Government Debt Management: The Long and the Short of It

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The Long and the Short of It*

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Abstract

The optimal Debt Management (DM) literature concludes that the covariance of long bond prices with fiscal deficits justifies a dominant role for long bonds and that when long and short bond positions are time varying they are negatively correlated. An important, but unheralded, assumption in this literature is that each period governments repurchase all outstanding long bonds and immediately reissue (r/r) new long bonds. We show that all these features are in sharp contrast to basic features of observed DM in the US where the share of short bonds is large and persistent, short and long bond positions are positively correlated and r/r operations have been very rare. From the DM literature one could derive the normative implications that governments should issue fewer short bonds and they should engage in r/r operations but these implications would only be valid if they are robust to reasonable variations in market settings. To investigate this we systematically examine optimal DM under various reasonable market frictions. We find that under incomplete markets and small transaction costs, calibrated to observed data, there is no role for repurchases and the share of short bonds should be significant and stable. Under no buyback long bonds introduce volatility of cash payments whilst short bonds are desirable as they smooth cash flows. We find a robust result that optimal DM under no buyback resembles the data much more closely than the conventional modelling assuming r/r. Solving incomplete market models with large dimensional state spaces is challenging so we introduce a computational method that enables the efficient global solution of optimal portfolio models under incomplete markets with multiple assets.

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1 Introduction

What type of debt should a government issue? A sizeable literature (see \textit{inter alia} Angeletos (2002) and Barro (2003)) studies debt management (DM) using the canonical framework of a Ramsey planner under full commitment, stochastic government expenditure and distortionary taxes and concludes that governments should focus on issuing long bonds. As the covariance between deficits and long bond prices is negative, both in the model and in the data, issuing long bonds ensures that the market value of debt falls when there is an adverse fiscal shock, allowing the government to effectively complete markets and smooth taxes.

Buera and Nicolini (2004) point out that the optimal portfolios emerging from this approach feature very large long term debt issuance, by several multiples of GDP, and governments investing in short term bonds. Faraglia, Marcet and Scott (2010) point out that introducing more elaborate microfoundations, such as habits and capital, leads to even more extreme portfolios: bond positions are even larger, more volatile and characterised by a negative correlation between issuance of short and long bonds.

The above papers focus on optimal portfolios in the case where governments are able to complete the markets and this plays a role in producing the extreme debt management policies that seem so at odds with observed DM amongst OECD countries. However the main finding of these papers - the importance of long bonds in optimal debt management - survives even when market completeness cannot be achieved due to financial frictions. For instance, Lustig et al. (2008) and Nosbusch (2008) obtain less extreme results by introducing a simple financial friction: they rule out the ability of governments to invest in private assets. They find that it is still optimal for governments to focus almost exclusively on long term debt as it provides useful fiscal insurance - short bonds should be issued only very rarely.\footnote{Lustig et al (2008) use a monetary model to demonstrate a further reason for long term debt: inflation as a tool to lower the debt burden when the deficit is high.} \footnote{All of these papers assume that governments are creditworthy and can fully commit to their tax plans. A few papers have moved away from these assumptions, of no default and full commitment. An older literature (Calvo (1988) and Blanchard and Missale (1994)) considers moral hazard factors that lead governments to issue short term debt. Broner, Lorenzoni and Schmukler (2013), Aguiar et al (2016) and Arellano and Ramnaranayan (2012), Acharya and Rajan (2013), explain the interaction between debt management and default. He and Xiong (2012) study the interplay between liquidity and credit risks in the corporate bond market but their findings can also be applied to government finances. Finally, Debortoli, Nunes and Yared (2016) show that modifying the Angeletos (2002) model to allow for lack of commitment leads to an increase in long term interest rates such that governments issue substantial amounts of short term debt as well as long term debt.}

As we document in Section 2, observed debt management by the US government is at odds with these recommendations. Firstly the US government issues substantial amounts of short term debt with an average share of debt under one year of 43%. This share is always far from zero, never falling below 24% in our sample. Furthermore, the portfolio share of short bonds is relatively stable and highly persistent. In addition, issuances of short and long term bonds tend to rise or decrease together, suggesting that governments tend to increase the stock of both long and short bonds as a response to a deficit shock.

There is another dimension in which observed debt policy deviates from the optimal DM literature. All the papers mentioned so far (and indeed essentially the whole academic literature on long bonds) maintain the assumption that each period governments repurchase the whole stock of previously issued bonds with this repurchase being financed by freshly issued long bonds. This can be described
as a repurchase/reissuance (r/r) operation. This assumption lends simplicity to the analysis as it reduces the number of state variables. However, as we show in Section 2 assuming a full r/r is widely at odds with the data as long bonds are rarely repurchased before maturity. This has also been documented for the US in Garbade and Rutherford (2007), for OECD in Marchesi (2006) and Blommestein et al (2012) and for the UK since 1876 in Ellison and Scott (2017). The only occasions where the US debt management authority has repurchased debt significantly before maturity have been during the 1920’s when the government ran 11 successive years of fiscal surpluses and during the 2000-2001 fiscal surpluses. However these instances were not actually r/r operations, they could be dubbed “pure buybacks”, since repurchases occurred without reissuance of similar maturity as the government was trying to reduce the stock of debt.

From the above studies in optimal policy one could draw the normative recommendation that governments should issue a much larger share of long bonds and engage in r/r operations. However, for the normative insights of a Ramsey model to be useful it is crucial that they remain robust in important directions. If optimal policy changes considerably by introducing reasonable bond markets frictions that concern Debt Management Offices then the above normative implications would be mute. In this paper we systematically compare the recommendations that emerge from optimal policy under various financial frictions. In this way we examine, first, if governments should indeed change their policy in the direction of the above normative recommendation; second, we study what are the key ingredients that are needed in order to build a useful theory of debt management that provides insights into the tradeoffs policymakers face. Thus our approach has both normative and positive elements.

The issue we address in this paper is the following: full buyback provides some fiscal insurance, but since it is achieved by very large repurchases and reissuances this strategy would be costly if there are transaction costs of any type. If we find that these costs outweigh the benefits the nature of optimal policy may change considerably. Indeed, our discussions with Debt Management Offices reveal considerable nervousness about the possibility of operating large scale r/r operations in practice, with concerns expressed over market disruption, transaction costs and fears that large scale purchases and issuances will adversely affect bond prices. This leads us to consider in this paper two questions around r/r operations, and more generally about debt buybacks: a) if a DMO chooses not to buy back debt does this have a substantive impact on optimal debt management? and b) why might a debt management office (DMO) choose not to buy back debt each period?

In Section 2 we outline observed features of US debt management including statistics on debt buybacks and discuss why in practice DMOs may be reluctant to perform r/r’s. In Section 3 we begin our analysis of a) - does the assumption of no buyback have a substantive impact on optimal debt management? Clearly if it doesn’t then b) is an uninteresting question. In Section 3 we outline two extreme models - one with r/r, where every period the government buys back all outstanding debt and then reissues an optimal portfolio and the second model where when the government issues debt it only buys back at the prespecified maturity date. The first model is motivated by the existing literature and the second model by the empirical evidence on debt management practice. This second

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3To be precise, in a model where only 10-period bonds are issued, an r/r involves replacing long bonds issued last year, that have now 9-years to maturity, by new 10-year bonds.

4We discuss later instances, such as Quantitative Easing, where other government agencies, not the Debt Management Office, repurchase debt before maturity.
"no buyback" version of the model is new to the literature.

The literature has to date possibly taken for granted that this distinction cannot matter too much, perhaps because buyback or no is irrelevant if markets are complete. But under incomplete markets no-buyback changes the timing of cash flows in the model, therefore it can substantially change the outcome. Under buyback bonds of all maturities issued in period $t$ pay something in period $t+1$ and nothing after so the timing of payments is the same for long and short bonds. The only difference is that the long bonds’ payoff, namely their price in the secondary market, is stochastic. It is in this setup that long bond prices correlate with the deficit and, as shown by Angeletos, Buera and Nicolini, provide fiscal insurance. No buyback changes the cash flow properties of short compared to long term bonds, since the latter give a payoff far into the future. Through specific analytical cases in Section 3 we show how buyback reduces the attractiveness of long bonds and motivates the issuance of short debt. Firstly under no buyback whilst long bonds still provide fiscal insurance this effect is attenuated. Secondly under no buyback issuing $N$ period bonds introduces $N$ cycles into taxes through lumpy rollovers of debt. By contrast issuing short term debt helps smooth cash flows and promotes better tax smoothing.

Whilst insightful these analytical examples need to be supplemented by calibrated simulations to gain a more robust understanding of the effect of introducing no buyback on optimal DM. Solving optimal debt management models under incomplete markets is computationally demanding. This is particularly true in our case as not only are we looking at multiple assets but also the set of available assets does not span the state space. Further under our assumption of no buyback it is necessary to keep track of all outstanding bonds and for large maturities this leads to a rapidly expanding state space. Since non-negativity constraints on bond issuance (no-lending) and other non-linearities are often crucial elements of incomplete markets equilibria, perturbation algorithms are not a good option. A significant contribution of this paper is therefore to be found in Section 4 where we outline a computational method, based on the Parameterized Expectations Algorithm (hereafter PEA) of den Haan and Marcet (1990) to solve for optimal portfolios globally. We confront two difficulties: i) the size of the state space is very large and ii) using a standard formulation of first order conditions the optimal portfolio choice is indeterminate. We solve the first issue by introducing the Condensed PEA and the second through the Forward States PEA. The Condensed PEA significantly reduces the size of the state space, by forming an initial solution to the model using a small size vector of core state variables, and subsequently finding a few linear combinations of the remaining state variables that summarize these variables efficiently. We also use this idea to introduce relevant non-linear terms of some higher order, as these are indeed necessary for a good approximation. Forward States circumvents the indeterminacy by approximating the integrand terms inside the expectations at $t$ with a function of $t+1$ state variables. As we show this is necessary to uniquely determine date-$t$ control variables (e.g. the bond portfolio) from the Euler equations. These numerical procedures are of interest per se as they are likely to be useful in many other applications involving large portfolios and in models with large state spaces.

In Section 5 we use this solution method to examine optimal DM when the government can issue both short and long term debt (in this case one- and ten-year bonds). We consider four different market environments: buyback/no-buyback combined with lending/no-lending. We find that the introduction of a no buyback constraint has substantial implications for optimal debt management
- now the government should make much greater use of short term debt, in some cases even more than long term debt; portfolio shares are much more stable and persistent and the government tends to increase or decrease the stock of both short and long run debt at the same time. Viewed in this light, Ramsey policy does not seem to urge governments to issue much larger amounts of long debt.

These findings then lead us to consider our second substantive issue around debt management - why would an optimising government not buyback its debt each period through an $r/r$ operation? Obviously both the conventional model of full $r/r$ each period and the model we introduce of no buyback each period are extreme cases. The full $r/r$ assumption each period enables the government to fully utilise the covariance of long bonds with fiscal deficit shocks but assumes that the transaction costs of buying back all debt and then reissuing each period are insignificant. Conversely the no buyback model assumes that transaction costs are sufficiently large that the government never buys back debt before maturity and so cannot fully exploit the fiscal insurance offered by long bonds. In the first part of Section 6 we perform a shadow cost computation of the utility loss of transaction costs by valuing the utility loss from buyback and no-buyback transactions using Lagrange Multipliers from our solutions in Section 5. Based on a calibration of transaction costs from US data, the shadow cost calculation shows that the no buyback solution leads to higher levels of welfare than the canonical $r/r$ assumption. In the second part of this section we consider the case where, allowing for transaction costs, the government decides each period the optimal amount of repurchases. We find that the government hardly ever chooses to buy back debt before its maturity date (and on the very rare occasions it does it does so because it is running large fiscal surpluses) and that it issues a portfolio of both short and long bonds much more broadly consistent with the basic facts displayed in Section 2. In addition to not buying back, there is now a substantial role for short debt, stability of portfolio shares of bonds issued and positive correlation of short and long bond positions.

