it is easy to show that the relations \( A_s = m_s - \sum_{r=1}^{s-1} m_{s-r} A_r \) and \( m_s = \sum_{r=0}^{s-1} m_r A_{s-r} \) also hold in the matrix case, where \( m_s \), defined as \( m_s = E_I(a^s) \), is of order \( J \times J \). Thus, it is straightforward to check that

\[
IRF_{AG}(t, h) = E_I\left(a^h\right) \delta. \tag{29}
\]

Thus, comparing (29) and (27), one can conclude that in the vector case, the average response to economic shocks is the same, regardless of the level of aggregation at which it is considered.

When correlation between some of the random variables entering the micro model is allowed, in general, additional terms enter the aggregate equation. For instance, the assumption most likely to be violated is that of independence between \( a \) and \( b \). Whenever this assumption is violated, it is obtained that (see Lewbel, 1994)

\[
Y_t = \sum_{s=1}^{\infty} A_s Y_{t-s} + B' X_t + \sum_{s=1}^{\infty} Cov(\alpha_s, b) X_{t-s} + u_t, \tag{30}
\]

where \( \alpha_1 = a \) and \( \alpha_{s+1} = (\alpha_s - E(\alpha_s)) a \), for \( s \geq 1 \). However, it is straightforward to check that the expression of the aggregate IRF in (13) is not altered by the addition of this new term and, hence, the relation between the micro and macro IRFs established in Section 3 is preserved.

If \( X_t \) is a simply a vector of ones, then, provided \( \sum_{s=1}^{\infty} Cov(\alpha_s, b) < \infty \), the new term is just an addition to the aggregate intercept. Otherwise, in order to have consistent estimates of the aggregate responses, an increasing number of lags of \( X_t \) should be included in the aggregate equation.

If the distributions of \( b \) and \( x_t \) are not independent, the aggregate model becomes,

\[
Y_t = \sum_{s=1}^{\infty} A_s Y_{t-s} + B' X_t + Cov(b, x_t) + u_t, \tag{31}
\]

and, as in the previous case, the expression of the aggregate IRF is not affected by this term. If the covariance between \( b \) and \( x_t \) is finite constant over time, the \( Cov(b, x_t) \) is also an addition to the aggregate constant term.

The next step would be to allow higher order lags at the disaggregate level. To simplify the exposition, we next analyze the AR(2) case. The AR(p) one, though notationally cumbersome,
can be analyzed in the same manner. Suppose the individual agents $i$ behave according to the model,

$$y_{it} = a_{1i}y_{it-1} + a_{2i}y_{i(t-2)} + u_{it}, \quad i = 1, ..., N, \quad t = 1, ..., T. \tag{32}$$

In addition to the assumptions stated in Section 3, we now require the existence of all the moments of $a_2$ and all the cross-moments between $a_1$ and $a_2$. Notice that we do not need to assume independence between $a_1$ and $a_2$. The aggregate model is obtained by taking expectations in (32),

$$Y_t = E_I(a_1 y_{t-1}) + E_I(a_2 y_{t-2}) + u_t, \quad t = 1, ..., T.$$

which can be written as an AR($\infty$) process

$$Y_t = \sum_{j=0}^{\infty} C_j Y_{t-j} + u_t. \tag{33}$$

In order to obtain the coefficients $C_j$, define $a_j = a_i, A_{jk} = E(a_{jk})$ and $a_{jk+1} = (a_{jk} - E(a_{jk})) a_j$, for $j = 1, 2$ and $k = 1, 2, ..., $ and notice that

$$E_I(a_1 y_{t-1}) = E_I(a_1 + E_I(a_1) - E_I(a_1) y_{t-1}) =$$

$$E_I(a_1) Y_{t-1} + E_I(\alpha_1 y_{t-2}) + E_I((a_1 - E(a_1)) a_2 y_{t-3}),$$

and

$$E_I(a_2 y_{t-2}) = E_I(a_2 + E_I(a_2) - E_I(a_2) y_{t-2}) =$$

$$E_I(a_2) Y_{t-2} + E_I((a_2 - E_I(a_2)) a_1 y_{t-3}) + E_I(\alpha_2 y_{t-4}),$$

Iterating the procedure above, after some algebra one can obtain that

$$C_1 = E_I(\alpha_{11}) = E_I(a_1)$$

$$C_2 = E_I(\alpha_{12}) + E_I(\alpha_{21}) = Var(a_1) + E_I(a_2)$$

$$C_3 = E_I(\alpha_{13}) + 2cov(a_1, a_2) = Cov(a_{1}^{2}, a_1) + 2cov(a_1, a_2)$$

$$C_4 = E_I(\alpha_{14}) + E_I(\alpha_{22}) + 3cov(a_1 a_2, a_1),$$

Next, we check that the disaggregate and aggregate IRFs are the same. The IRF for each of the micro-units can be easily computed from definition (3) and, taking expectations, it is obtained

