Almost Mutually Best in Matching Markets: Rank Gaps and Size of the Core*

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Abstract

This paper studies the one-to-one two-sided marriage model (Gale and Shapley, 1962). If agents' preferences exhibit mutually best (i.e., each agent is most preferred by her/his most preferred matching partner), there is a unique stable matching without rank gaps (i.e., in each matched pair the agents assign one another the same rank). We study in how far this result is robust for matching markets that are "close" to mutually best. Without a restriction on preference profiles, we find that natural "distances" to mutually best neither bound the size of the core nor the rank gaps at stable matchings. However, for matching markets that satisfy horizontal heterogeneity, "local" distances to mutually best provide bounds for the size of the core and the rank gaps at stable matchings.

Keywords: matching, mutually best, horizontal heterogeneity, stable matching, core, rank gaps.

JEL–Numbers: C78.

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1 Introduction

We study the one-to-one two-sided marriage model (Gale and Shapley, 1962) with the same number of men (or workers) and women (or firms). Each agent has strict and complete preferences over the agents on the other side of the market. We assume that being matched is preferred to being unmatched (and choosing some outside option). Hence, each agent's preferences can be represented by a ranking of the agents on the other side of the market. The set of stable matchings coincides with the core and is the central solution concept in matching theory. A matching is stable if there is no blocking pair, i.e., a man and a woman that would prefer being matched to one another rather than keeping their current mates. Gale and Shapley (1962) show that the core is always non-empty.

However, the existence of stable matchings neither precludes conflict within matched pairs nor across market sides. While stability is equivalent to the absence of justified envy (i.e., there is no pair of a man and a woman who mutually prefer each other to their current mates and therefore justifiably envy another member of their own market side), it does allow for large rank gaps within a matched pair (i.e., a large difference in the ranks matched partners assign to each other) and between partners that are assigned to a given agent at different stable matchings. But rank-gaps within matched pairs may make matchings that are stable for a given set of market participants less robust in a dynamic setting. E.g., if marriage or cohabitation is the result of a sequential search process as in Adachi (2003), it is expected that search yields stable matchings in the limit of vanishing search frictions. Empirical studies in economics and sociology indicate that assortative mating is important for a successful duration of a marriage– men and women prefer to be matched to someone of a similar type regarding age, ethnicity, religious denomination, social attitude, or physical attractiveness.[1] But if long-lasting relationships feature assortative mating, we expect partners to rank each other similarly with respect to these attributes in these relationships. So, even if a snapshot of the social network of an individual exhibits stable matchings with a large rank-gap, these matchings are unlikely to last (because the individual tends to resolve the relationship as soon as the social network offers superior options). Likewise, large rank gaps between partners that are assigned to a given agent in different stable matchings impose problems for centralized market design. If ranks assigned to partners at different stable matchings differ (i.e., the marriage problem exhibits multiple stable matchings), there is no mechanism that

[1]See, e.g., Chiappori and Salanie (2016) for an overview and Frimmel et al. (2013, p.908), for a general discussion. More specifically, Hitsch et al. (2010) use data on user attributes and interactions from an online dating site to estimate individuals' preferences. Stable matchings computed for estimated preferences are similar to the actual matches achieved by the dating site. Moreover, out-of-sample predictions exhibit assortative mating patterns similar to those observed in actual marriages. Interestingly, Hitsch et al. (2010, p.161) demonstrate that assortative mating is largely driven by preferences that favor a partner who is similar rather than a vertical differentiation of potential matching partners.

finds a stable matching with (true) preference revelation being a dominant strategy for all participating men and women (e.g., Roth, 1982).

The above-mentioned sources of conflict vanish, however, on specific domains of preference profiles. For instance, stable matchings are unique *and* have no rank gaps, if preference profiles satisfy *vertical heterogeneity* (e.g., Eeckhout, 2000), i.e., there is a strict and objective ranking of men by women and of women by men. In the Computer Science literature, each of these two rankings is called a master list (e.g., Irving et al., 2008 and Manlove, 2013). On this domain, the unique stable matching is assortative: if a woman ranks her mate k^{th}, then he also ranks her k^{th}. Similarly, there is a unique stable matching and there are no rank gaps, if preference profiles satisfy *mutually best*, i.e., each man's most preferred woman also prefers him most. For instance, mutually best is satisfied if (1) agents are located in some (geographical, socio-economic, etc.) space, (2) for each agent, agents from the other side are more preferred if the distance to the agent is smaller, and (3) for each agent i there is a unique agent from the other side of the market, say i', such that i and i' are each other's closest agent from the other side of the market (see, e.g., Burdett and Wright, 1998; Kiyotaki and Wright, 1989; Smith, 2002). On this domain, the unique stable matching matches mutually best pairs and thus induces rank gaps of size zero.

But are rank gaps small if the preference profile is "close" to these domains? In other words, is the distance of a preference profile to one of these domains a bound for rank gaps within matched pairs at a given stable matching and/or rank gaps between partners at different stable matchings? Holzman and Samet (2014) study this question for the domain of vertical heterogeneity. They show that rank gaps within matched pairs at a stable matching and the size of the set of stable matchings (i.e., the core) are bounded by measures of the diversity of rankings (i.e., the distance to vertical heterogeneity).[2] The size of the core is measured by the distance between the man- and woman-optimal stable matching, i.e., the rank gap that exists between each agent's least preferred and most preferred mate among those that are obtained at stable matchings. Loosely speaking, if a preference profile is "close" to vertical heterogeneity, then the core is "small" and rank gaps at stable matchings are "small." In our paper, we carry out a similar analysis for the domain of mutually best.

Unlike the results in Holzman and Samet (2014), our findings crucially depend on the domain from which we "approach" mutually best and the type of distance measure. Observe that the domain of mutually best is the intersection of two domains: horizontal heterogeneity and the domain in Eeckhout (2000, Theorem 1). *Horizontal heterogeneity* is the condition that all men most prefer different women and all women most prefer different men.[3] The con-

[2] Jaramillo et al. (2019) prove similar results for a class of roommate problems that is not logically related to the class of marriage problems studied in Holzman and Samet (2014).

[3] In Eeckhout (2000, Corollary 4) "horizontal heterogeneity" refers to our condition of mutually best. Our domain of horizontal heterogeneity contains the domain with the same name in Eeckhout (2000) but also contains profiles where all men prefer different women and all women prefer different men but where not all

dition in Eeckhout (2000, Theorem 1), which is sufficient but not necessary for the uniqueness of stable matchings, requires there to be a pair of mutually most preferred agents on recursively defined subsets.[4] Following Clark (2006), we will refer to this condition as *Eeckhout's sequential preference condition* (SPC). This condition is satisfied in marriage problems with vertical heterogeneity and is implied by the generalized increasing difference (GID) condition in Legros and Newman (2007).[5] For a problem that satisfies horizontal heterogeneity, the following are equivalent: the problem satisfies Eeckhout's SPC; the problem satisfies mutually best; and the problem has a unique stable matching. The Venn-diagram in Figure 1 summarizes the logical relations between the discussed domains.