In Section 7 we turn to a number of issues to examine the robustness of our results. A relatively unexamined feature of observed DM is the fact that bonds pay a fixed semi annual coupon. The existence of coupons becomes more important under no buyback, as coupons can be thought of as a security design aimed at mitigating the no buyback restriction. We also explore the effects of introducing a third bond under no buyback. We finally introduce a simple form of callable bonds. In all these cases we find very little influence on our debt management implications - governments should still issue long term bonds to achieve fiscal insurance but need to issue short term bonds to smooth cash flows, short and long bond positions correlate positively. Finally we look at issues of accuracy of solutions and a final section concludes.

2 US Government Debt Management

This section documents stylised facts about the U.S Treasury’s management of marketable government debt$^5$ held by the public over the period 1955-2015. We use these facts as guidance for the interpretation and robustness of the optimal policy recommendations arising from Ramsey models and for the ingredients that are important in an incomplete market model of DM.

$^5$Our focus is on the Treasury’s debt management practice and so we leave aside the Federal Reserve and its balance sheet. We do so because ours is a real model with no central bank and we are interested in the operation of debt management and how it affects fiscal policy. We are therefore assuming that the Federal Reserve’s transactions in government debt are about the operation of monetary policy not debt management.
The full details of our data and calculations are contained in Appendix A1. We use data from the CRSP about gross government debt issued. As a reference, we classify as "short" debt payments due in less than one year and as "long" debt payments due in over a year. Most of our stylised facts are quoted based on converting bonds into a zero coupon form (eg coupon payments are treated as a separate bond with a maturity date set to when the coupon is paid) so when we refer to “short” or “long” term bonds we are referring to redemption and coupon payments that are due in either before or after one year.

Figure 1 shows the share of the market value of short bonds as a proportion of the total market value of U.S government debt and reveals:

**Fact 1** Portfolio shares of long and short maturities are both substantial.

The share of short term debt is on average 43% and ranges between 24% and 57%.

**Fact 2** Portfolio shares are never close to zero or negative.

**Fact 3** Portfolio shares are highly stable over time, exhibit low standard deviation and high serial correlation.

The first order serial correlation of the share is 0.94 and its standard deviation 0.078.

**Fact 4** Short debt is positively correlated with long term debt.

The ratio of short debt (in market value) over GDP comoves positively with the same ratio for long debt, the correlation is 0.86. In other words, the government issues both short and long debt in response to a deficit shock.

All of these facts are in sharp contrast to the optimal DM recommendations from Ramsey models (i.e Angeletos (2002), Buera and Nicolini (2004), Faraglia, Marcet and Scott (2010)) which usually produce very large issuance of long bonds, large short (negative) positions on short bonds, and, in models with time-varying bond positions, considerable volatility and a negative correlation between short and long issuance. These facts are also unlike the models with non-negativity constraints on bond issuances, as in Nosbusch (2008) and Lustig et al. (2008), where the share of short bonds is often equal to zero.

Our focus in this paper is on models of incomplete markets and in this environment the timing of cash flows matters. This is what motivates our remaining stylised facts which focus on the timing of cash flows by the Treasury - specifically around when they buy back bonds from investors.

Figure 2 shows the total issuance of government debt (long and short) each period over the total stock of debt held and illustrates the following.\(^7\)

\(^6\)In calculating this correlation we divide by GDP to remove non-stationarity. The high correlation remains under alternative detrending techniques. For example, using a linear trend we obtain a correlation of 0.84 and with a linear quadratic trend the correlation is 0.79. We conclude that the high correlation of long and short debt is robust to detrending.

\(^7\)See the appendix for details on how the series displayed in Figure 2 was constructed.
Fact 5 Total (gross) bond issuance each period is a fraction of the stock of outstanding debt.

More specifically, total gross issuance is never larger than 60%. This is in sharp contrast to the optimal debt literature which assumes full $r/r$ each period. Under that assumption the above ratio will fluctuate around 100%.^8^ To better understand government behaviour around buying back debt before maturity we examine the dates at which the US Treasury^9^ has bought back bonds over our sample period. Consider first the case of non-callable bonds. As shown in Table 1, for the whole sample 99.8% of all long maturity government debt is redeemed either at maturity or within a year of its stated maturity date (and 98.86% one quarter before maturity). As mentioned in the introduction, this practice is not confined to the US but is standard practice across the OECD. This leads us to state

Fact 6 Non-callable long bonds are effectively redeemed only at their maturity date and not before.

We have found in presenting this fact to research economists that the extent to which governments rarely buy back debt before maturity is not widely known. Instead we have found a general presumption that the buyback assumption found in the optimal DM literature is not so far from real world practice. The justification normally given is either the widely known buybacks of government debt that occurred between 2000-2001 (and also the repurchases of debt in the 1920s) or the previously widespread use of callable bonds. It should be stressed that in order to include these periods of buybacks we used our CRSP database extended as far back as the 1920s to calculate Fact 6. Further whilst the government did repurchase outstanding long bonds before maturity in these periods this was due to large and persistent budget surpluses and was done to avoid running down issues of short maturity bills.^11^ In other words, this is an example of a repurchase and not a $r/r$.

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^8^Assuming that on average the primary deficit is in balance and that there is no nominal GDP growth.

^9^Spelling out the reasons we focus on the US Treasury and leave aside the Fed is important at this point. In the wake of the Financial Crisis of 2007-9 the Federal Reserve embarked on major purchases of outstanding government debt. As these were clearly made as part of the operation of non-conventional monetary policy (widely held as having their effect either through the size of the central bank’s balance sheet or their impact on long term yields) and ours is a non-monetary model we feel ignoring these transactions is legitimate given our focus on the liability side of the central government’s balance sheet i.e gross debt management. Given that QE was focused on purchases of long term debt extending our model to include monetary factors could further bolster the attractiveness of issuing long term bonds. However note that QE is aimed at increasing the price of long bonds and so this motivation for issuing long term bonds is the very opposite of the standard fiscal insurance channel which is based on falls in long bond prices in response to adverse shocks. As shown in Ellison and Scott (2017) the tendency for long bonds to rise in value after a financial crisis due to low interest rates suggests that this is the very opposite of the fiscal insurance channel normally stressed for long bonds in debt management. This just reiterates how monetary models will introduce additional channels compared to those in the standard real model that we and others address.

^10^Though we observe issuances of bonds and repurchases as far back as 1925, there are missing bond records in the CRSP in the 20s, 30s and 40s. This unavoidable issue constrains us to using data from the 1950s for most of our empirical work. However, since one of the few occasions where buybacks did take place was in the 1920s and to include as many $r/r$ operations as possible we use the whole CRSP sample to construct Table 1. We cannot therefore claim that the estimates reported in Table 1 ‘fully capture’ the 1920s buybacks, but they do partly capture the episode.

^11^Garbade and Rutherford (2007) document the details of the 2000-2001 buyback. The buyback, defined as a consensual transaction between the Treasury and the private sector, involved purchases of the following securities: 1. callable bonds maturing between February 2010 and November 2014 (this consists of very long term debt issued in the 80s), 2. non-callable bonds maturing between February 2015 and 2019, 3. non-callable bonds maturing between 2019 and 2022 or 2023, and 4. non-callable bonds maturing between 2022 or 2023 and November 2027. Garbade and Rutherford (2007) document that roughly 14% of long term government debt (in market value) was cancelled as the result of these buybacks operations.
Further, close examination of government behaviour around callable bonds is also not supportive of the buyback assumption. Although unused since 1985 the US has in the past made extensive use of callable bonds. Callable bonds are basically long maturity government debt which embed an option for the government to redeem the principal (at par) prior to maturity. The period prior to maturity containing the dates when the bond can be redeemed (or recalled) is dubbed the "call window". As shown in Figure 3 in 1955 around 50% of long bonds outstanding were callable although this proportion declined to around 10% by the early 70s before rising once again. The last column of Table 2 shows the fraction of every issuance of callable bonds which has been redeemed prior to the maturity date. Note that aside from a few cases in the late 50s and the early 60s it is typical for all callable debt to be bought back, i.e. for the government to exercise the option to redeem it before the bond matures.

Given the magnitude of callable bonds, especially in the first half of our sample, and the fact that they were nearly always redeemed before maturity, it may seem at first glance that the "buyback" assumption considered in previous papers on optimal DM may be relevant. However closer inspection shows this not to be the case.

Firstly, in nearly every case the callable bonds were bought back within the call window and often the first date in the call window. Table 3 counts the call windows for all callable bonds issued by the US during this time period. The first row shows that there were three five-year callable bonds issued and they all had a call window which started two years from their maturity date, i.e. once the bonds had been outstanding for three years. It shows that all ten-year callable bonds could be recalled only two years prior to maturity at the earliest, and so on. Furthermore, Figure 4 shows the year that callable bonds have been redeemed within the call window, for different call windows. We see that around 80% of debt is repurchased at the first opportunity in the call window across maturities, and the remaining debt is repurchased within a year of the stated original redemption date.

Whilst debt managers exercise call options, they do not buyback callable bonds before the callable window starts, and since call windows are close to maturity it means that most callable long bonds are recalled close to their maturity date. Further, an important feature of the buyback of callable bonds is that it is made at par and not the prevailing market price for bonds which is an important distinction from the usual $r/r$ assumption.

Therefore we have

**Fact 7** Most callable bonds in the US are redeemed at their first call date. For long bonds the first call date is most commonly close to the redemption date.

Facts 5-7 suggest that when the Treasury issues bonds they tend to stay in private hands until maturity. Only callable bonds, which are a declining fraction of long debt, were redeemed before maturity but even in this case they were redeemed close to maturity and at par and not market value.
Therefore actual DM is very different from the buyback assumption found in models on optimal DM, where all long bonds are assumed to be repurchased one period after being issued.

In a section aimed at summarising the practical operation of debt management it is worth considering what market features might make \( r/r \)'s practically nonexistent. In seeking an answer to this question we have held various informal discussions with debt management officers. In answer to us asking "why don't you engage in \( r/r \)'s?" the response is either surprise (often more akin to bafflement and on more than one occasion we have been asked “why would we do that?”) or disappointment at the question (in the vein of 'those academics, always thinking in a parallel world ...'). In short, their answer was that DMO’s are mainly worried about issuing cheaply, and for this they need to promote bond market stability. The responses here were broadly consistent with models such as Greenwood and Vayanos (2010), Gorton, Levellen and Metrick (2012), Guibaud et al (2013), Greenwood et al (2015) and Quinn and Roberds (2017) which emphasise investors being attracted to liquid and safe assets where bonds function as money or investors have a strong preference for particular habitats. In this environment large repurchases or sales are costly to manage and may disrupt the market. Only in a debt reduction environment might buybacks be needed in order to maintain a desirable mix of maturities. However this is not strictly speaking an \( r/r \) operation but a "pure repurchase" without reissuance.

In general we can think of three different reasons why \( r/r \) may be costly and all three we capture under the term “transaction costs”. The first are simply the resources required to run the government’s debt management office e.g buildings, personnel, equipment. These either seem fixed in nature or costs that the Treasury would have to pay anyway to run its issuance operations and so are unlikely to influence whether buyback or no buyback is optimal. The second category is due to the existence of bid-ask spreads (as documented by Amihud and Mendelson (1991) and Engle et al (2012)) and brokerage fees. The existence of a bid-ask spread will make full scale \( r/r \) every period more expensive by creating a wedge between the buying and selling price. The third category of transaction costs arise from price pressures and the belief that the supply and demand curves for government bonds are not perfectly elastic (perhaps for the clientele reasons given above) so issuing or purchasing more of specific bonds will influence the market price. If the demand for bonds of particular maturities is not perfectly elastic then issuing or purchasing large quantities will drive the price against the government and lead to more expensive borrowing. Lou, Yan and Zhang (2013), Breedon and Turner (2016) and Song and Zhu (2016) all derive estimates of how bond prices are influenced by large scale purchases or issues of government debt. As with bid-ask spreads, the existence of these auction effects will add additional costs to \( r/r \) which may make buyback suboptimal. In a later section we will calibrate these costs and see if they can explain, in the context of a Ramsey model, why governments do not perform \( r/r \) operations but for now we simply note that DMOs tend

\[12\] In a related paper, Faraglia, Marcet, Oikonomou and Scott (2017) we outline a three period model where the government has superior information around future public finances than the investor. This informational asymmetry, in a similar spirit to Myers and Majluf (1984), generates the result that the government chooses not to buy back previously issued debt before maturity. Doing so triggers a bond market shut down as investors believe that the government is trying to reschedule its debt ahead of poor public finances.

\[13\] A particular concern of debt managers is around “auction failures” and the fear they produce market instability. Given the size of government debt there are frequent issues of new debt and an orderly market requires these to be sold at or near prevailing market prices. Large scale issues which go unsold or create market volatility are perceived as very damaging. This fear naturally produces a conservatism in issuance and a reluctance to do \( r/r \) as that would increase the number and scale of auctions and increase the probability of an auction failure.
to offer these facts as a reason for why Facts 5-7 show little evidence for buyback at any other times other than prespecified maturity or call dates or during periods of sustained high fiscal surpluses.