\footnote{Notice that these coefficients are different from the ones provided by Lewbel (1994) (Equations 6’ and 16) since those expressions contain several typos.}
that

\[ IRF_{dis}(t,1) = E_I(a_1); \quad IRF_{dis}(t,2) = E_I(a_1^2) + E_I(a_2); \]
\[ IRF_{dis}(t,3) = E_I(a_1^3) + 2E_I(a_1a_2); \quad IRF_{dis}(t,4) = E_I(a_1^4) + 3E_I(a_1^2a_2) + E_I(a_2^2), \]

and, in general,

\[ IRF_{dis}(t,h) = E_I(\text{IRF}(t,h)) = E_I(a_1\text{IRF}(t,h-1) + a_2\text{IRF}(t,h-2)). \] (35)

It is not difficult, although algebraically tedious, to check that the aggregate IRF associated with the aggregate process (32) coincides with the one obtained from averaging the individual IRFs. For instance, consider the first values of the aggregate IRF that are given by

\[ IRF_{AG}(t,1) = C_1 = E_I(a_1) = E_I(\text{IRF}(t,1)) = IRF_{dis}(t,1) \]
\[ IRF_{AG}(t,2) = C_1^2 + C_2 = E_I(a_1^2) + E_I(a_2) = E_I(\text{IRF}(t,2)) = IRF_{dis}(t,2) \]
\[ IRF_{AG}(t,3) = C_1^3 + 2C_1C_2 + C_3 = E_I(a_1^3) + 2E_I(a_1a_2) = E_I(\text{IRF}(t,3)) = IRF_{dis}(t,3) \]
\[ IRF_{AG}(t,4) = C_1^4 + 3C_1^2C_2 + 2C_1C_3 + C_4 = E_I(a_1^4) + 3E_I(a_1^2a_2) + E_I(a_2^2) = E_I(\text{IRF}(t,4)) = IRF_{dis}(t,4), \] etc. (36)

Hence, the aggregate IRF is just the expected value of the individual IRFs, as shown in Section 3 for the simple AR(1) case. The same result also holds when more general AR(p) dynamics are considered.

6. EMPIRICAL ILLUSTRATION

Monetary authorities and central banks are very interested in knowing how sluggishly inflation returns to its long-run equilibrium level after the arrival of a shock. It has been recently argued that inflation persistence tends to be higher, the higher the aggregation level. For instance, the Inflation Persistence Network, created by the European Central Bank with the aim of analyzing the patterns of inflation persistence, concluded that there is clear evidence of large differences across sectors and “that measures of the degree of inflation persistence increase with the level of aggregation. Individual or highly disaggregate price series are, on average, much less persistent than aggregate ones” (see Angeloni et al., 2007).\(^\text{17}\) These conclusions were based on a number

\(^{17}\)These conclusions are based on a number of studies that can be found in the network’s webpage at http://www.ecb.int/events/conferences/html/inflationpersistence.en.html.
of studies, e.g., Altissimo et al. (2006, 2007), Bilke (2005), Lünnemann and Mathä (2004) and others. Similar findings have been reported for U.S. inflation data (see Clark, 2006).

To show whether measures of the degree of persistence are highly compatible across aggregation levels, sectoral and aggregate U.S. inflation data has been analyzed. We use a similar data set to Clark (2006), namely, the price index for all components of consumption, provided by the Bureau of Economic Analysis. As in that paper, the aggregate variable, denoted as Level 1 of aggregation, is core-inflation, which excludes food and energy. At the disaggregate level, three different layers have been examined. Level 2 of aggregation contains the 11 most aggregate categories of core-inflation, Level 3 breaks these components into its 46 most aggregate ones while Level 4 contains 109 disaggregate prices (see Clark, 2006 for details). The data is quarterly and covers the period 1976 to 2002.\(^{18}\)

Where does sectoral inflation heterogeneity come from? It can be easily introduced into a model of sticky prices, as in Rotemberg (1987), where it is assumed that each firm faces a quadratic cost of changing its price. As is well known, the dynamics of prices in that model are given by

\[
p_t^i = \vartheta p_{t-1}^i + (1 - \vartheta) p_t^*.
\]

(37)

where \(p\) and \(p^*\) represent the actual and optimal level of the prices of firm \(i\) and \(\vartheta\) is a parameter that captures the extent to which imbalances are remedied in each period. Taking first differences and rearranging terms leads to the best-known form of the Partial Adjustment Model (PAM),

\[
\Delta p_t^i = \vartheta \Delta p_{t-1}^i + \nu_t^i,
\]