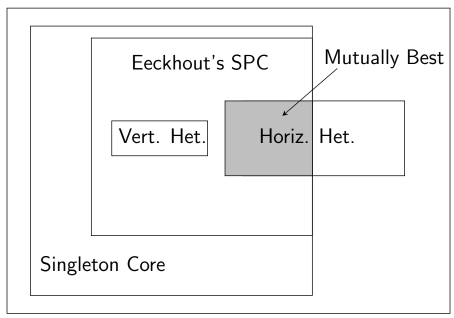


Figure 1: Venn-diagram of domains of preference profiles.

Given a marriage problem, to quantify the extent to which the preference profile violates mutually best (i.e., its "distance" from the domain of mutually best), we introduce two types of measures. The first type of measure is a "local" distance which first computes for each agent the rank with which his/her most preferred mate reciprocates and then aggregates these ranks by using generalized means. The starting point of our second type of measure, which we call "global" distance, is that for mutually best preferences there is a feasible matching at which each agent is matched to his/her most preferred mate. Our "global" distance focuses on feasible matchings and is based on how close we can get to the most preferred mates by minimizing some generalized mean.

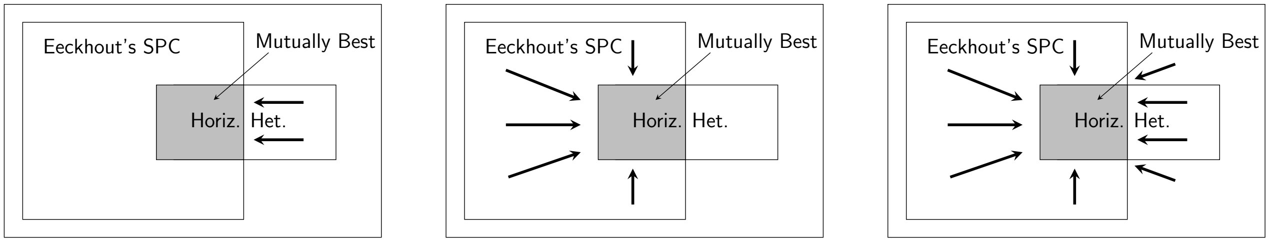
As our goal is to determine whether the size of the core and rank gaps at stable matchings are bounded by the distance of the preference profile to the domain of mutually best, the Venn-diagram in Figure 1 suggests to study and "approach" mutually best from three different domains: horizontal heterogeneity, Eeckhout's SPC domain, and the general domain (i.e.,

most preferred mates form mutually best pairs.

[4]The condition requires that agents can be relabeled $m_1, \ldots, m_n$ and $w_1, \ldots, w_n$ such that for each k, man m_k and woman w_k are mutually most preferred among agents $m_k, \ldots, m_n$ and $w_k, \ldots, w_n$.

[5]GID is necessary and sufficient for positive assortative matching if utility is at least partially transferable between matching partners as in Legros and Newman (2006) and is sufficient but not necessary in the non-transferable utility case (Legros and Newman, 2006).

without any restrictions)— see Figure 2.



(a) From the domain of horizontal heterogeneity.

(b) From the domain of Eeckhout's sequential preference condition.

(c) Without domain restrictions.

Figure 2: Approaching mutually best from three different domains.

Our main findings are as follows.

(a) On the domain of horizontally heterogeneous problems: rank gaps at stable matchings and the size of the core are bounded in terms of local distances (Propositions 1 and 3) but not global distances (Propositions 2 and 4).

(b) On the domain of problems that satisfy Eeckhout's sequential preference condition: the size of the core is trivially bounded (as there is a unique stable matching) but rank gaps at stable matchings are not bounded in terms of local or global distances (Proposition 5).

(c) On the general domain: rank gaps at stable matchings and the size of the core are not bounded in terms of local or global distances (Proposition 6 and Corollaries 2 and 3 to results in (a) and (b)).

Our measures for the distance of a preference profile from mutually best provide several links to the Computer Science literature. First, determining the global (generalized) distance is equivalent to the problem of finding a minimum weight bipartite graph with a perfect matching in the classical job assignment problem. Thus, global distances can be computed using the Hungarian Algorithm (Kuhn, 1955; Munkres, 1957). Second, special cases of our global distance have been extensively studied in the Computer Science literature (see, e.g., Manlove, 2013, p.23). For instance, a matching that minimizes the average distance from mutually best is referred to as an *egalitarian* matching and a matching that minimizes the maximal distance from mutually best is referred to as a matching with *minimal regret.* Since this branch of the literature focuses on egalitarian and minimum regret matchings within the set of *stable* matchings, we redo our analysis with an accordingly adapted alternative definition of the global distance to mutually best. The alternative definition allows to utilize well-established algorithms to determine egalitarian and minimum regret stable matchings etc. and thus the alternative global distance. Moreover, it enables us to strengthen our findings with respect to bounds for rank gaps and the size of the core: In contrast to result (b), the alternative global distance bounds rank gaps whenever there is a unique stable matching.

The remainder of the paper is organized as follows. In Section 2, we describe the model and define rank gaps and the size of the core. In Section 3, we introduce restricted domains of problems and local/global distances to mutually best. In Section 4, we state and prove our main findings. In Section 5, we provide an analysis using global distances that are based on stable rather than feasible matchings. Section 6 concludes.

2 The model

In the presentation of the model we closely follow Holzman and Samet (2014). There are two finite disjoint sets of **men** M and **women** W of equal cardinality $n \in \mathbb{N}$ with $n \geq 2$. Each man $m \in M$ has a strict preference relation $\succ_m$ over W, and each woman $w \in W$ has a strict preference relation $\succ_w$ over M. In particular, it is assumed that all agents on the other side are "acceptable," i.e., each agent prefers to be matched to any agent from the other side rather than remaining unmatched (and choosing some outside option). For each $m \in M$, man m's preferences can be represented by a **ranking**, i.e., a bijection $r_m : W \to \{1, \ldots, , n\}$ such that for $w, w' \in W$, $r_m(w) < r_m(w')$ if and only if $w \succ_m w'$. The integer $r_m(w)$ is the **rank** of w in man m's ranking. Hence, more preferred agents have a smaller rank. Similarly, for each $w \in W$, woman w's preferences can be represented by a ranking r_w. Let $r = (r_i)_{i \in M \cup W}$ be the profile of rankings. For each $i \in M \cup W$, let $t(i)$ denote i's most preferred agent, i.e., $r_i(t(i)) = 1$. Fix M and W. A **(marriage) problem** (Gale and Shapley, 1962) is given by r.

A **matching** is a function $\mu : M \cup W \to M \cup W$ such that for all $m \in M$ and all $w \in W$, $\mu(m) \in W$, $\mu(w) \in M$, and $\mu(m) = w \Leftrightarrow \mu(w) = m$. If $\mu(m) = w$, we say that m and w are matched to one another and that they are (each other's) mates at μ. Let $\mathcal{M}$ denote the set of matchings. For convenience we often denote a matching by its set of matched pairs.