3 The Model

In this section we compare the extreme assumption of full \( r/r \) with the opposite extreme where the government each period does no buyback but just lets bonds stay in the hands of private investors until reaching their redemption date. In Section 6.3 we will examine a model where each period the government can choose how much to repurchase. If we found that the case of buyback produces similar portfolio recommendations as no buyback then pursuing the modelling of transaction costs and the complexity of partial buybacks would be an unnecessary distraction.

For both the case of buyback and no buyback our benchmark model is a Ramsey policy equilibrium with perfect commitment and two bonds.\(^\text{14}\) Essentially it can be seen as adding a long bond to the model of Aiyagari et al. (2002). We also follow the literature and consider the existence of a non-negativity constraint on bond issuance. The aim throughout is to see what recommendations for debt management emerge from this different environment that possibly captures important features of bond markets.

We assume a single representative household whose preferences over consumption, \( c_t \), and leisure, \( x_t \), are given by \( E_0 \sum_{t=0}^{\infty} \beta^t [u(c_t) + v(x_t)] \), where \( u \) and \( v \) are strictly increasing and strictly concave functions and \( 0 < \beta < 1 \) is the discount factor.

The economy produces a single good that cannot be stored. The household is endowed with \( T \) units of time that it allocates between leisure and labour. Technology for every period \( t \) is given by:

\[
(1) \quad c_t + g_t = (T - x_t)
\]

where \( g_t \) represents government expenditure which is assumed to be stochastic and exogenous and is the only source of uncertainty in the model. The representative firm maximizes profits. Both the household and the firm take prices and taxes as given.

The government engages in the following activities to finance spending: First, it levies distortionary taxes \( \tau_t \) on labor income and second, it issues debt. Bond issuance of the government at period \( t \) is a vector \( b_t = (b_t^S, b_t^N) \) where \( N \) denotes the long and \( S \) the short bond. Both are real, zero-coupon, riskless bonds: the short (long) bond promises to pay one unit of consumption in \( S (N) \) periods with certainty, we take the integer \( S \geq 1 \) to be much lower than the integer \( N \). Let \( p_t^i \) be the price of a bond of maturity \( i = 1, \ldots, N \) with \( p_t^0 = 1 \).

In the standard case with \textit{buyback} the period-\( t \) government budget constraint can be written as:

\[
(2) \quad \sum_{i=\{S,N\}} b_t^i p_t^i = \sum_{i=\{S,N\}} b_{t-1}^i p_t^{i-1} + g_t - \tau_t (T - x_t)
\]

The left side of this equation is the value of the bond portfolio issued this period. The first term

\(^\text{14}\)Our benchmark model is purposefully simplistic allowing the government to issue debt in short and long bonds and in Section 7 we will also consider a third asset. Though government portfolios are indeed more complex in reality, the work of Piazzesi and Schneider (2010) shows that parsimonious portfolios, with a few zero coupon bonds, can span payoffs of more complex portfolios.
on the right side is the market value of debt outstanding and \( g_t - \tau_t(T - x_t) \) is the primary deficit. Notice that with this constraint the government is assumed to perform a full repurchase/reissuance operation \((r/r)\) every period.

In the case where government debt is held to maturity i.e no buyback, then the government’s budget constraint becomes:

\[
\sum_{i \in \{S,N\}} b_t^i p_t = \sum_{i \in \{S,N\}} b_{t+1}^i + g_t - \tau_t(T - x_t).
\]

The left hand side of (3) once more corresponds to the market value of new debt issued in period \( t \) but the first term on the right hand side now measures not the total value of debt outstanding but instead the total value of debt maturing that period. Under no buyback although the government only issues bonds of maturity \( S \) and \( N \) the maturity of these issued bonds declines each period. At any point in time there are therefore bonds of any maturity \( i \leq N \) outstanding under no buyback e.g long bonds issued today will eventually become short bonds as they approach their redemption date. This provides a mechanical channel whereby for fixed issuance strategy e.g a given proportion of long/short bonds, the overall portfolio shows a greater reliance on short bonds than under the case of full buyback.

In order to solve models of optimal debt management it is standard to assume that \( b_t^i \) satisfies some ad hoc limits\(^{15}\) (see Aiyagari et al. (2002)). For the standard case of buyback the assumption is made that debt evolves between an upper and lower bound e.g

\[
\frac{M_i}{\beta^i} \leq b_t^i \leq \frac{\overline{M}_i}{\beta^i}.
\]

In the case where \( M_i < 0 \) the government can purchase private bonds but this is ruled out when \( \overline{M}_i = 0 \) (as in Lustig et al. (2008) and Nosbusch (2008)). We shall refer to the latter as the “No Lending” case. Notice that in (4) both the upper and lower bounds of maturity \( i \) are scaled by the steady state price of debt for that maturity \( p^i = \beta^i \) (see below); this facilitates the interpretation of the \( M \)’s as they are in units of the (steady state) market value of debt issued each period.

In the case of no buyback we have to modify these constraints as the amount of debt outstanding per period is no longer given by debt issued that period. Then, given the issuances between \( t \) and \( t - N + 1 \), the market value of debt in \( N \) bonds still outstanding using steady state prices is:

\[
\sum_{j=1}^{N} \beta^i b_{t-N+j}^i.
\]

Therefore we normalize the debt constraints for \( i = \{S, N\} \) as

\[
\frac{M_i}{\sum_{j=1}^{N} \beta^j} \leq b_t^i \leq \frac{\overline{M}_i}{\sum_{j=1}^{N} \beta^j}.
\]

Note that, for the same \( M \)’s, this puts the same limits as for the case of buyback on the steady state value of debt in each bond, and for the total market value of debt.

\(^{15}\)As in Aiyagari et al. (2002) we assume for simplicity that the debt limits that the government faces are tighter than the consumer’s debt limits, thus the consumer is always at the margin.
3.1 Ramsey Problem

As is standard in the Ramsey policy literature we assume the government chooses tax and bond policies knowing the implied equilibrium quantities and seeking to maximize household utility. We first summarize the competitive equilibrium in a few equations.

The consumer budget constraint is analogous to (2) or (3). From the maximization problem of the consumer, equilibrium bond prices satisfy $p_t^i = \beta^t E_t \left( \frac{u_{c,t+i}}{u_{c,t}} \right)$, where $u_{c,t} \equiv u'(c_t)$ and taxes satisfy $\tau_t = 1 - \frac{v_{c,t}}{u_{c,t}}$. Substituting these conditions and using the budget constraint under buyback we obtain the implementability conditions

$$L, L_t \sum_{i \in \{S,N\}} b_t^i E_t \left( \beta^t u_{c,t+i} \right) = \sum_{i \in \{S,N\}} b_{t-1}^i E_t \left( \beta^{t-1} u_{c,t+i-1} \right) + g_t u_{c,t} - (u_{c,t} - v_{x,t})(g_t + c_t)$$

for all $t$ a.s. As argued in Aiyagari et al. (2002) it is not possible to simplify further under incomplete markets, (6) has to be imposed for all $t$. For the no buyback case we get

$$L, L_t \sum_{i \in \{S,N\}} b_t^i E_t \left( \beta^t u_{c,t+i} \right) = \sum_{i \in \{S,N\}} b_{t-1}^i u_{c,t} + g_t u_{c,t} - (u_{c,t} - v_{x,t})(g_t + c_t).$$

Using standard arguments we have that $\{c_t, b_t^S, b_t^N\}$ is a competitive equilibrium if and only if it satisfies the implementability constraint (6) (or (7)) and debt limits (4) (or (5)) almost surely for all $t$. The Ramsey equilibrium solves a planner’s problem choosing sequences $\{c_t, b_t^S, b_t^N\}$ to maximize the household’s utility subject to the implementability constraint and debt limits a.s. for all $t$. The Lagrangean for the planner’s problem under buyback is:

$$L = E_0 \sum_{t} \beta^t \left[ u(c_t) + v(T - c_t - g_t) + \lambda_t \left( \sum_{i \in \{S,N\}} b_t^i \beta^t u_{c,t+i} - \sum_{i \in \{S,N\}} b_{t-1}^i \beta^{t-1} u_{c,t+i-1} - g_t u_{c,t} + (u_{c,t} - v_{x,t})(g_t + c_t) \right) \right] + \sum_{i \in \{S,N\}} \xi_{L,t}^i \left( \frac{M_i}{\beta_i} - b_t^i \right) + \sum_{i \in \{S,N\}} \xi_{U,t}^i \left( b_t^i - \frac{M_i}{\beta_i} \right).$$

Here $\xi_{L,t}^i$ and $\xi_{U,t}^i$ denote the multipliers on the lower and upper bounds respectively and $\lambda_t$ is the multiplier of (6).

Under no buyback the corresponding Lagrangean is

$$L = E_0 \sum_{t} \beta^t \left[ u(c_t) + v(T - c_t - g_t) + \lambda_t \left( \sum_{i \in \{S,N\}} b_t^i \beta^t u_{c,t+i} - \sum_{i \in \{S,N\}} b_{t-1}^i u_{c,t} - g_t u_{c,t} + (u_{c,t} - v_{x,t})(g_t + c_t) \right) \right] + \sum_{i \in \{S,N\}} \xi_{L,t}^i \left( \frac{M_i}{\beta_i} - b_t^i \right) + \sum_{i \in \{S,N\}} \xi_{U,t}^i \left( b_t^i - \frac{M_i}{\beta_i} \right).$$

As noted by Angeletos (2002) and Buera and Nicolini (2004), in the case that $g_t$ is a Markov process taking only two possible values and assuming $M$’s are sufficiently large in absolute value the optimal debt management strategy under buyback provides full insurance, i.e. $\lambda_t$ is constant and $\xi_{j,t}^i = 0$ in (8). Further the optimal bond portfolio is time invariant and the complete market outcome
is achieved in terms of prices, allocations and taxes. As explained in the introduction this occurs
because long bond prices and the primary deficit are negatively correlated. However, we assume
that \( g_t \) has a continuum of possible values and further that the debt limits may be binding. In this
case the negative correlation of deficits and long bond prices is not sufficient to fully offset the
effects of spending shocks on the government budget. Therefore, in our setup the complete market
allocation is not reached, the multiplier \( \lambda_t \) and the bond portfolios are time varying and the debt
limit multipliers \( \xi \) are not necessarily equal to zero.