(38)

with \(\nu_t^i = (1 - \vartheta) \Delta p_t^*\). The parameter \(\vartheta\) is a function of the adjustment costs and describes the speed of the adjustment. Hence, firms facing different costs could present different speeds of adjustment so that

\[
\Delta p_t^i = \vartheta^i \Delta p_{t-1}^i + \nu_t^i.
\]

(39)

To build a price index, aggregation over a large number of individual prices is considered. The aggregate inflation index is defined as

\[
\Delta P_t = E_t(\Delta p_t).
\]

\(^{18}\)Clark (2006) analyzed the period 1959-2002. Nevertheless, in order to avoid problems derived from the existence of structural breaks around the 1973 crisis, which would have a great impact on persistence estimates (see Perron, 1989), we have preferred to avoid this period by considering only post-crisis data.
If sectoral data is generated as in (37), $\Delta P_t$ can be represented as an AR process whose coefficients can be consistently estimated provided the number of autoregressive lags increases at a sufficient rate with the sample size (Berk, 1974).

To compute the impulse responses, AR($p$) processes have been estimated where the order $p$ has been chosen according to the AIC.$^{19}$ Table I presents the sum of the first $h$ values of the impulse responses computed at different aggregation levels, where $h = \{4, 8, 12, 16$ and $20\}$, that is, from 1 to 5 years after the shock occurs. Also, the SAC is reported for the four aggregation levels. Figure 1, in turn, depicts the four IRFs. Confidence intervals have been computed using bootstrap methods.

In agreement with the theoretical results, impulse responses computed from aggregate and sectoral data are very close at any horizon. The aggregate IRF is less smooth than the individual IRFs, which is not surprising since the latter are computed as averages of the individual IRFs. Nevertheless, the four functions present approximately the same values and the same pattern of decay, so they imply a very similar degree of persistence, in line with the results presented in Table 1.

The values of the SAC, however, vary considerably across aggregation levels, reproducing what has been found in previous studies. SAC values range from the 0.66 corresponding to Level 4 of aggregation (with a confidence interval of (0.64, 0.68)) to the 0.86 for Level 1 (with a C.I. of (0.82, 0.92)). Therefore, from only looking at these figures, one would conclude that the response to a shock is higher, the higher the level of aggregation at which it is measured when, in fact, the analysis of the IRFs points in the opposite direction.

(Figure I about here)

(Table I about here)

7. CONCLUSION

This paper has shown that the effect of an aggregate shock over time on an economy whose agents present linear heterogeneous dynamics is the same regardless of the aggregation level at which the impact is measured. More specifically, the average of the individual responses equals

$^{19}$A maximum number of 20 and 10 AR lags was set for aggregate and sectoral prices, respectively.
the aggregate IRF. The same relation also holds for some scalar measures of persistence such as
the CIR and the HL. However, it has also been shown that other popular measures of persistence,
such as the sum of the autoregressive coefficients, the LAR or the first autocorrelation have very
poor properties in this framework. In particular, they tend to yield higher values the higher the
aggregation level considered, in situations where the response to a shock over time is constant
across aggregation levels. This helps to explain why different shock response behavior has been
reported in many empirical papers when considering different aggregation levels. An empirical
application using U.S. inflation data that illustrates the theoretical results has also been provided.
REFERENCES


APPENDIX

This Appendix presents the results of a small Monte Carlo experiment that has two goals. Firstly, to show that fairly standard time series techniques allow one to obtain similar shock response estimates at different aggregation levels and, secondly, to illustrate that what matters in order to obtain a highly persistent aggregate process is not the degree of heterogeneity of the individual units but, rather, the fact that they are, on average, highly persistent too.

The following experiment has been considered. A collection of $N$ heterogeneous individuals $y_{it}$ behaving as

$$y_{it} = a_i y_{i,t-1} + \rho_i u_t + \varepsilon_{it}, \ i = 1, ..., N,$$

have been generated, where $u_t \sim N(0, 1), \varepsilon_{it} \sim N(0, \sigma_i^2), \sigma_i^2 \sim U(0.5, 1.5), \rho \sim U(0.5, 1.5)$. Different distributions of $a$, that will be specified below, have been considered to capture different degrees of persistence. The aggregate data, $\bar{Y}_{Nt}$, is constructed as

$$\bar{Y}_{Nt} = \frac{\sum_{i=1}^{N} y_{it}}{N},$$

where $N=1000$ has been set in all experiments and two sample sizes have been considered, namely, $T = \{200, 400\}$. AR(1) and AR(k) models have been fitted to $y_{it}, i = 1, ..., N$, and to $\bar{Y}_{Nt}$ (or to $(1-L)\bar{Y}_{Nt}$, see below) respectively, where $k$ has been chosen according to the AIC. The number of replications was equal to 1000.