A pair (m, w) blocks a matching μ if $r_m(w) < r_m(\mu(m))$ and $r_w(m) < r_w(\mu(w))$. A matching μ is **stable** if no pair blocks it. It is well-known that the core equals the set of stable matchings. Gale and Shapley (1962) showed that each marriage problem has a stable matching. In fact, they proved the existence of a **man-optimal** (and woman-pessimal) stable matching μ_M such that for all $m \in M$ and all $w \in W$, $r_m(\mu_M(m)) = \min\{r_m(\mu(m)) : \mu \text{ is stable}\}$ and $r_w(\mu_M(w)) = \max\{r_w(\mu(w)) : \mu \text{ is stable}\}$. Similarly, there exists a **woman-optimal** (and man-pessimal) stable matching μ_W.

In order to carry out an analysis that goes beyond averages and maxima, we recall the definition of a generalized mean. Let $p \in (0, \infty)$. For each multiset[6] $\{x_1, \ldots, x_\ell\}$ that consists of $\ell \in \mathbb{N}$, $\ell \geq 1$ non-negative numbers $x_1, \ldots, x_\ell \in \mathbb{R}_+$, we define its **($p$-)generalized mean**

[6]Unlike a set, a multiset allows for multiple instances for each of its elements.

by

$$\boldsymbol{M_p}(\{x_1, \dots, x_\ell\}) \equiv \sqrt[p]{\frac{1}{\ell}\sum_{k=1}^{\ell} x_k^p}\;.$$

Note that M_1 is the arithmetic mean. It is also known (see, e.g., Bullen, 2003) that for each multiset $\{x_1, \dots, x_\ell\}$ of non-negative numbers,

$$\lim_{p\to\infty} M_p(\{x_1, \dots, x_\ell\}) = \max\{x_1, \dots, x_\ell\}. \tag{1}$$

For this reason it is convenient to define the **(∞-)generalized mean**:

$$\boldsymbol{M_\infty}(\{x_1, \dots, x_\ell\}) \equiv \max\{x_1, \dots, x_\ell\}.$$

Deriving results for p-generalized means with $p \in (0, \infty)$ therefore allows to establish proofs for ∞-generalized means as limit results. This significantly simplifies our exposition relative to the approach in Holzman and Samet (2014) who considered averages and maxima separately.

Rank gaps. The rank gap of a pair $(m, w) \in M \times W$ is given by $|r_m(w) - r_w(m)|$. To aggregate the rank gaps of all n pairs at a matching we use generalized means. More specifically, for each $p \in (0, \infty]$, the **(p-aggregated) rank gap at matching μ** for problem r is defined as the p-generalized mean of all rank gaps, i.e.,

$$\boldsymbol{\Gamma_p(\mu)} \equiv M_p\left(\bigcup_{(m,w)\in\mu} \{|r_m(w) - r_w(m)|\}\right).$$

For $p = 1$ and $p = \infty$, the p-aggregated rank gap boils down to the average rank gap and the maximum rank gap, respectively. Observe that if for some $p \in (0, \infty]$, $\Gamma_p(\mu) = 0$, then for each $p \in (0, \infty]$, $\Gamma_p(\mu) = 0$.[7]

Size of the core. To measure the size of the core, we consider for each agent $i \in M \cup W$ the range of different ranks obtained at stable matchings which, by side-optimality of μ_M and μ_W, equals $|r_i(\mu_M(i)) - r_i(\mu_W(i))|$. To aggregate over all $2n$ agents we use again generalized means. Specifically, for $p \in (0, \infty]$, the **(p-)size/diameter of the core of problem r** is defined by

$$\boldsymbol{D_p(r)} \equiv M_p\left(\bigcup_{i\in M\cup W} \{|r_i(\mu_M(i)) - r_i(\mu_W(i))|\}\right).$$

Observe that if for some $p \in (0, \infty]$, $D_p(r) = 0$, then for each $p \in (0, \infty]$, $D_p(r) = 0$.

[7] Jaramillo et al. (2019) call a matching μ rank-fair if $\Gamma_1(\mu) = 0$ or, equivalently, for all $p \in (0, \infty]$, $\Gamma_p(\mu) = 0$.

3 Domains and distances

3.1 Restricted domains of problems

A marriage problem r satisfies **horizontal heterogeneity** if all women most prefer different men and all men most prefer different women, i.e., $i \neq j \implies t(i) \neq t(j)$. Let $\boldsymbol{H}$ denote the class of problems that satisfy horizontal heterogeneity. For each marriage problem that satisfies horizontal heterogeneity, matching each woman to her most preferred man yields the woman-optimal stable matching, and similarly, matching each man to his most preferred woman yields the man-optimal stable matching.

A marriage problem r satisfies **Eeckhout's sequential preference condition (SPC)** if there is a pair of mutually most preferred agents on recursively defined subsets, i.e., there is an ordering of M, say w.l.o.g. $m_1, \ldots, m_n$, and an ordering of W, say w.l.o.g. $w_1, \ldots, w_n$, such that for all $k, l = 1, \ldots, n$ with $l > k$, $r_{w_k}(m_k) < r_{w_k}(m_l)$ and $r_{m_k}(w_k) < r_{m_k}(w_l)$. Let $\boldsymbol{E}$ denote the class of problems that satisfy Eeckhout's sequential preference condition. Eeckhout (2000, Theorem 1) showed that for each $r \in E$, there is a unique stable matching.

Finally, a marriage problem r satisfies **mutually best** if for each $i \in M \cup W$, $t(t(i)) = i$. Let $\boldsymbol{MB}$ denote the class of problems that satisfy mutually best. For each problem $r \in MB$, the rank gap at the unique stable matching is 0. Note $MB = H \cap E$. It is easy to see that the domains H and E are logically unrelated. Finally, a problem $r \in H$ has a unique stable matching if and only if $r \in E$. To see this, note that there is a unique stable matching if and only if the two side-optimal stable matchings coincide, which for $r \in H$ occurs if and only if the top preferences of the two sides are reciprocal, i.e., the problem satisfies mutually best.

3.2 Distances to mutually best

We consider two natural types of "distances." The first type is that of local distances, i.e., measures that consider the distance from mutually best separately for each agent.

Local distances. Let r be a marriage problem. For each $p \in (0, \infty]$, the **local (p-)distance from mutually best at r** is

$$\boldsymbol{\Delta^l_p(r)} \equiv M_p \left(\bigcup_{i \in M \cup W} \{r_{t(i)}(i) - 1\} \right).$$

Note that for $p = 1$ and $p = \infty$, $\Delta^l_p(r)$ yields the average and the maximum of the numbers $\{r_{t(i)}(i) - 1\}_{i \in M \cup W}$, respectively. Moreover, for each $r \in MB$ and each $p \in (0, \infty]$, $\Delta^l_p(r) = 0$.

The distances have a "local" flavor as they compute for each agent the rank with which his/her most preferred mate reciprocates. The computation of local distances is easy as it only requires finding for each agent i the rank of i according to the ranking of agent $t(i)$.