### 3.2 Optimality Conditions under Buyback

In the case of buyback the first order conditions are:

\[
\begin{align*}
u_{c,t} - v_{x,t} + \lambda_t [u_{cc,t} c_t + u_{c,t} + v_{xx,t} (c_t + g_t) - v_{x,t}] + \\
+ u_{cc,t} \sum_{i \in \{S, N\}} (\lambda_{t-i} - \lambda_{t-i+1}) b_{t-i}^i = 0
\end{align*}
\]

\[
\beta^t E_t (u_{c,t+i} \lambda_t - u_{c,t+i} \lambda_{t+1}) + \xi_{L,t}^i - \xi_{U,t}^i = 0 \quad \text{for } i = S, N.
\]

Equation (10) represents the first order condition of consumption and (11) the analogous condition
with respect to \( b_t^i \). For the case of loose debt limits we have \( \xi_{L,t}^i = \xi_{U,t}^i = 0 \). Then, using the
arguments of Aiyagari et al. (2002) we see that (11) states that \( \lambda_t = E_t u_{c,t+i} \lambda_{t+1}/E_t (u_{c,t+i}) \) which
evolves as a risk-adjusted random walk with two risk measures, namely \( u_{c,t+i}/E_t (u_{c,t+i}) \) for
\( i = S, N \).

Extending the argument in Marcet and Marimon (2012) the optimal solution has a recursive
formulation where the optimal tax schedule may be written as:

\[
\begin{align*}
\tau_t &= \tau(X_t) \text{ for } (12) \\
X_t &= (g_t, \lambda_{t-1}, \lambda_{t-2}, \ldots, \lambda_{t-N}, b_{t-1}^S, \ldots, b_{t-S}^S, b_{t-1}^N, \ldots, b_{t-N}^N) \text{ for a time-invariant function } \tau(\cdot) \text{ as long as we constrain } \lambda_{-1} = \ldots = \lambda_{-N} = 0. \text{ The state space for this solution is clearly very large as the dimension of } X_t \text{ is } 2N + S + 1. \text{ In the very simple model below, when we take } S = 1 \text{ and } N = 10, \text{ this amounts to 22 state variables. As explained in detail in Faraglia et al. (2016), the reason } N \text{ lags of } \lambda \text{ and } b \text{ are needed is because when the government issues debt it makes a promise to change future taxes in order to influence the future marginal utility of consumption and so in this way favourably influences current bond prices. Solving this model is computationally demanding because of the magnitude of the state space and in the next section we outline a new computational method which offers an efficient solution procedure.}
\end{align*}
\]

### 3.3 Optimality Conditions under No Buyback

The above summarises results already known in the literature whereas the focus of this section is the
implications of assuming no buyback. Under no buyback the first order conditions for the optimum
(for consumption and bonds respectively) are:

\[
\begin{align*}
&\quad u_{c,t} - v_{x,t} + \lambda_t [u_{cc,t}c_t + u_{ct,t} + v_{xx,t}(c_t + g_t) - v_{x,t}] \\
&\quad + u_{cc,t} \sum_{i \in \{S,N\}} (\lambda_{t-i} - \lambda_t) b_{t-i}^i = 0 \\
\end{align*}
\]

\[
\beta^t E_t (u_{c,t+i} \lambda_t - u_{c,t+i+1} \lambda_{t+1}) + \xi_{i,t}^t - \xi_{i,t}^{i+1} = 0 \quad \text{for} \quad i = S, N
\]

Now we have that off corners \( \lambda_t = E_t u_{c,t+S} \lambda_{t+S}/E_t (u_{c,t+S}) \) and \( \lambda_t = E_t u_{c,t+N} \lambda_{t+N}/E_t (u_{c,t+N}) \). The only difference between the derivative with respect to bonds here (15) and the one under buyback (11) is that we now have \( \lambda_{t+i} \) instead of \( \lambda_{t+1} \). Therefore under no buyback \( \lambda_t \) shows more complex dynamics displaying (risk-adjusted) cycles of periodicity \( S \) and \( N \). As we discuss in detail below this is because under no buyback long bonds provide less possibilities for fiscal insurance. Although the first order conditions are different the state vector is the same as under buyback so the optimal allocation satisfies (12)-(13).

Now we use the first order conditions to derive some analytic results on the impact of no buyback.

### 3.3.1 Reduced Effectiveness of Fiscal Insurance

Consider first the issue of fiscal insurance, that is the ability for declines in bond prices to offset adverse government expenditure shocks and so support tax smoothing. This still remains in the case of no buyback but is reduced in strength under standard conditions. To see why this is so it is useful to reformulate the budget constraint. Let \( s_t = \tau_t (T - x_t) - g_t \) be the primary surplus in \( t \). Adding and subtracting the value of non-matured bonds \( \sum_{i \in \{S,N\}} \sum_{j=1}^{i-1} p_{t}^{i-j} b_{t-j}^{i} \) on both sides of (3), using \( p_{t}^i = E_t (p_{t}^{i+j} p_{t}^{j}) \) and the equilibrium values for \( p_{t}^i \) we can express the intertemporal budget of the government as

\[
E_t \sum_{j=0}^{\infty} \beta^j \frac{u_{c,t+j}}{u_{c,t}} s_{t+j} = \sum_{i \in \{S,N\}} \sum_{j=1}^{i} p_{t}^{i-j} b_{t-j}^{i}
\]

which gives the usual condition that the discounted sum of expected future surpluses has to equal the current market value of total debt. The intertemporal budget constraint in (16) shows a hedging effect of issuing long term debt analogous to the buyback model. Suppose there is a rise in \( g_t \) above the steady state level. This causes the left side of this equation to go down. Since long bond prices go down when the primary deficit increases the right side of (16) will compensate if \( b_{t-j}^{N} \) are positive. Therefore we may expect that issuing long bonds helps build a portfolio that absorbs shocks to the primary deficit and so helps alleviate tax volatility. However notice that because of no buyback what matters is the whole sequence of long bonds issued over the past \( N \) periods. Long bonds issued several periods ago are no longer such long bonds and so their price shows less of a decline. In the case of buyback where debt could be reissued each period long bonds retain their maturity and so provide enough fiscal insurance. In the case of no buyback this effect is attenuated as previously issued long bonds get closer to their maturity date.
3.3.2 N cycles

As well as affecting the efficacy of fiscal insurance, long bonds under no buyback introduce an additional problem - they add $N$ cycles to taxes and so contribute to tax volatility rather than tax smoothing. To illustrate this point consider the following special case:

i) no short bonds can be issued or purchased, that is $M_S = M_i = 0$;

ii) $g_i$ is deterministic and higher in the first period: $g_0 > g_1 = g_2 = \ldots$;

iii) debt limits on the long bond are not binding

iv) $b_{t-i}^N = \bar{b}$ for $i = 1, \ldots, N$;

Equation (15) for $b_t^N$ gives:

(17) \[ \lambda_t = \lambda_{t+N} \quad \text{for} \quad t = 0, 1, \ldots . \]

Furthermore $\lambda_1 = \lambda_2 = \ldots = \lambda_{N-1}$ \footnote{To prove this notice that (17) and (14) imply $c_i = c_{i+N}$ for $i = 1, 2, \ldots, N - 1$ and all $t = 1, 2, \ldots$. Together with (18) this implies $u_{c,i}(\tau_i(T - x_i) - g_i) = u_{c,i}\bar{b}(1 - \beta^N)$, therefore $c_i = \bar{\tau}$ and $\lambda_i = \bar{\lambda}$, for $i = 1, 2, \ldots, N - 1$.}

Putting all this together implies that $\lambda$ has a simple $N$-period cycle.

\[ \lambda_{tN} = \lambda_0 \quad \text{for} \quad t = 1, 2, \ldots \]

\[ \lambda_{tN+i} = \bar{\lambda} \quad \text{for} \quad i = 1, \ldots, N-1, \quad \text{and} \quad t = 0, 1, 2, \ldots . \]

Equation (14) implies that this cycle also arises for consumption and taxes.

The evolution of taxes in this example is shown in Figure 5 assuming $b_{t-i}^N = 0$ for $i = 1, \ldots, N$. The dashed line represents taxes when the government issues just a three year bond ($N = 3$), the crossed line for a ten year bond ($N = 10$) and the solid line for when just a one year bond is issued. Clearly taxes are more volatile under no buyback if only a long bond is issued. The higher tax needed in period $t = 0$ because of high $g_0$ reverberates every $N$ periods even if there are no further high values of $g$. This in turn causes a large increase in taxes in future periods at intervals of $N$ periods ahead, while the high $g_0$ has no effect on taxes in the intervening periods. Obviously the longer the maturity of long bonds the greater the volatility in taxes at longer frequencies.

To understand the reason for this result notice that through forward iteration on the budget constraint (3) we have

(18) \[ \sum_{j=0,N,2N,\ldots} \beta^j u_{c,t+j}(\tau_{t+j}(T - x_{t+j}) - g_{t+j}) = u_{c,t}b_{t-N}^N \quad \text{for all} \quad t. \]

To emphasize, notice the summation index is over $j = 0, N, 2N, \ldots$. This shows that if only one long bond is issued taxes can only be compensated at $t + N, t + 2N, \ldots$ and intervening periods become disconnected. Given the large value of $g_0$ there is a rise in taxes and issuance of debt in $t = 0$. But the new debt issued at $t = 0$ has to be redeemed in $N$ years at which point taxes have to increase to pay the accumulated interest and further debt has to be issued. It is pointless to increase taxes in intervening periods when spending is back at the steady state value. More precisely, it is not possible to reduce $\tau_0$ by increasing, for example, $\tau_1$ or $\tau_2$. The additional tax revenue at $t = 2$ cannot be utilized to reduce the debt accumulated at $t = 0$ if only long bonds are issued under no buy back.
An increase in $\tau_2$ would only reduce $\tau_{2+N}$ and therefore induce even more volatility in the intervening periods.

[Figure 5 About Here]

This example is chosen to illustrate a stark result but the finding is robust. This is an implication of the fact that $\lambda_{t+1}$ is replaced by $\lambda_{t+i}$ in going from (11) to (15), generating an $N$-period cycle in $\lambda$. One key result of this paper is that debt management can offset this $N$-period lumpiness through issuing short term debt. Short debt helps offset tax volatility by distributing debt payments in between these $N$-period cycles. In the simple example above, issuing short bonds (i.e. allowing $-M_S = M_S > 0$, actually achieves the complete market outcome. Short bonds under no buyback have additional smoothing properties over long bonds in general.

The above discussion suggests that assuming no buyback will influence optimal DM as it makes long bonds less effective at providing fiscal insurance, induces $N$ cycles in taxes and provides a tax smoothing role for short bonds. What isn’t of course clear from this section is whether these channels are quantitatively significant and for that we need to turn to simulations. However, as mentioned above, solving Ramsey models under incomplete markets with multiple bonds is a computationally challenging task and even more so under the assumption of no buyback. In the next section we introduce two new computational methods that help significantly to produce numerical solutions of this model.

4 The Solution Method

In solving our model we apply the widely used Parameterized Expectations Algorithm (PEA) of den Haan and Marcet (1990). Solving our model requires introducing two modifications to PEA. The first modification is necessary because the state vector $X_t$ in our model may be very large requiring a method to reduce the state space. The second modification is required because using PEA in the standard way yields a system of equations that is indeterminate. We refer to the first modification as Condensed PEA and to the second as the Forward States PEA. Whilst our focus is on a problem of government debt management these computational methods have much wider applicability.