The values of $a$ in (40) have been generated according to a $U(-0.5, 0.95)$, a Beta(1,0.8), and a Beta(1, 0.3) distributions. Figure 2 presents histograms of $a$ corresponding to one of the replications in the Monte-Carlo study for each of the distributions considered. The sample standard deviation of $a$ is 0.41, 0.29 and 0.28, respectively and thus, the degree of heterogeneity of $a$ is higher in the uniform than in the beta cases. However, the corresponding histograms show, the beta distributions have bigger probability mass in the neighborhood of 1, giving rise to processes that are, on average, more persistent. This fact is reflected on the properties of $\bar{Y}_{Nt}$. For the case where $a$ follows the uniform distribution, the aggregate process is I(0) while it is long-memory stationary (with an integration order $d = 0.2$) and non-stationary (with $d = 0.7$), for the Beta(1,0.8) and Beta(1,0.3) distributions, respectively, (see Zaffaroni, 2004, for details).

Figures 3 contains the time path of three IRFs corresponding for the case where $a \sim U(-0.5, 0.95)$. The solid line represents the true impulse response implied by model (40). The dashed line represents the IRF estimated by fitting AR(1) models to the disaggregate data, obtaining
each of the individual IRFs and then, computing their average. Finally, the dashed-dotted line depicts the estimates of the aggregate IRF. Figures 4 and 5 represent similar functions as Figure 3, corresponding this time to the cases where $a \sim \text{Beta}(1,0.8)$ and $a \sim \text{Beta}(1, 0.3)$, respectively. For the Uniform and the Beta(1,0.8) distributions, the aggregate data has been estimated in levels, while for the Beta(1,0.3) one, first differences have been taken prior to carrying out the estimation because of the non-stationarity character of $\bar{Y}_{Nt}$. In this case, the IRF$_{AG}$ has been calculated by cumulating the obtained values of the IRF.

The IRF estimated from aggregate data, $\hat{IRF}_{AG}$, is always close to the true IRF, even for moderate sample sizes, for all the distributions of $a$ considered. With respect to the IRF estimated with disaggregate data, $\hat{IRF}_{dis}$, it approximates very well the true IRF (and thus, it is also very close to $\hat{IRF}_{AG}$) in Figures 3 and 4. However, it is remarkable that for the Beta(1,0.3) case, $\hat{IRF}_{dis}$ is considerably below the true IRF and $\hat{IRF}_{AG}$. This is due to the well known small-sample downward bias that affects OLS estimates when they are computed on highly persistent data. This problem does not affect the estimates obtained with aggregate data because the estimation has been carried out in first differences.
Fig 2. Histograms of $a$
Fig 3. True and estimated IRFs, $a \sim U(-0.5, 0.95)$. 
Fig 4. True and estimated IRFs, $a \sim Beta(1, 0.8)$.
Fig 5. True and estimated IRFs, $a \sim Beta(1, 0.3)$
TABLES AND FIGURES

Fig I. Impulse responses of US inflation computed at different aggregation levels

**Table I. Persistence Measures**

<table>
<thead>
<tr>
<th></th>
<th>Aggregate</th>
<th>Disagg.: Level 2</th>
<th>Disagg.: Level 3</th>
<th>Disagg.: Level 4</th>
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<tbody>
<tr>
<td>( \sum_{i=1}^{h} IRF_{AGG}(t,h) )</td>
<td>1.77</td>
<td>2.07</td>
<td>1.88</td>
<td>1.81</td>
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<tr>
<td>( \sum_{i=1}^{h} IRF_{dis,1}(t,h) )</td>
<td>3.06</td>
<td>3.32</td>
<td>3.04</td>
<td>2.91</td>
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<tr>
<td>( \sum_{i=1}^{h} IRF_{dis,2}(t,h) )</td>
<td>4.01</td>
<td>4.26</td>
<td>3.90</td>
<td>3.72</td>
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<td>( \sum_{i=1}^{h} IRF_{dis,3}(t,h) )</td>
<td>4.58</td>
<td>5.03</td>
<td>4.59</td>
<td>4.33</td>
</tr>
<tr>
<td>( \sum_{i=1}^{h} IRF_{dis,4}(t,h) )</td>
<td>4.86</td>
<td>5.62</td>
<td>5.13</td>
<td>4.81</td>
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</table>

**SAC at different Aggregation Levels**

<table>
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<tr>
<th></th>
<th>0.86</th>
<th>0.83</th>
<th>0.71</th>
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<tbody>
<tr>
<td></td>
<td>(0.82,0.92)</td>
<td>(0.76,0.84)</td>
<td>(0.65,0.72)</td>
<td>(0.64,0.68)</td>
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