Example 1. [Local distances.]

Consider problem r in Table 1. Here, each column represents the ranking of an agent. For instance, the 3rd entry of the 2nd column of the table on the left hand side shows that $r_{m_2}(w_4) = 3$. The bold-faced entries depict agents i in the ranking of $t(i)$. By looking at the bold-faced entries, we obtain $\Delta_1^l(r) = \frac{1}{8}(0+2+3+0+0+2+0+3) = \frac{5}{4}$ and $\Delta_\infty^l(r) = r_{m_1}(w_1) - 1 = 4 - 1 = 3$. For other values of $p \in (0, \infty)$, $\Delta_p^l(r)$ can be calculated similarly.

m_1	m_2	m_3	m_4
$\boldsymbol{w_4}$	$\boldsymbol{w_2}$	w_3	w_4
w_2	w_3	w_4	w_2
$\boldsymbol{w_3}$	w_4	w_1	w_3
$\boldsymbol{w_1}$	w_1	w_2	w_1

w_1	w_2	w_3	w_4
m_1	$\boldsymbol{m_2}$	m_1	$\boldsymbol{m_1}$
m_3	m_3	m_2	m_2
m_2	m_4	$\boldsymbol{m_3}$	m_3
m_4	m_1	m_4	$\boldsymbol{m_4}$

Table 1: Problem r in Example 1. Bold-faced entries indicate how most preferred agents reciprocate.

◇

Note that local distances do not require the function $i \mapsto t(i)$ to constitute a well-defined matching. In order to measure the extent to which a problem fails to satisfy mutually best from a global point of view, we consider how close we can get to the most preferred mates for all agents simultaneously by a well-defined matching.

Global distances. Similarly to local distances, we consider global distances based on generalized means. Formally, the **global (p-)distance from mutually best at r**, is

$$\boldsymbol{\Delta_p^g(r)} \equiv \min_{\mu \in \mathcal{M}} M_p \left(\bigcup_{i \in M \cup W} \{ r_i(\mu(i)) - 1 \} \right).$$

Note that for $p = 1$ and $p = \infty$, $\Delta_p^g(r)$ yields the minimum over all matchings of the average and the maximum of the numbers $\{r_i(\mu(i)) - 1\}_{i \in M \cup W}$, respectively. Moreover, for each $r \in MB$ and each $p \in (0, \infty]$, $\Delta_p^g(r) = 0$.

To compute the global distance $\Delta_p^g(r)$ for $p \in (0, \infty)$, we can create a bipartite graph (V, E) with $V \equiv M \cup W$, $E \equiv M \times W$, and where each edge $(m, w) \in E$ has weight

$$(r_m(w) - 1)^p + (r_w(m) - 1)^p.$$

The global distance from mutually best is the weight of the graph's minimum-weight (perfect) matchings, which can be computed efficiently using the Hungarian algorithm (cf. Kuhn, 1955; Munkres, 1957).

The computation of $\Delta_\infty^g(r)$ can be done as follows. For each $k = 1, \ldots, n$, create a bipartite graph (V, E_k) with $V \equiv M \cup W$ and $E_k \equiv \{(m, w) \in M \times W : r_m(w), r_w(m) \leq k\}$. Then, $\Delta_\infty^g(r) + 1$ equals the smallest k for which (V, E_k) has a perfect matching. Binary search can be used to speed up finding $\Delta_\infty^g(r)$.

Remark 1. [Global distances based on stable matchings.]

The global distance above is defined as a minimum over all matchings. It is possible to introduce further restrictions and focus on a subset of matchings. One approach along this line, common in the Computer Science literature, is to take the minimum over all *stable* matchings (see, e.g., Manlove, 2013). For the sake of exposition we provide the details of the parallel analysis following this alternative approach in a separate section (Section 5). ◇

Example 2. [Local and global distances are uncorrelated.]

We will show that local and global p-distances are uncorrelated, i.e., for each $p \in (0, \infty]$, there are problems r and r' such that[8]

$$\Delta_p^l(r) < \Delta_p^l(r') \quad \text{and} \quad \Delta_p^g(r) > \Delta_p^g(r'). \tag{2}$$

Let $p \in (0, \infty)$. First, consider problem r in Table 2. One immediately verifies that $\Delta_p^l(r) = \sqrt[p]{\frac{1}{6}(2 \cdot 1 + 2 \cdot 2^p)}$. Next, note that at each matching (stable or not), on each side one agent gets his/her most preferred mate, one agent gets his/her second most preferred mate, and one agent gets his/her least preferred mate. Hence,

$$\Delta_p^g(r) = \min_{\mu \in \mathcal{M}} M_p \left(\bigcup_{i \in M \cup W} \{r_i(\mu(i)) - 1\} \right) = \sqrt[p]{\frac{1}{6}(2 \cdot 1 + 2 \cdot 2^p)}.$$

m_1	m_2	m_3	w_1	w_2	w_3
w_1	w_1	w_1	m_1	m_1	m_1
w_2	w_2	w_2	m_2	m_2	m_2
w_3	w_3	w_3	m_3	m_3	m_3

Table 2: Problem r.

m_1	m_2	m_3	w_1	w_2	w_3
w_1	w_2	w_3	m_3	m_1	m_2
w_3	w_1	w_2	m_2	m_3	m_3
w_2	w_3	w_1	m_1	m_2	m_1

Table 3: Problem r'.

Second, consider problem r' in Table 3. One immediately verifies that $\Delta_p^l(r') = \sqrt[p]{\frac{1}{6}(1 + 5 \cdot 2^p)}$. Hence,

$$\Delta_p^l(r) < \Delta_p^l(r'). \tag{3}$$

Moreover,

$$\sqrt[p]{\frac{1}{6}(1 + 2 \cdot 2^p)} = M_p \left(\bigcup_{i \in M \cup W} \{r'_i(\mu_M(i)) - 1\} \right) \geq \Delta_p^g(r').$$

Hence,

$$\Delta_p^g(r) > \Delta_p^g(r'). \tag{4}$$

[8] In fact, we prove a stronger result: (i) there are problems r and r' such that for each $p \in (0, \infty)$, $\Delta_p^l(r) < \Delta_p^l(r')$ and $\Delta_p^g(r) > \Delta_p^g(r')$ and (ii) there are problems r and r' such that $\Delta_\infty^l(r) < \Delta_\infty^l(r')$ and $\Delta_\infty^g(r) > \Delta_\infty^g(r')$.

Inequalities (3) and (4) yield (2).