4.1 The Conventional PEA Approach

We first describe how a standard application of PEA would proceed in this model. For the sake of simplicity we focus on solving the model under buyback described in Section 3.2 when debt limits

\footnote{Sometimes, in order to reduce the dimensionality of the state space, the literature has assumed bonds consist of geometrically decaying coupons. One justification for this simplification is that the decay may capture a given portfolio with decaying weights on higher bond maturities outstanding. Since the objective of this paper is to aim precisely at explaining portfolio choices taking as fixed the weights of the bond portfolio seems self defeating. Furthermore, Faraglia, Marcet, Oikonomou and Scott (2016) show, in the context of models of optimal fiscal policy, that this approach is at best a weak approximation whilst Hilscher, Raviv, and Reis (2014) show that actual portfolio of bonds outstanding is not geometrically decaying but hyperbolic.}

\footnote{To conserve space we mention here only the principles of these methods. In the online Appendix B we describe the technical aspects of their implementation. Faraglia, Marcet, Oikonomou and Scott (2014) provide a detailed description of how to solve many optimal fiscal policy problems with this extended PEA.}
are not binding. Given the vector $X_t$ our aim is to solve the system of equations (6), (10) and (11) to obtain the current value of consumption $c_t$, the bond quantities $b_i^t$, $i = S, N$, and the multiplier $\lambda_t$. Parameterized expectations requires approximating the terms $E_t(u_{c,t+i})$ and $E_t(\lambda_{t+1}u_{c,t+i})$ with functions of the state vector $X_t$, in other words:

\begin{equation}
E_t(u_{c,t+i}) = \Phi^i(X_t, \gamma^i) \quad \text{and} \quad E_t(\lambda_{t+1}u_{c,t+i}) = \Psi^i(X_t, \delta^i) \quad i = S, N
\end{equation}

where $\Phi^i$ and $\Psi^i$ belong to a class of functions such that $\Phi^i(\cdot, \gamma^i)$ and $\Psi^i(\cdot, \delta^i)$ can approximate the conditional expectations arbitrarily well. We will take $\Phi^i$ and $\Psi^i$ to be polynomials of a given order so $\gamma^i$ and $\delta^i$ will be coefficients on the variables in $X_t$ as well as their squares, cubes, cross-products and so on, depending on the order of the approximating polynomial of $\Phi^i$ and $\Psi^i$ that is used.\(^{19}\)

The system (6), (10) and (11) has four equations that we hope will give a solution for the four variables $(c_t, b_{S}^t, b_{N}^t, \lambda_t)$ given the parameterized expectations. In Section 4.3 we discuss how to set up this system so that $(c_t, b_{S}^t, b_{N}^t, \lambda_t)$ can be conveniently solved for.

PEA then iterates to find parameter values $\gamma^{i,f}$ and $\delta^{i,f}$ that satisfy the following fixed point property: the series for $\{c_t, b_{S}^t, b_{N}^t, \lambda_t\}$ generated by $(\gamma^{i,f}, \delta^{i,f})$ is such that $\Phi^i(X_t, \gamma^{i,f})$ and $\Psi^i(X_t, \delta^{i,f})$ are the best predictors of the objects inside the conditional expectations (19) among any other $\gamma^i$, $\delta^i$.

### 4.2 The Condensed PEA

Despite the simplicity of our model the state vector is very large. For the case where the government issues one- and ten-year bonds (i.e $S = 1$, $N = 10$) the state vector $X_t$ has 22 elements. Allowing the government to issue all maturities between 1 and 10 increases the length of $X_t$ to 67, as every maturity $m$ adds $m$ lags of bond quantities to the state vector. Since debt limits play a role in our model perturbation methods are not appropriate as they cannot approximate well the solution both near and away from the debt limits, so we strive to approximate the non-linear solution globally. In this situation a state vector of such dimension is difficult to handle even for our relatively basic model.

However, there are reasons to believe that, for most models, the dimensions of the state vector can be effectively reduced. With so many state variables our numerical methodology has a tendency towards close collinearity in the elements of $X_t$. Furthermore, in models with incomplete markets both $\lambda$ and $b$ have near unit roots.\(^{20}\) This means that the regressions used to compute parameters $\gamma$ and $\delta$ are nearly undefined and in plain PEA this often leads the algorithm to either circle indefinitely or even diverge.

However this multicollinearity is in a way encouraging: it suggests that in the optimal solution many elements of $X_t$ influence the conditional expectation only slightly. Therefore it is likely that the “relevant” information in $X_t$ can be condensed in a few state variables to obtain a good approximation. After all, we know some models where this is exactly right. For example, under complete markets all past bond issuances can be “condensed” in total wealth, which is the only relevant state

\(^{19}\)For the sake of clarity we represent the approximating functions using ordinary polynomials though it should be noted that the technique may be applied to orthogonal polynomials (such as Chebyshev, Hermite and Legendre families). We utilize polynomials that are additively separable in the state variables as this allows us to calculate the coefficients with linear methods.

\(^{20}\)See Aiyagari et al. (2002).
variable. In our case we may expect this to be approximately true.

Furthermore, in PEA we only need that the variable "condensing" $X_t$ have good properties in predicting the objects inside the conditional expectation of the equilibrium conditions. In the solutions we compute each of the elements of $X_t$ determine the simulation through their role in the budget constraints and other optimality conditions - only the role of $X_t$ in the conditional expectations is condensed.

Intuitively, the dimensionality of $X_t$ can be reduced for two reasons: first because many elements of $X_t$ may be perfectly correlated with the rest of the states, and second because they may be (nearly) irrelevant in predicting the objects they should predict along the optimal solution.\footnote{Reiter (2009) addresses a related issue in solving dynamic models with heterogeneous agents. He applies techniques used in control theory to reduce the dimensionality of the agents’ distribution of wealth.}

More specifically, in solving the buyback model we approximate the expectation

$$E_t(u_{c,t+i})$$

which appears in the implementability constraint (6) and the first order condition (11).\footnote{Of course, the remaining conditional expectations that appear in the equilibrium conditions must be handled with this procedure as well.}

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We partition $X_t$ into two parts: a subset of $n$ state variables $\{X_{t}^{\text{core}}\} \subset \{X_t\}$, where $n$ is small and an omitted subset of state variables $\{X_{t}^{\text{out}}\} = \{X_t\} - \{X_{t}^{\text{core}}\}$. Although in our later application the approximating function $\Phi^i(\cdot, \gamma^i)$ includes higher order terms in the solution we study below for the sake of the exposition we illustrate assuming $\Phi^i(\cdot, \gamma^i)$ is linear.

The idea is to first solve the model including only $X_{t}^{\text{core}}$ as state variables and find a fixed point $\gamma_{i,f,\text{core}}$ when only $X_{t}^{\text{core}}$ is included in $\Phi^i$. We subsequently define the prediction error:

$$\phi_{t+i} \equiv u_{c,t+i} - \Phi^i(X_{t}^{\text{core}}, \gamma_{i,f,\text{core}})$$

If this is a good approximation then $E_t(u_{c,t+i}) \simeq \Phi^i(X_{t}^{\text{core}}, \gamma_{i,f,\text{core}})$ and the error $\phi_{t+i}$ would be linearly unpredictable with $X_{t}^{\text{out}}$. In this case we would claim the solution with core variables is a good approximation. But if $X_{t}^{\text{out}}$ is correlated with $\phi_{t+i}$ it means that some elements of $X_{t}^{\text{out}}$ help predict $u_{c,t+i}$ above the prediction provided by $X_{t}^{\text{core}}$. We then find the linear combination $X_{t}^{\text{out}}$ that has the highest predictive power for $\phi_{t+i}$, say $X_{t}^{\text{out}} \cdot \alpha$, we add this linear combination (only one more variable) to the set of state variables in $\Phi^i$, solve the model again with $(X_{t}^{\text{core}}, X_{t}^{\text{out}} \cdot \alpha)$ as state variables, find a new fixed point $\gamma_{i,f,1}$ with one more element, check if $X_{t}^{\text{out}}$ can predict the new error $\phi_{t+i}$ and possibly add new linear combinations of $X_{t}^{\text{out}}$. Once we find that $X_{t}^{\text{out}}$ does not have any linear predictive power for the prediction errors we claim that we have found a sufficient summary of the whole state vector $X_t$.

We now provide a more formal definition of Condensed PEA. Given a selection for core variables

**Step 1** Parameterize the expectation as

$$E_t(u_{c,t+i}) = \Phi^i(X_{t}^{\text{core}}, \gamma_{i,\text{core}}).$$

Since we consider for now linear $\Phi^i$ (with a constant term) we have $\gamma_{i,\text{core}} \in R^{n+1}$. Find $\gamma_{i,f,\text{core}}$ that satisfies the usual PEA fixed point i.e where the series generated by $\Phi^i(X_{t}^{\text{core}}, \gamma_{i,f,\text{core}})$
predicts $u_{c,t+i}$ better than with any other $\gamma^{t,\text{core}}$.

The next step orthogonalizes the information in $X^\text{out}_t$. This will be helpful to give good initial conditions for the next iteration and to arrive at a well conditioned fixed point problem in Step 4.

**Step 2** Using a simulation of $T$ periods, for a large $23$ $T$, run a regression of each element of $X^\text{out}_t$ on the core variables. That is, letting $X^\text{out}_{j,t}$ be the $j$-th element, we now run the regression

$$X^\text{out}_{j,t} = (1, X^\text{core}_t) \cdot \omega^1_j + v^1_{j,t}$$

$\omega^1_j \in R^{n+1}$ for $j = 1, 2, \ldots, 2N + S + 1 - n$ and calculate the residuals

$$(23) \quad X^{\text{res},1}_{j,t} = X^\text{out}_{j,t} - (1, X^\text{core}_t) \cdot \omega^1_j.$$ 

It is clear that $X^{\text{res},1}$ adds the same information to $X^\text{core}$ as $X^\text{out}$ does, but $X^{\text{res},1}$ has the advantage of being orthogonal to $X^\text{core}$.

**Step 3** Find the first linear combination $\alpha^1 \in R^{2N+S+1-n}$ through the following OLS regression:

$$(24) \quad \alpha^1 = \arg \min_{\alpha} \sum_{t=1}^{T} \left( u_{c,t+i} - (1, X^\text{core}_t) \cdot \gamma^{i,f,\text{core}} - X^{\text{res},1}_t \cdot \alpha \right)^2.$$ 

If introducing $X^{\text{res},1}_t \cdot \alpha$ does not reduce significantly the sum of squared residuals relative to the solution with only $X^\text{core}$ we claim the core solution is sufficiently accurate and stop. Otherwise there is evidence that more state variables should be added to the solution and we go to the next step.

**Step 4** Apply PEA adding $X^{\text{res},1}_t \cdot \alpha^1$ as a state variable, i.e. parameterizing the conditional expectation as

$$E_t(u_{c,t+i}) = \Phi^t(X^\text{core}_t, X^{\text{res},1}_t \cdot \alpha^1, \gamma^t)$$

where $\gamma^t \in R^{n+2}$. Find a fixed point $\gamma^{t,f}$ for this parameterized expectation. Because $\gamma^{i,f,\text{core}}$ is a fixed point and since $X^\text{core}_t$ and $X^{\text{res},1}_t$ are orthogonal and the linear combination $\alpha^1$ has high predictive power, in order to find the fixed point $\gamma^{t,f}$ it makes sense to start iterations with the initial conditions

$$\gamma^{t,f}_{(n+2)\times1} = \left( \begin{array}{c} \gamma^{i,f,\text{core}} \\ 1 \end{array} \right).$$

Go to Step 2 with $(X^\text{core}_t, X^{\text{res},1}_t \cdot \alpha^1)$ in the place of $X^\text{core}_t$, check if a new linear combination reduces squared residuals, etc.

A couple of remarks are in order. First, note that the Condensed PEA proposed in this section is designed to deal with a very large number of state variables. Our focus is on debt management and more broadly portfolio models but the method should be useful in many other applications with

---

23This definition assumes we are interested in the steady state distribution. This step could be modified in the usual way (i.e. running the model with many short samples) to take into account transitions. See, for example, Faraglia, Marcet, Oikonomou and Scott (2014) for a detailed description.
high-dimensional states including models with many sectors or heterogeneous agents. Second, note that in the presence of many state variables the literature has often solved dynamic economic models by adding state variables one by one in some “order” until the next variable does not materially influence the solution. For example, if many lags are needed the typical approach is to add the first lag, then the second lag, and so on. If at some step the solution changes very little it is claimed that the solution is sufficiently accurate. But it is easy to find reasons why this argument may fail. For instance, maybe the variables further down the list are more relevant, as is the case in our model since a simple inspection of (10) suggests that the \( N \)-th lags of both \( b^N \) and \( \lambda \) play a special role in the solution. Also, it can be that a linear combination of the remaining variables makes a difference but these variables do not make a difference one by one. The condensed PEA gives a chance to all state variables to make a difference in the solution in only one step and it will pick up the relevance of combinations of state variables (for example, capturing that under complete markets only total wealth matters).

4.3 Forward States PEA

As we discussed above a key step in PEA relies on solving for equilibrium variables using given functions \( \Psi^N(\cdot, \delta^N) \) and \( \Phi^N(\cdot, \gamma^N) \). In particular, off corners the system of four equations (6), (10) and (11) should deliver a solution for the four variables \( (c_t, b^S_t, b^N_t, \lambda_t) \). However, because of the multiplicity of assets the system (6), (10) and (11) is not well determined. Note that the two Euler equations imply

\[
\lambda_t = \frac{\Psi^S(X_t, \delta^S)}{\Phi^S(X_t, \gamma^S)} \quad \text{and} \quad \lambda_t = \frac{\Psi^N(X_t, \delta^N)}{\Phi^N(X_t, \gamma^N)}.
\]

Since the vector \( X_t \) contains only predetermined variables, (25) gives us two equations to solve for the variable \( \lambda_t \), so this multiplier is overdetermined while the values for bond holdings are indeterminate.\(^{24}\) Note that this is not a fundamental indeterminacy in the model, it is only an indeterminacy of the particular way PEA solves this problem. We overcome this through the following modification.

4.3.1 Solution through Forward States (FS)

Our proposal is to formulate conditional expectations as functions of current values of state variables. We accomplish this using the following two steps. First, instead of approximating (19) we approximate

\[
E_t(u_{c,t+i-1}) = \Phi^i(X_t, \gamma^i) \quad \text{and} \quad E_t(\lambda_t u_{c,t+i-1}) = \Psi^i(X_t, \delta^i) \quad i = S, N,
\]

where the only difference with the parameterization in (19) is that we have subtracted -1 from the subindices of the variables inside the conditional expectation. Second, we invoke the law of

\[^{24}\text{Marcet and Singleton (1999) and den Haan (1995) (MSDH) already identified this problem in related models. Applying their procedure to the current model is done as follows: replace } E_t(u_{c,t+1}) \text{ by } E_t(u_{c,t+1}/H(\zeta, b^1_t))/H(\zeta, b^1_t) \text{ where } H \text{ is some function with fixed parameters } \zeta, \text{ invertible in } b^1_t, \text{ and } H > 0. \text{ Then parameterize } E_t(u_{c,t+1}/H(\zeta, b^1_t)) = \Phi^S(X_t, \gamma^S) \text{. The multiplier can be recovered with the second equation in (25) and bond holdings from } H(\zeta, b^1_t) = \Phi^S(X_t, \gamma^S)/\Psi^S(X_t, \gamma^S) \text{. When we used this approach for the current model the algorithm diverges or circles indefinitely. We discuss in the last footnote of this section the reason why FS may work better.}\]
iterated expectations to write, for example, $E_t(u_{c,t+i}) = E_t(\Phi^i(X_{t+1}, \gamma^i))$ in order to approximate (11). Similarly, $E_t(\Psi^i(X_{t+1}, \delta^i))$ approximates $E_t(u_{c,t+i}\lambda_{t+1})$. Substituting these expressions in the system of first order conditions we get that (25) becomes

$$
\lambda_t = \frac{E_t(\Phi^i(X_{t+1}, \delta^i))}{E_t(\Phi^i(X_{t+1}, \gamma^i))} \quad \text{for} \quad i = S, N
$$

(26)

$$
\sum_{i \in \{S,N\}} b^i_t \beta^i E_t(\Phi^i(X_{t+1}, \gamma^i)) = \sum_{i \in \{S,N\}} b^i_{t-1} \beta^{i-1} \Phi^i(X_t, \gamma^i) + g_t u_{c,t} - (u_{c,t} - v_x, t)(g_t + c_t)
$$

(27)

Now current $b$'s do enter the right side of (26), therefore these equations plus (10) determine $(c_t, b^S_t, b^N_t, \lambda_t)$ given $\Psi^i(\cdot, \delta^i)$ and $\Phi^i(\cdot, \gamma^i)$ and the first order conditions with respect to $b^i_t$ hold.\(^{25}\)

In the online Appendix we describe further the details of applying this procedure in the model at hand.\(^{26}\)

5 Optimal Debt Management

Having outlined our solution method we now turn to examine numerically optimal DM under four different market scenarios: full buyback and no buyback, each combined with loose lending constraints (i.e. large $|M_s|$, $|M_i|$) and no-lending ($M_s = 0$). The no lending constraint $M_s = 0$ follows a number of DM papers (e.g Lustig et al. (2008) and Nosbusch (2008)) and is clearly consistent with the stylised facts of Section 2. A number of candidate possible explanations spring to mind such as the uninsurable risk involved in holding private assets and controversies over exactly which private assets the government should buy.\(^{27}\)

To calibrate all four models we follow Marcet and Scott (2009). We choose $\beta = 0.95$ and set utility $u(c_t) + v(x_t) = \log(c_t) + \eta \frac{(x_t)^{\gamma} - 1}{\gamma}$ and use a time endowment $T = 100$. We choose a value of $\gamma = 2$ and target a value for $\eta$ so that on average the household’s leisure is 30% of the time endowment; with taxes that balance the budget at the deterministic steady state, this gives $\eta = 12.857$. Finally, our

\[ E_t \Phi^i(X_{t+1}, \gamma^i) = \int \Phi^i(g', \lambda_t, \ldots, b^N_{t-N+1}, \gamma^i) f_{g' | g_t} dg' \]

analytically, we give the formula in the appendix. This expression shows how this term depends on $b^N_t, b^S_t$ in addition to predetermined variables.

\(^{25}\)In particular we compute

\(^{26}\)In a previous footnote we mentioned that the approach of MSDH did not work in practice for the current paper, while FS works in the very many versions of the model that we have tried. We do not have a theorem that FS works better than MSDH in general, but we can offer two reasons why it may behave better: first the function $H$ in MSDH is arbitrary, it has to be such that its realized value correlates significantly with marginal utility, it has to be well-approximated by PEA. It is difficult to know beforehand which function $H$ has such properties. FS avoids such an arbitrary choice. Second, our approximation to $E_t(u_{c,t+1})$ under FS depends at most on lags $t - N + 1$, it does not depend on lags dated $t - N$. Obviously, the true solution also has this property. We contend that in this way FS imposes more closely features of the true solution in the numerical approximation and this lends more stability to the algorithm.

\(^{27}\)Of course the Fed and Treasury have purchased private assets during the financial crisis. This seems not to do with debt management but more tackling financial market disruptions that we, as does the rest of the literature, abstract from. Introducing self-fulfilling debt crises, as in Conesa and Kehoe (2015), in our model would justify that in periods when a debt crisis may occur the government purchases its own bonds while in other periods bond issuance is governed by the mechanisms we describe in the paper.
parameterization of the stochastic process for spending shocks is:

\[ g_t = (1 - \rho g) \bar{g} + \rho g g_{t-1} + \epsilon_t. \]

We set \( \rho_g = 0.95, \sigma^2 = 1.44 \). We further truncate the value of spending so that it always lies within an interval of 15% to 35% of steady state GDP. \( \bar{g} \) is chosen so that the ratio of spending to output is 25% in the deterministic steady state.

We consider two maturities \( S = 1 \) and \( N = 10 \), in other words we focus on the case where the government issues a one and a ten year bond. As discussed previously, the bounds \( M_i \) and \( \bar{M}_i \) are in units of steady state market value of debt and we set the upper bounds \( \bar{M}_i \) equal to 100% of GDP. This implies that the government can issue debt equal to a maximum of 200% GDP if both bonds are at their upper bound.\(^{28}\)

In Table 4 we show the key moments to summarize Facts 1 to 4, namely, the serial correlation, standard deviation and the mean of the share of short bonds over total debt (denoted \( S_t \)) and the correlation of the market value of short debt with long debt normalized by output. We show results for each of the four cases considered and for comparison purposes we also show U.S data in the first line.

### 5.1 Buyback

Consider first the case of buyback with lending i.e. loose borrowing constraints. We summarize the output of this model in Figure 6 and the second row of Table 4. In Figure 6 we show a simulated part of the optimal portfolio plotting the market value of short and long bonds for a typical realization of the shocks in \( g_t \).\(^{29}\)

\[ \text{[ Figure 6 About Here ]} \]

\[ \text{[ Table 4 About Here ]} \]

The simulations show clearly that the predictions of the complete market models of Angeletos, Buera and Nicolini carry over to the case of incomplete markets. In periods the long bonds are away from the upper bound the government "issues long and saves short". On average, the market value of short bonds in the sample equals -22.9 and the value of long debt is 45.8 so that in an "average" sense the optimal policy here echoes the standard complete market result.

However in contrast to the case of complete markets the optimal portfolio is time varying.\(^{30}\) As is clear from Figure 6 and the large variance of the share of short bonds (e.g. Table 4) the bond shares

\(^{28}\)In our simulations the government never hits this upper bound for total debt. In our simulations the overall debt level is rarely as high as 120-130% of GDP. Under buyback and when lending is permitted, long term bonds may hit their upper bound constraint and short bonds their lower bound.

\(^{29}\)The moments displayed in Table 4 are constructed from simulating the model 1000 times over 60 years. For each sample we feed to the model the initial conditions for the debt to GDP ratio and the share of short maturity debt we find in the data (see online appendix for further details). The sample shown in Figure 6 starts from zero total debt to make it comparable to the benchmark of ABN. We show 350 periods to make the figure readable.

\(^{30}\)In Angeletos, Buera and Nicolini the value of the shares of short bonds are not exactly constant but their variance is near zero since the position is constant and the price has very high serial correlation.
change substantially over time. This is because, as is standard under incomplete markets, total debt varies through time as it performs the role of a buffer smoothing out shocks. Combined with the sensitivity of optimal positions to changes in initial wealth that carries over from complete markets this leads to the variation in optimal portfolio shares.

In summary: the general insights of standard Ramsey models prevail - on average $b^N_t > b^S_t$, debt positions are large and $b^N_t, b^S_t$ are negatively correlated. As expected the moments for this model in the second line of Table 4 are very different from the data reported in the first line.

Now consider no-lending, where $M_i = 0$. A typical simulation is displayed in Figure 7 based on the same sequence of spending shocks as Figure 6. The solid line represents one year debt and the dashed line ten year debt.

Figure 7 shows that it is still optimal to issue mostly long term debt under buyback and no lending. Short bonds are often close to their lower limit and indeed in some periods we have $b^1_t = 0$. The third row of Table 4 shows the key moments for the model. The average share of short bonds is at a low 12%, with $b^1_t = 0$ in 13.1% of the periods and is less than 10% more than half of the time. Recall that in the data the minimum share short bonds reached was 24%.

The intuition for this result is that since it is now impossible to build the "issue long, save short" portfolio that provides fiscal insurance, the government gets as close as possible to this portfolio by setting short bonds close to zero. For high levels of total debt we see the government issuing both short and long debt so that overall there is a weakly positive correlation between them. The cause of this weak positive correlation is the existence of the separate debt limits on each bond. As debt rises the government, for precautionary reasons, issues short term debt in order to minimise the risk of reaching the debt limit on long bonds in the future.

Our conclusion is that in this case fiscal insurance concerns still dominate, leading the government to prefer to issue mostly long bonds. The average share of short debt is relatively minor and concentrated in times when total government debt is high.

5.2 No Buyback

As we mentioned in the introduction our paper has both normative and positive aspects: we systematically compare our models with the data and to the extent that large differences arise one may conclude that actual policy should change. As the optimal buyback no-lending policy is at odds with the data (i.e the first and third lines in Table 4 are very different) several normative recommendations emerge: governments should issue a much larger proportion of long bonds to achieve fiscal insurance;

\[\text{See for example Aiyagari et al (2002). When markets are incomplete the government in the long run accumulates savings for precautionary reasons, so that it can fund future adverse shocks without having to raise taxes. Aiyagari et al (2002) show one example where it can be proved analytically that government savings go to a very large number, so that the government can implement the first best in the long run. It is a common feature of optimal dynamic contracts that if the planner can implement the first best in the long run a martingale convergence theorem leads the economy to this first best as time goes by. Albanesi and Armenter (2012) give a set of sufficient conditions for this to arise. In our model the government cannot implement the first best even in the long run, but the model still has a tendency to show a large amount of savings, the average market value of total debt to GDP ratio in the sample that we use to solve the model is roughly -50 percent.}\]

\[\text{This intuition was already mentioned in Nosbusch (2008) and Lustig et al. (2008).}\]
short bonds should be issued only when debt is already very large; the government should repurchase previously issued bonds.