Next, let $p = \infty$. First, consider problem r in Table 4. One immediately verifies that $\Delta^l_\infty(r) = 1$. Moreover, $\Delta^g_\infty(r) = 2$. To see this, suppose $\Delta^g_\infty(r) \leq 1$. Since r does not satisfy mutually best, $\Delta^g_\infty(r) \neq 0$. So, $\Delta^g_\infty(r) = 1$. Then, there is a matching μ where each agent's match has rank at most 2. Given that man m_4 has rank 3 or 4 for women w_1, w_2, and w_4, we have $\mu(m_4) = w_3$. But then, using similar arguments, $\mu(m_3) = w_2$. However, $r_{m_3}(w_2) = 3$, i.e., at μ not each agent's match has rank at most 2. Hence, $\Delta^g_\infty(r) \neq 1$. So, $\Delta^g_\infty(r) > 1$. Since at matching $\{(m_1, w_4), (m_2, w_1), (m_3, w_2), (m_4, w_3)\}$ each agent's match has rank at most 3, it follows that $\Delta^g_\infty(r) = 2$.

m_1	m_2	m_3	m_4	w_1	w_2	w_3	w_4
w_1	w_2	w_3	w_3	m_1	m_2	m_3	m_1
w_4	w_3	w_1	w_2	m_2	m_3	m_4	m_2
w_3	w_1	w_2	w_1	m_3	m_4	m_1	m_3
w_2	w_4	w_4	w_4	m_4	m_1	m_2	m_4

Table 4: Problem r.

m_1	m_2	m_3	m_4	w_1	w_2	w_3	w_4
w_1	w_1	w_1	w_1	m_1	m_1	m_1	m_1
w_2	w_2	w_3	w_4	m_2	m_2	m_3	m_4
w_3	w_3	w_2	w_2	m_3	m_3	m_2	m_2
w_4	w_4	w_4	w_3	m_4	m_4	m_4	m_3

Table 5: Problem r'.

Second, consider problem r' in Table 5. One immediately verifies that $\Delta^l_\infty(r') = 3$ and $\Delta^g_\infty(r') = 1$ (consider matching $\{(m_1, w_1), (m_2, w_2), (m_3, w_3), (m_4, w_4)\}$). Therefore $\Delta^l_\infty(r) = 1 < 3 = \Delta^l_\infty(r')$ and $\Delta^g_\infty(r) = 2 > 1 = \Delta^g_\infty(r')$. So, (2) holds again. ◇

In the next section our aim is to determine whether rank gaps at stable matchings and the size of the core are bounded by means of local and global distances from mutually best. For this reason it is important to first check whether either type of distances is always smaller than the other type. If this is the case, then bounds in terms of one type of distances yield bounds in terms of the other type of distances as well.

Lemma 1. *Let r be a problem where all men most prefer different women or all women most prefer different men. Then, for each $p \in (0, \infty]$, $\Delta^l_p(r) \geq \Delta^g_p(r)$.*

Proof. Assume, without loss of generality, that all men most prefer different women. Then, assigning each man m to his most preferred woman $t(m)$ yields the stable matching μ_M. Let $p \in (0, \infty)$. We have

$$\Delta^l_p(r) \quad = \quad \sqrt[p]{\frac{1}{2n} \sum_{i \in M \cup W} (r_{t(i)}(i) - 1)^p}$$

$$\geq \sqrt[p]{\frac{1}{2n}[\sum_{m\in M}(1-1)^p + \sum_{m\in M}(r_{t(m)}(m)-1)^p]}$$
$$= \sqrt[p]{\frac{1}{2n}\sum_{i\in M\cup W}(r_i(\mu_M(i))-1)^p}$$
$$\geq \min_{\mu\in\mathcal{M}} \sqrt[p]{\frac{1}{2n}\sum_{i\in M\cup W}(r_i(\mu(i))-1)^p} = \Delta_p^g(r).$$

Moreover, from the above and (1), $\Delta_\infty^l(r) \geq \Delta_\infty^g(r)$ as well. □

Corollary 1. [Local and global distances on domain of horizontal heterogeneity.]
Let $r \in H$. Then, for each $p \in (0,\infty]$, $\Delta_p^l(r) \geq \Delta_p^g(r)$.

Remark 2. Corollary 1 shows that if for problems $r \in H$ we could establish an upper bound in terms of $\Delta_p^g(r)$ for rank gaps or the size of the core, then we immediately would have similar bounds in terms of $\Delta_p^l(r)$. Unfortunately, it will turn out that on H we cannot establish bounds in terms of $\Delta_p^g(r)$ for rank gaps (Proposition 2) and the size of the core (Proposition 4). ◇

As the following lemma shows, there is no similar clear-cut relation between local and global distances on the domain of Eeckhout's sequential preference condition (and hence also not on the general domain).

Lemma 2. [Local and global distances on domain of Eeckhout's SPC.]
There are problems $r, r' \in E\backslash H$ such that for each $p \in (0,\infty]$, $\Delta_p^l(r) < \Delta_p^g(r)$ and $\Delta_p^l(r') > \Delta_p^g(r')$.

Proof. First, consider problem $r \in E\backslash H$ in Table 4. We claim that for each $p \in (0,\infty]$, $\Delta_p^l(r) < \Delta_p^g(r)$. To see this, let $p \in (0,\infty)$. One immediately verifies that $\Delta_p^l(r) = \sqrt[p]{\frac{1}{8}(1^p+1^p)}$. Next, by simple but cumbersome checking of cases, one verifies that at each feasible matching (stable or not), at least one agent is matched to his/her second-ranked mate (or worse) and at least one other agent is matched to his/her third-ranked mate (or worse). Hence,

$$\Delta_p^g(r) = \min_{\mu\in\mathcal{M}} M_p\left(\bigcup_{i\in M\cup W}\{r_i(\mu(i))-1\}\right) \geq \sqrt[p]{\frac{1}{8}((2-1)^p+(3-1)^p)}.$$

Therefore, $\Delta_p^l(r) < \Delta_p^g(r)$. Finally, from the calculations in Example 2 it follows that $\Delta_\infty^l(r) = 1 < 2 = \Delta_\infty^g(r)$. Hence, the first inequality in the statement holds.

Second, consider problem $r' \in E\backslash H$ in Table 6. We claim that for each $p \in (0, \infty]$, $\Delta_p^l(r') > \Delta_p^g(r')$.

m_1	m_2	m_3	m_4
w_1	w_2	w_3	w_3
w_3	w_1	w_2	w_4
w_2	w_3	w_1	w_1
w_4	w_4	w_4	w_2

w_1	w_2	w_3	w_4
m_3	m_1	m_3	m_3
m_1	m_3	m_2	m_4
m_2	m_2	m_1	m_1
m_4	m_4	m_4	m_2

Table 6: Problem r'.

To see this, let $p \in (0, \infty)$. One immediately verifies that $\Delta_p^l(r') = \sqrt[p]{\frac{1}{8}(1 + 3 \cdot 2^p + 2 \cdot 3^p)}$. Next, for the unique stable matching $\mu^* = \{(m_1, w_1), (m_2, w_2), (m_3, w_3), (m_4, w_4)\}$, we have

$$\Delta_p^g(r') \leq M_p \left(\bigcup_{i \in M \cup W} \{r_i'(\mu^*(i)) - 1\} \right) = \sqrt[p]{\frac{1}{8}(2 \cdot 1 + 2 \cdot 2^p)}.$$

Therefore, $\Delta_p^l(r') > \Delta_p^g(r')$. Finally, from the above and (1) it follows that $\Delta_\infty^l(r') = 3 > 2 = \Delta_\infty^g(r')$. Hence, the second inequality in the statement holds. □

4 Bounds on rank gaps and size of core

In Sections 4 and 5 we present and prove our main results. Table 7 provides a summary. Each entry in the table indicates whether rank gaps or the size of the core are bounded by local/global distances (Section 4) or the alternative global distances (Section 5) to mutually best. Here, B stands for "bounded" and N stands for "not bounded." A number x in the table refers to the corresponding Proposition x. A mere B indicates that the bound is trivial. The brackets () indicate that the result follows as a corollary to some other entry in the same column.

	rank gaps			size of core		
domain	local	global	alt. global	local	global	alt. global
H	B1	N2	N2	B3	N4	N4
E	N5	N5	B7	B	B	B
any	(N)	(N)	(N)	N6	(N)	(N)

Table 7: Summary of main results.

4.1 Approaching mutually best on the domain of horizontal heterogeneity

Consider a marriage problem that satisfies horizontal heterogeneity. If the problem satisfies mutually best, then there is a unique stable matching without rank gaps. Below we explore what happens with the size of the core and the rank gaps at stable matchings if the problem does not satisfy mutually best. The first question we tackle is whether rank gaps are bounded in terms of the distance to mutually best. It turns out that the answer depends on whether local or global distances are used.

Proposition 1. [On $\boldsymbol{H}$, rank gaps are bounded by local distances to $\boldsymbol{MB}$.] *For each problem $r \in H$, each stable matching μ, and each $p \in (0,\infty)$, $\Gamma_p(\mu) \leq \sqrt[p]{2}\,\Delta_p^l(r)$ and $\Gamma_\infty(\mu) \leq \Delta_\infty^l(r)$.*

Proof. Let $p \in (0,\infty)$. Then,

$$\begin{aligned}
\Gamma_p(\mu) &= \sqrt[p]{\frac{1}{n}\sum_{(m,w)\in\mu} |r_m(w) - r_w(m)|^p} \\
&\leq \sqrt[p]{\frac{1}{n}\sum_{(m,w)\in\mu} \max\{(r_m(\mu_W(m)) - 1)^p, (r_w(\mu_M(w)) - 1)^p\}} \\
&\leq \sqrt[p]{\frac{1}{n}\sum_{(m,w)\in\mu} (r_m(\mu_W(m)) - 1)^p + \frac{1}{n}\sum_{(m,w)\in\mu} (r_w(\mu_M(w)) - 1)^p} \\
&= \sqrt[p]{\frac{1}{n}\sum_{i\in M\cup W} (r_{t(i)}(i) - 1)^p} \\
&= \sqrt[p]{2}\,\Delta_p^l(r),
\end{aligned}$$

where the second equality follows from $r \in H$. Hence, for each $p \in (0,\infty)$, $\Gamma_p(\mu) \leq \sqrt[p]{2}\,\Delta_p^l(r)$. Then, from (1) it follows that $\Gamma_\infty(\mu) \leq \Delta_\infty^l(r)$. □

Proposition 1 says that on the domain of problems that satisfy horizontal heterogeneity, if for a given problem either of the local distances to mutually best is small, then no stable matching can exhibit large rank gaps.

However, as the following result shows, rank gaps are not bounded in terms of the *global* distances to mutually best.

Proposition 2. [On $\boldsymbol{H}$, rank gaps are *not* bounded by global distances to $\boldsymbol{MB}$.] *For each $\alpha > 0$, there is a problem $r \in H$ and a stable matching μ such that for each $p \in (0,\infty]$, $\Gamma_p(\mu) > \alpha\Delta_p^g(r)$.*

Proof. Let $n \geq 3$ be an integer such that $n - 1 > \alpha$. Consider problem r in Table 8.

m_1	m_2	m_3	$\dots$	m_n
w_1	w_2	w_3	$\dots$	w_n
$\underline{w_2}$	$\underline{w_3}$	$\underline{w_4}$	$\dots$	$\underline{w_1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
w_n	w_1	w_2	$\dots$	w_{n-1}

w_1	w_2	w_3	$\dots$	w_n
m_2	m_3	m_4	$\dots$	m_1
$\underline{m_n}$	$\underline{m_1}$	$\underline{m_2}$	$\dots$	$\underline{m_{n-1}}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
m_1	m_2	m_3	$\dots$	m_n

Table 8: Problem r exhibits "mutually second-best" (underlined matching).

Note that $r \in H$. Let μ be a side-optimal stable matching. Then, $\Gamma_p(\mu) = n - 1$.

Let $p \in (0, \infty)$. Let μ^* be the stable matching where each agent is matched to his/her second most preferred mate. Then,

$$\begin{aligned}
\Delta_p^g(r) &= \min_{\mu \in \mathcal{M}} M_p \left(\bigcup_{i \in M \cup W} \{r_i(\mu(i)) - 1\} \right) \\
&\leq M_p \left(\bigcup_{i \in M \cup W} \{r_i(\mu^*(i)) - 1\} \right) \\
&= \sqrt[p]{\frac{1}{2n}(1^p + \cdots + 1^p)} \\
&= 1.
\end{aligned}$$

So, $\Delta_p^g(r) \leq 1$. Hence, $\Gamma_p(\mu) > \alpha \Delta_p^g(r)$. Finally, from the above and (1), $\Gamma_\infty(\mu) = n - 1 > \alpha \geq \alpha \Delta_\infty^g(r)$ as well. □

Next, we turn to the size of the core.

Proposition 3. [On $\boldsymbol{H}$, size of core is bounded by local distances to $\boldsymbol{MB}$.] *For each problem $r \in H$ and each $p \in (0, \infty]$, $D_p(r) = \Delta_p^l(r)$.*

Proof. Note that for each $m \in M$,

$$|r_m(\mu_M(m)) - r_m(\mu_W(m))| = r_m(\mu_W(m)) - 1 = r_{t(\mu_W(m))}(\mu_W(m)) - 1.$$

Therefore, taking into account that $r \in H$,

$$\{|r_m(\mu_M(m)) - r_m(\mu_W(m))|\}_{m \in M} \quad \text{and} \quad \{r_{t(w)}(w) - 1\}_{w \in W}$$

are identical multisets. Similarly,

$$\{|r_w(\mu_W(w)) - r_w(\mu_M(w))|\}_{w \in W} \quad \text{and} \quad \{r_{t(m)}(m) - 1\}_{m \in M}$$

are identical multisets. Hence,

$$D_p(r) = M_p \left(\bigcup_{i \in M \cup W} \{|r_i(\mu_M(i)) - r_i(\mu_W(i))|\} \right) = M_p \left(\bigcup_{i \in M \cup W} \left\{ r_{t(i)}(i) - 1 \right\} \right) = \Delta_p^l(r)$$

follows. □

As the following result shows, it is not possible to replace local distances by global distances in Proposition 3: the size of the core is not bounded in terms of the *global* distances to mutually best.