But the buyback assumption was introduced in the literature for convenience not because it describes actual bond markets. Motivated by empirical observations of Facts 5-7 it is of interest to introduce no buyback in the model. If we should find that introducing no buyback helps match the first line of Table 4 and if we can justify that no buyback is closer to the data then the above normative recommendations would be mute.

We now describe the optimal portfolio under no buyback. Later we discuss why no buyback may arise. Notice that in this model at any point in time there are bonds of all maturities between 1 and 10 outstanding in the market, so that ten-year bonds issued nine years ago are now short bonds. We take this properly into account in the statistics we compute below.

The typical realizations with loose debt limits are displayed in Figure 8 and with no lending in Figure 9. Under lending the most striking difference with buyback is the strong positive correlation between issued short and long term debt. The usual incomplete market result that governments should accumulate assets still holds in this case. However, under no buyback the government funds a larger deficit by issuing both short and long run debt at the same time. As explained in Section 3.3.2, the reason is that under no buyback issuing only long term debt is less effective at providing fiscal insurance and adds volatility in taxes through $N$ cycles. The implication is that short bonds are a valuable asset in mitigating these cycles. The result is a much stronger co-movement of short and long bond issuance.

When we add to no buyback the No-Lending constraint the tendency for short and long debt to co-move is strengthened further but now short term debt plays a much more substantial role (with an average portfolio share of 48%). Under no lending the share of short term debt is also less volatile and more persistent than under buyback.

The fourth and fifth rows in Table 4 confirm that under no buyback the moments of the data are matched quite closely. Since we are engaged in comparing statistics in the model and the data it is of interest to make this comparison systematic by using standard inference. Table 5 shows t-statistics for the null hypothesis that the various statistics reported from our simulations are equal to those for US data in our sample period. The t-statistics are simply the difference of the model and data moment divided by the standard deviation of the moment. The standard deviations of the moments are shown in the first row of Table 5. The t-statistics are shown in rows 2 to 7 of the Table.

Our aim in calculating t-statistics is not to see if our model can explain the data. Our modelling of both the economy and the bond market are extremely simple and easy to falsify. Instead the purpose of Table 5 is to provide some form of statistical gauge to our previous observation that buyback is very far from the data while no buyback is much closer. Table 5 suggests that once allowance is made for

\[ \text{We estimate the standard deviation of each moment considered in Table 4 from the data. For this purpose we use the fact that, for a mean zero stationary and ergodic process } x_t, \text{ we have } \frac{\sum_{j=-\infty}^{\infty} E(x_t x_{t-j})}{\sqrt{T}} \rightarrow N(0, S_w) \text{ in distribution. We estimate } S_w = \sum_{j=-\infty}^{\infty} E(x_t x_{t-j}) \text{ using the Newey-West statistic. The asymptotic standard deviation of functions of moments (such as correlations) are found with the delta method.} \]
no buyback and no lending the discrepancy between Ramsey recommendations and observed practice is very small, in fact the government should issue more short term debt. Therefore, if no buyback is an optimal strategy for debt managers then the policy recommendations arising from buyback (namely, as we mentioned earlier, to issue much more long bonds, to repurchase often previously issued bonds) are, indeed, mute.

5.2.1 In-sample model fit

An approach that is often used to visualize the goodness of fit of a dynamic model is to check if a variable determined by the model compares well with the data. For this purpose one plugs in the observed values for the state variables into the model’s law of motion and then compares the resulting value of the endogenous variable with the data. In our case we would like to compare the short share $S_t$ determined by the model with the data. For this purpose we should solve out the short bond issuance using the law of motion

$$S_t = S(X_t)$$

where $S(\cdot)$ is the time-invariant policy function determining the share implied by the model solution.

One difficulty in our case is that $X_t$ contains non observables, namely, the lagged $\lambda$’s. Therefore we replace $S(\cdot)$ by an approximate law of motion that is a function only of observables. These observables are selected with an eye to using variables that are stationary, so that their data counterpart is a reasonable value to be plugged into the model’s law of motion. For this purpose we use model simulations to regress $S_t$ on its first and second lags, two lags of the market value of total debt over GDP, the current value and first lag of the spending over GDP series. We also include higher order terms of these variables to produce a ‘good fit’ to the model’s policy function. This was done separately for the buyback and no buyback models, since the DM policies differ across these two models. The regression is performed with all the periods and realizations used in constructing the moments reported in Table 4.\(^{34}\) Second, we feed the ‘data state variables’ to each model’s approximate policy function to obtain the graphs shown in Figure 10. The data line for $S_t$ is the same as in Figure 1, but because of the inclusion of two lags in the regressions the sample starts in 1957 in Figure 10.

As can be seen the fit of the model under no buyback tracks the short share quite well both in terms of its level as well as its movements in response to government spending shocks.\(^{35}\) Given the extreme simplicity of the model economy the fit of the no buyback model is surprisingly good. The fit of the model under buyback is much poorer. The correlation between the series produced by the buyback model and the data series equals 0.4. The analogous correlation between the no buyback model and the data equals 0.88.

\(^{34}\)To capture better the part of the state space that is active for the CRSP sample, we only use in the regression observations with market value to GDP ratio between 20 percent and 70 percent.

\(^{35}\)The fit is not perfect - there is an obvious ‘lag’ in the data relative to the model. We attribute this to the fact that spending does not respond contemporaneously to higher output in the US (see for example the considerable literature on identifying spending shocks in SVARs).
6 Optimal Bond Repurchases and Transaction Costs

The previous section considered optimal DM when the government was constrained to either repurchase as much as possible (buyback), or not to repurchase at all (no-buyback). Both are extreme restrictions on government behaviour and, as we have shown, optimal DM differs substantially between them. The question then is which set of assumptions is the most plausible for an analysis of optimal DM. Viewed from the perspective of the buyback model the fact that governments actually do not engage in $r/r$’s may appear self-limiting. Indeed, it turns out that for the calibrated model of the previous section the consumer utility is higher under buyback than under no-buyback, lending support to the view governments should engage in large buybacks, issue far fewer short bonds and focus on fiscal insurance through long bond issuance.

However, as discussed at the end of Section 2, there are many features of government bond markets that makes $r/r$ costly. The standard debt management literature assumes governments can issue and repurchase debt at the same market price but the existence of transaction costs means that in practice this is not feasible. Even if transaction costs are relatively small, since the increase in utility from repurchases is also small (see previous footnote), it may be that buyback is too costly in terms of utility and, in that case, the no buyback model offers the best basis for policy recommendations. Of course depending on the level of transaction costs the optimal behaviour may be an intermediate position with the government sometimes buying back some of the debt. It is to these issues that we now turn. In this section we perform three exercises. The first is to use various studies of the US government debt market to calibrate a reduced form function capturing the various transaction costs we discussed at the end of Section 2. The second is to use these estimates of transaction costs and perform a shadow cost analysis of the two extreme cases of full buyback and no-buyback and see whether the fiscal insurance benefits of buyback outweigh the transaction costs involved in $r/r$. Finally we move away from the two extreme cases and allow for the government to choose each period how much of debt to repurchase or leave outstanding.

6.1 Transaction Costs Function and Calibration

At the end of Section 2 we outlined a number of factors that will lead $r/r$ operations to be costly. These factors will be reflected in bid-ask spreads and auction effects, whereby the more of a specific bond the government wishes to issue or purchase the more the price shifts against the government. It is beyond the scope of this paper to introduce explicit microfoundations for these effects in the model. Instead our interest is in establishing whether concern over these costs should substantially influence optimal policy. To do so we extend our model to allow for a reduced form transactions cost function. We denote issuance costs by $T^i(b^i_t)$ and the cost of repurchasing a bond that was issued at $t−1$ by $T^R(R_t)$. Assuming these costs are ad valorem e.g the bid-ask spreads are a percentage of

\[ E_0 \sum_{t=0}^{\infty} \beta^t \left[ u \left( (1 + \omega) c_t^{NBB} \right) + v \left( x_t^{NBB} \right) \right] = E_0 \sum_{t=0}^{\infty} \beta^t \left[ u \left( c_t^{BB} \right) + v \left( x_t^{BB} \right) \right] \]

turns out to be $\omega = 0.00416$.
price, gives total transaction costs at \( t \) as

\[
\mathcal{T}_{ot_t} = \sum_{i \in \{S,N\}} p_i^t b_i^t T^i(b_i^t) + p_{t-1} R_t \mathcal{T}^R(R_t).
\]

Based on our discussion in Section 2 we specify the following functions for transaction costs:

\[
T^i(b_i) = \alpha_0^i + \alpha_1^i b_i^t \quad \text{for } i = S, N \quad \text{and} \quad \mathcal{T}^R(R_t) = \alpha_0^R + \alpha_1^R R_t. \quad 37
\]

The bid-ask spread margin is independent of the scale of purchases and so will be reflected in the intercept terms \( \alpha_0 \) whilst the auction effects pin down the slope effects. The fact that the costs are linear in issuance/repurchases means that the term \( T^i(b_i) \) which appears in the total costs is linear quadratic. Assuming a linear quadratic function is a standard specification in the literature on transaction costs and captures the notion, common in our conversations with debt managers, that price pressures increase the larger the transaction.\( ^{38} \)

As usual in the transaction cost literature we assume that these costs are in terms of hours worked so that feasibility now requires\( ^{39} \)

\[
c_t + g_t + \mathcal{T}_{ot_t} = T - x_t
\]

Amihud and Mendelson (1991) calculate that bid asks spreads and brokerage fees amount to 0.0381 percent of the price for bonds and 0.0099 percent for Treasury bills. This gives us estimates of \( \alpha_0^S = 0.000099 \) and \( \alpha_0^N = \alpha_0^R = 0.000381 \) (given the face value of a bond is 1 in our model and bid-ask spreads are symmetrical on buyers and sellers). To calibrate the slope terms \( \alpha_1 \) we use the estimates of Lou, Yan and Zhang (2013) such that yields are affected by 3 basis points on average due to auction effects on issuance/repurchases.\( ^{40} \) Their estimate is common across all maturities.

The fact the impact is calibrated in terms of yearly yields and the costs \( \mathcal{T} \) above are paid only at issuance means that we need to translate the 3 bps estimate into issuance costs \( \mathcal{T} \). The effect of issuance/purchase is larger on longer maturity bonds (the impact on bond prices is proportional to the impact on yield multiplied by the duration of the bond). This calibration means that the steady state yields of bonds, after taking into account auction effects, is \( 1/\beta + 0.0003 - 1 \), i.e auction effects increase the cost of issuing bonds across all maturities by 3 basis points on average. The annualized yield plus auction costs implied by the above transaction cost function is \((\frac{1}{p^t(1-\alpha_1^ib_i^t)})^{1/i}\) - 1, where \( p^t \) and \( b_i^t \) are averages. Equating these expressions and rearranging gives an estimate of \( \alpha_1^ib_i^t = 1 - (1 + 0.0003\beta)^{-1} \). For \( i = 10, 9, 1 \) this gives average auction costs \( (\alpha_1^ib_i^t) \) of 0.0028, 0.0026 and 0.000284 respectively. To calibrate the slope terms we need to divide these average auction effects through by the average issuance in our simulations. Since the no-buyback model of Section 5.2 matches the data reasonably well we use average issuances in that model to arrive at the following estimates for the slope coefficients: \( \alpha_1^S = 0.000021, \alpha_1^N = 0.001 \) and \( \alpha_1^R = 0.000926. \quad 41 \)

\( ^{37} \) If \( b_i^t \) would be allowed to be negative this function would present a kink at \( b_i^t = 0 \), giving rise to non-smooth solutions that would be hard to compute. Since we impose \( b_i^t \geq 0 \) in this subsection there is no kink, solutions are smooth, and all costs considered are indeed positive if we take \( T^i \geq 0 \).