Proposition 4. [On H, size of core is *not* bounded by global distances to MB.]
For each $\alpha > 0$, there is a problem $r \in H$ such that for each $p \in (0, \infty]$, $D_p(r) > \alpha \Delta_p^g(r)$.

Proof. Let $n \geq 3$ be an integer such that $n - 1 > \alpha$. Consider again problem $r \in H$ in Table 8. Let $p \in (0, \infty]$. We know from the proof of Proposition 2 that $\Delta_p^g(r) \leq 1$. Since $D_p(r) = n - 1 > \alpha$, the statement follows. □

4.2 Approaching mutually best on the domain of Eeckhout's sequential preference condition

For each problem that satisfies Eeckhout's sequential preference condition, there is a unique stable matching so that the diameter of the core is trivially bounded.

The following result shows that on the domain of Eeckhout's sequential preference condition rank gaps are bounded in terms of distances to mutually best.

Proposition 5. [On E, rank gaps are *not* bounded by distances to MB.]
For each $\alpha > 0$, there is a problem $r \in E$ such that for each stable matching μ and each $p \in (0, \infty]$, $\Gamma_p(\mu) > \alpha \Delta_p^l(r), \alpha \Delta_p^g(r)$.

Proof. Consider problem r in Table 9 where $n > \max\{\alpha + 2, 3\}$. It is easy to see that $r \in E$. Let $p \in (0, \infty)$. For the unique stable matching μ, which is underlined in Table 9, we have $\Gamma_p(\mu) = \sqrt[p]{\frac{1}{n}(n-2)^p}$.

m_1	m_2	m_3	m_4	m_5	…	m_n
$\boldsymbol{w_1}$	w_1	$\boldsymbol{w_3}$	$\boldsymbol{w_4}$	$\boldsymbol{w_5}$	…	$\boldsymbol{w_n}$
$\boldsymbol{w_2}$	w_2	⋮	⋮	⋮	⋮	⋮
⋮	⋮					

w_1	w_2	w_3	w_4	w_5	…	w_n
$\boldsymbol{m_1}$	m_1	$\boldsymbol{m_3}$	$\boldsymbol{m_4}$	$\boldsymbol{m_5}$	…	$\boldsymbol{m_n}$
$\boldsymbol{m_2}$	⋮	⋮	⋮	⋮	⋮	⋮
⋮	m_2					

Table 9: Problem $r \in E$ in Proposition 5.

From the bold-faced entries in Table 9 it follows that $\Delta_p^l(r) = \sqrt[p]{\frac{1}{2n}(1^p + 1^p)} = \frac{1}{n^{1/p}}$. Similarly, from the encircled matching in Table 9 it follows that $\Delta_p^g(r) \leq \frac{1}{n^{1/p}}$. Hence,

$$\Gamma_p(\mu) = \frac{n-2}{n^{1/p}} > \alpha \frac{1}{n^{1/p}} \geq \alpha \Delta_p^l(r), \alpha \Delta_p^g(r) \quad \text{for all} \quad p \in (0, \infty). \tag{5}$$

Finally, from (1) and (5) one obtains

$$\Gamma_\infty(\mu) = n - 2 > \alpha \geq \alpha \Delta_\infty^l(r), \alpha \Delta_\infty^g(r),$$

which completes the proof. □

4.3 Approaching mutually best without domain restrictions

As shown in Proposition 5, rank gaps are not bounded by distances to mutually best for problems in the domain of Eeckhout's sequential preference condition. This immediately implies the following first negative result on the general domain of marriage problems.

Corollary 2. [Rank gaps are *not* bounded by distances to $\boldsymbol{MB}$.]
For each $\alpha > 0$, there is a problem r such that for each stable matching μ and each $p \in (0, \infty]$, $\Gamma_p(\mu) > \alpha\Delta^l_p(r), \alpha\Delta^g_p(r)$.

Similarly, as Proposition 4 demonstrates that the size of the core is not bounded by global distances to mutually best on the domain of horizontal heterogeneity, we get the second negative result on the general domain of marriage problems.

Corollary 3. [Size of core is *not* bounded by global distances to $\boldsymbol{MB}$.]
For each $\alpha > 0$, there is a problem r such that for each $p \in (0, \infty]$, $D_p(r) > \alpha\Delta^g_p(r)$.

Given Corollaries 2 and 3, the only remaining question for approaching mutually best from the general domain is: is the size of the core bounded in terms of *local* distances to mutually best? Our last result, below, answers also this question in the negative.

Proposition 6. [Size of core is *not* bounded by local distances to $\boldsymbol{MB}$.]
For each $\alpha > 0$ and each $p \in (0, \infty]$, there is a problem r such that $D_p(r) > \alpha\Delta^l_p(r)$.

Proof. Let $p \in (0, \infty)$. Let $n > 4$ be an integer such that

$$\sqrt[p]{[6 + 2^p + (n-1)^p]} > \alpha \sqrt[p]{[5 + 3 \cdot 2^p]}. \tag{6}$$

Consider problem r in Table 10. The man-optimal stable matching μ_M is underlined, while the woman-optimal stable matching μ_W is depicted in circles.

m_1	m_2	m_3	m_4	m_5	$\dots$	m_n
$\underline{w_2}$	w_2	$\underline{w_3}$	$\underline{w_4}$	($\underline{\boldsymbol{w_5}}$)	$\dots$	($\underline{\boldsymbol{w_n}}$)
($\boldsymbol{w_1}$)	$\underline{w_1}$	w_4	($\boldsymbol{w_3}$)	$\vdots$	$\vdots$	$\vdots$
$\vdots$	($\boldsymbol{w_4}$)	($\boldsymbol{w_2}$)	$\vdots$			
	$\vdots$	$\vdots$				

w_1	w_2	w_3	w_4	w_5	$\dots$	w_n
(m_1)	(m_3)	(m_4)	(m_2)	($\underline{\boldsymbol{m_5}}$)	$\dots$	($\underline{\boldsymbol{m_n}}$)
m_3	$\underline{\boldsymbol{m_1}}$	$\underline{\boldsymbol{m_3}}$	$\underline{\boldsymbol{m_4}}$	$\vdots$	$\vdots$	$\vdots$
m_4	$\boldsymbol{m_2}$	$\vdots$	$\vdots$			
$\vdots$	$\vdots$					
$\underline{m_2}$						

Table 10: Problem r in Proposition 6.

From the two side-optimal stable matchings and the bold-faced entries it follows that

$$D_p(r) = \sqrt[p]{\frac{1}{2n}[6 \cdot 1 + 1 \cdot 2^p + 1 \cdot (n-1)^p]} \text{ and } \Delta^l_p(r) = \sqrt[p]{\frac{1}{2n}[5 \cdot 1 + 3 \cdot 2^p]}\ , \tag{7}$$

respectively. Hence, in view of (6), $D_p(r) > \alpha\Delta^l_p(r)$.

Finally, using (1) and (7) or directly the definitions of D_∞ and Δ^l_∞, we find that for problem r in Table 10 with $n > \max\{4, 2\alpha+1\}$, $D_\infty(r) > \alpha\Delta^l_\infty(r)$. □

5 Global distances based on stable matchings

Global distances to mutually best are defined as minima over all matchings. One reason to look at these global distances was that local distances are based on the function $i \mapsto t(i)$ that typically does not constitute a well-defined matching.

While the approach of global distances based on the full set of matchings is a first natural step up from local distances, it is of interest to put further restrictions on the set of matchings. In particular, one interesting alternative approach along this line is to take the minimum over all *stable* matchings.[9] This alternative approach connects with several notions that have been studied mostly in the Computer Science literature, see, e.g., Manlove (2013). In this section, we discuss this connection and provide the details of the analysis using the alternative approach.

Let r be a problem and $p \in (0, \infty]$. The (alternative) global (p-)distance from mutually best at r is defined as

$$\boldsymbol{\tilde{\Delta}^g_p(r)} \equiv \min_{\text{stable } \mu} M_p\left(\bigcup_{i\in M\cup W}\{r_i(\mu(i)) - 1\}\right). \tag{8}$$

Obviously, $\tilde{\Delta}^g_p(r) \geq \Delta^g_p(r)$. Moreover, for each $r \in MB$, $\tilde{\Delta}^g_p(r) = 0$.

For each $p \in (0, \infty)$, a stable matching that is a minimizer in (8) is usually referred to as a minimum weight stable matching.[10] In the particular case of $p = 1$, a stable matching that is a minimizer in (8) is a so-called egalitarian stable matching. Finally, if $p = \infty$, then a stable matching that is a minimizer in (8) is a so-called minimum regret stable matching. The alternative global distances remain easy to compute. Manlove (2013, Theorem 1.14) summarizes known computational results and provides further references.

Equipped with the alternative global distances, one can carry out an analysis that parallels the analysis in Sections 3 and 4. Below we discuss which arguments have to be replaced (wherever necessary) and which result changes (and how). *All other results in Sections 3 and 4 remain completely unchanged with the alternative global distances, i.e., in these statements $\Delta^g_\infty(r)$ can be replaced by $\tilde{\Delta}^g_\infty(r)$ and the same proofs/arguments apply.*

- Example 2 and proof of Lemma 2 (first part only). In the analysis for $p = \infty$, the only (inconsequential) change is that $\tilde{\Delta}^g_\infty(r) = 3$ for problem r in Table 4.[11]

[9] We thank a reviewer for suggesting this alternative approach.

[10] See Manlove (2013, p.23) for definitions and a more detailed discussion.

[11] At the unique stable matching $\{(m_1, w_1), (m_2, w_2), (m_3, w_3), (m_4, w_4)\}$ agents m_4 and w_4 are matched to their 4^{th}–ranked mates.

- Proposition 5. The negative result for Δ^g on domain E turns into a positive result for $\tilde{\Delta}^g$ on a larger domain:

  Proposition 7. [For each problem with a unique stable matching, rank gaps are bounded by alternative global distances to $\boldsymbol{MB}$.]
  For each problem r with a unique stable matching μ and each $p \in (0,\infty)$, $\Gamma_p(\mu) \leq \sqrt[p]{2}\,\tilde{\Delta}^g_p(r)$ and $\Gamma_\infty(\mu) \leq \tilde{\Delta}^g_\infty(r)$.

  Proof. Let $p \in (0,\infty)$. It is easy to see that for each pair of real numbers $x, y \geq 1$, $|x-y| \leq (x-1)$ or $|x-y| \leq (y-1)$, so that

  $$|x-y|^p \leq (x-1)^p + (y-1)^p.$$

  Hence, for each $(m,w) \in \mu$,

  $$|r_m(w) - r_w(m)|^p \leq (r_m(w)-1)^p + (r_w(m)-1)^p.$$

  Therefore,

  $$\begin{aligned} \Gamma_p(\mu) &= \sqrt[p]{\frac{1}{n}\sum_{(m,w)\in\mu} |r_m(w) - r_w(m)|^p} \\ &\leq \sqrt[p]{\frac{1}{n}\sum_{(m,w)\in\mu} (r_m(w)-1)^p + (r_w(m)-1)^p} \\ &= \sqrt[p]{2}\, M_p\left(\bigcup_{i\in M\cup W}\{r_i(\mu(i))-1\}\right) \\ &= \sqrt[p]{2}\,\tilde{\Delta}^g_p(r), \end{aligned}$$

  where the last equality follows from the fact that μ is the unique stable matching. Hence, for each $p \in (0,\infty)$, $\Gamma_p(\mu) \leq \sqrt[p]{2}\,\tilde{\Delta}^g_p(r)$. Then, from (1) it follows that $\Gamma_\infty(\mu) \leq \tilde{\Delta}^g_\infty(r)$. □

- Corollary 2. The corollary now follows from Proposition 2 (alternative global distances) and Proposition 5 (local distances).

A final conclusion from the parallel analysis is that while the approach of global distances based on the full set of matchings seems a first natural step up from local distances, an advantage of the approach of global distances based on stable matchings is that it strengthens some of the negative results in Section 4 (Propositions 2 and 4 and Corollary 3).

6 Conclusion

For the analysis of matching problems, it is arguably not only crucial to consider the existence of stable matchings but also the corresponding conflict within matched pairs (as measured by the rank gaps at stable matchings) and across market sides (as measured by the diameter of the core).

While Holzman and Samet (2014) demonstrate that preference profiles that are close to vertical heterogeneity (i.e., agents of each market side almost agree on the ranking of the other market side) yield a small core and stable matchings with small rank gaps, our study indicates that a generalization of these findings beyond the case of vertical heterogeneity is not straightforward.

Rather than considering vertical heterogeneity, we start out with what can be thought of as the opposite of vertical heterogeneity: profiles that exhibit mutually best (i.e., for each man, his most preferred woman also considers him her best mate). As in the case of vertical heterogeneity, mutually best implies that there is a unique stable matching which moreover has no rank gaps. But in contrast to vertical heterogeneity, the fact that a profile almost satisfies mutually best does not necessarily mean that it has a small core nor that its stable matchings exhibit small rank gaps. While mutually best is a restrictive domain, it only pays attention to the most preferred agents. As a consequence, profiles close to mutually best can still vary substantially and hence exhibit very different results with respect to rank-gaps and the set of stable matchings. Intuitively, this is what drives our "negative" results. Only if the profile satisfies horizontal heterogeneity, results similar to those of Holzman and Samet (2014) can be established for local distances (Propositions 1 and 3 being the counterparts of Theorems 1, 2, 3, and 4 in Holzman and Samet, 2014). As is shown in Propositions 2 and 4, these results do not extend to global distances based on feasible matchings. In contrast, Proposition 7 shows that results do extend to global distances based on stable matchings whenever there is a unique stable matching.

Finally, it should be noted that, qualitatively, none of our (positive and negative) results depend on the particular generalized mean that is used to measure distances, rank gaps, and the size of the core.

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