\( ^{38} \) Support for this is also to be found in Breedon and Turner (2016) Table A2.1.

\( ^{39} \) Not all transaction costs require resources to be deducted from the resource constraint. Our findings in this section remain if no such deduction is made.

\( ^{40} \) Lou et al actually give a range of 2-3 bps. These estimates are broadly similar to Breedon and Turner (2016) Table 2 and substantially less than many estimates of the impact of QE on yields.

\( ^{41} \) We calibrate repurchases assuming that the auction effects are symmetric in buying and selling. We therefore use estimates for \( i = 9 \) to calibrate the repurchase auction effects one year after a ten year bond has been issued. Point estimates in Breedon and Turner (2016) suggest that repurchase auction effects may actually be larger than issuance effects. However given the importance of no buyback in our model we make the more conservative assumption that
6.2 A Shadow Cost Calculation

As mentioned before, for the calibration used here, the utility of the optimal allocation is indeed higher under buyback. But since the buyback strategy involves many more purchases and sales it is not clear if the benefit of fiscal insurance under buyback will dominate if transaction costs exist. We now make an approximate “shadow” calculation of the loss in utility due to transaction costs.\footnote{We thank Dimitri Vayanos for suggesting this calculation.}

This calculation has the virtue of being independent of the precise way that we model repurchases. Since the transaction costs are small and they indicate a larger government expenditure, the lagrange multipliers of each solution translate transaction costs into an approximate utility loss.

Let superindex $BB$ denote the solution with buyback of Section 5.1 and $NBB$ the solution under no buyback in Section 5.2.

A transaction cost plays the same role as higher $g_t$ both in the government budget constraint and in the utility term $\beta^t v(T - c_t - g_t)$. The total marginal utility loss of higher $g_t$ in $t$ is

$$\tilde{\lambda}_t^{BB} + \beta^t v'(T - c_t^{BB} - g_t^{BB} - T)$$

where $\tilde{\lambda}_t^{BB}$ is the “plain” lagrange multiplier of the government budget constraint. Since we have normalized the plain lagrange multipliers in the usual way,\footnote{The “plain” lagrange multiplier would be the one that would come out of a standard Lagrangean, namely}

we have that the lagrange multipliers of Section 5.1 are given by

$$\tilde{\lambda}_t^{BB} = \beta^t u^{BB}_{c,t}.$$ 

Therefore the total shadow transaction costs of buyback, in term of utility, is

$$Total^{BB} = E_0 \sum_{t=0}^{\infty} \beta^t (\tilde{\lambda}_t^{BB} u^{BB}_{c,t} + v^{BB}_{x,t}) T^{BB}.$$ 

The total costs of the no buyback strategy $Total^{NBB}$ are found similarly, using the lagrange multipliers of the no-buyback solution and $R_t = 0$.

Letting $U^i = E_0 \sum_{t=0}^{\infty} \beta^t [u(c_t^i) + v(x_t^i)]$ for $i = BB, NBB$ denote the total utility net of transaction costs for each policy is $U^i - Total^i$. Then, we want to find the scaling factor $\chi$ such that if the transaction cost functions would be $\chi (T^R, T^S, T^N)$ the government is indifferent between the two strategies. This factor is given by

$$\chi = \frac{U^{BB} - U^{NBB}}{Total^{BB} - Total^{NBB}}.$$ 

If it turns out that $\chi < 1$ it means that, given the approximate utility calculation, the transaction the two effects are symmetric and our results are clearly robust to attributing higher effects to repurchases.\footnote{The "plain" lagrange multiplier would be the one that would come out of a standard Lagrangean, namely}
costs of buyback outweigh their benefit.\textsuperscript{44}

We find $\chi = 0.07$. In other words, once we take into account transaction costs the buyback strategy is more costly than no buyback. This would be so even if transaction costs would only be one tenth of their calibrated values.

### 6.3 A Model of Optimal Bond Repurchases

The previous calculation suggested that if one considers the two extremes, buyback or no-buyback, the former is not a good strategy once transaction costs are taken into account. We now examine the same issue by studying a model where repurchases of any amount are allowed and the optimal level of repurchases is decided each period. The budget constraint of the government is as follows

\begin{equation}
\sum_{i \in \{S,N\}} p_i^t b^t_i (1 - T^i(b^t_i)) = b^S_{t-S} + b^N_{t-N} - R_{t-N+1} + p^1_{t-N} R_t (1 + T^R(R_t)) + g_t - \tau_t (T - x_t) \tag{29}
\end{equation}

\begin{equation}
0 \leq b^t_i \leq \frac{M_i}{\sum_{j=1}^{\beta_j}}, \quad 0 \leq R_t \leq b^N_{t-N-1} \tag{30}
\end{equation}

There are three differences relative to the models of Sections 3.2 and 3.3: i) we introduce transaction costs $T^i$ ii) repurchases $R_t$ appear as a cost in period $t$ and iii) repurchases $R_{t-N+1}$ appear as an income at $t - N$, the amount of long bonds that mature at $t$ is now $(b^N_{t-N} - R_{t-N+1})$. The buyback model of section 3.2 imposes $R_t = b^N_{t-N-1}$, the no-buyback model of Section 3.3 imposes $R_t = 0$ whilst the current model allows any value in between.

We assume only the government pays all transaction costs, hence the consumer/investor budget constraint is unchanged relative to previous sections. This simplifies the model. Since we look at optimal policy the government will take into account the presence of all transaction costs anyway, hence results should be very similar even if consumers pay part of the transaction costs. Another assumption in the above model is that we only consider repurchases of long bonds issued in the previous period. This keeps as close as possible to the standard optimal DM literature and considerably simplifies the analysis.\textsuperscript{45}

The Lagrangean is straightforward to write and is shown in the appendix. Endogenous repurchases complicate the simulations in various dimensions: obviously now we have an additional decision variable $R_t$. Furthermore, since $R_{t-N+1}$ enters the budget constraint at $t$ it might seem that we now have to add $N - 1$ lags of $R$ to the state variables $X_t$ so we end up with $S + 3N$ state variables. However, after some manipulations one can show that a sufficient set of state variables is

$$X_t = \left[ g_t, (B_{t-i}, B\lambda_{t-i})^S_{i=1}, (B^\text{net}_{t-i}, B\lambda^\text{net}_{t-i})^{N-S+1}_{i=1}, \lambda_{t-N}, b^N_{t-N} \right]$$

\textsuperscript{44}Details on the approximations can be found in the online appendix.

\textsuperscript{45}Since we will find that buyback of any amount is very rare there will be bonds outstanding of many maturities in this model. Hence we could entertain a model where bonds of any maturity could be repurchased. This would complicate the analysis as there would be $N - 2$ more decision variables. By considering only immediate repurchases we are maximizing the possible fiscal insurance benefits of long bonds, since the price of one-year-old long bonds is the one that provides the largest amount of fiscal insurance as it is the longest bond outstanding.
where

\[ B_{t}^{\text{net}} \equiv b_{t-1}^{N} - R_{t} \]
\[ B_{t} \equiv b_{t}^{S} + B_{t-N+1+S}^{\text{net}} \]
\[ B \lambda_{t}^{\text{net}} \equiv \lambda_{t-1}(1 - T^{N})b_{t-1}^{N} + \lambda_{t}(1 + T^{R})R_{t} \]
\[ B \lambda_{t} \equiv \lambda_{t}(1 - T^{S})b_{t}^{S} + B \lambda_{t-N+1+S}^{\text{net}} \]

see appendix. Therefore we have "only" 1 + 2N state variables.

6.3.1 Simulation Results

Solving our model with transaction costs calibrated as above but allowing the government to choose how much debt to repurchase each period effectively generates the no buy back situation. The government chooses to repurchase only very rarely and even then only during periods of strong debt reduction due to large government surpluses - similar to the US in the 1920s and 2000-1. In Figure 11 we plot the ratio of the market value of debt to GDP, and the absolute level of repurchases using the same sample for \( g \) as in Figures 6 to 9. Notice that repurchases are a tiny fraction of GDP and that indeed the government repurchases debt only in periods where debt falls sharply. The moments are as reported in the final row of Table 4 and they match closely the analogous objects under no buyback and no lending\(^{46}\) and those of the US data very well. The t-statistics reported in Table 5 confirm that the model is close to the data.

This confirms the main point of the paper, namely, that once we take into account small transaction costs resembling those found in the data a portfolio with a substantial share of short bonds and no repurchases achieves higher utility than the portfolios under buyback that resemble the recommendations arising from effectively complete market models. Therefore the basic Facts 1-7 described in Section 2 can be matched by a model where debt management is decided optimally. More critically allowing for transaction costs means no buyback is a preferred strategy and the assumption that optimal DM requires a reliance on long bonds is no longer accurate with short bonds playing a significant smoothing role.

7 Robustness and Accuracy

In this section we explore the robustness of our main results to the introduction of various relevant features. First of all, as US government bonds tend to pay a fixed semi-annual coupon we introduce coupons in the model to see if this alters our findings. Second, we introduce a third bond in the analysis enabling us to talk about short, medium and long issuance. Third, given that callable bonds have been used in the sample period we study a model consistent with Fact 7 where bonds are recalled before, but close to, maturity. Finally we explore the accuracy of our solutions.

\(^{46}\)For brevity we do not show the portfolio of long and short bonds in a separate figure. However, as the results in Table 4 show the behavior of the portfolio in the optimal repurchase model is very close to the no buyback model, e.g. Figure 9
7.1 Coupons

A further stylised fact of US debt management is that long term bonds in the US pay constant semi-annual coupons. The existence of coupons means that a bond’s duration (the measure of how long it takes to recoup the price paid for a bond in terms of its cash flow) is distinct from its maturity, a distinction that doesn’t exist for the zero coupon bonds we have considered so far. Whilst under complete markets the fact that bonds pay coupons is unimportant, under incomplete markets the timing of cash flows matter and so the effects of coupons are non-trivial.

Coupons can also be thought of as a means of lessening or even overcoming the problem of no buyback that has been our focus. If the government attaches to each bond a sequence of fixed coupon payments this makes long bonds closer to short bonds (reduces their duration) and therefore the consequences of no buyback are less severe. In this section we examine to what extent coupon payments overcome the $N$-period cycles of Section 3.3.2 and attenuation of fiscal insurance. In other words, are coupons a way of using security design to make long bonds more attractive?

Consider a case where long bonds issued in $t$ pay a (possibly time dependent) coupon denoted by $\kappa_t$. A bond then pays a constant amount $\kappa_t$ from periods $t + 1$ to $t + N$ and in addition it pays the principal (normalized to unity) at $t + N$.

It is easy to show that the competitive equilibrium price of this bond is

$$q^N_t = \kappa_t \sum_{j=1}^{N} \beta^j E_t \left( \frac{u_{c,t+j}}{u_{c,t}} \right) + \beta^N E_t \left( \frac{u_{c,t+N}}{u_{c,t}} \right),$$

i.e. the price is now the sum of prices of zero coupon bonds of maturity $j < N$ weighted by the coupon payments plus the value of the bond repayment. Obviously, in the steady state (with constant coupons and consumption) the bond price becomes equal to $\kappa \sum_{j=1}^{N} \beta^j + \beta^N$.

According to the CRSP data, bonds issued by the US government trade close to par when they are issued, namely $q^N_t \approx 1$. To choose coupons that are consistent with this observation we set $\kappa_t = \kappa = \frac{1-\beta}{\beta}$ for all $t$. It turns out that in this case bonds trade close to par, our simulations yield $q^N_t$ close to 1 in all periods.

7.1.1 The Ramsey Program with Coupons

We now find the optimal policy assuming that long bonds pay a yearly coupon $\kappa$ and assuming no buyback. Debt limits are:

$$b^N_t \in \left[ \frac{\bar{M}_N}{\sum_{j=1}^{N} \beta^j + \kappa \sum_{j=1}^{N} \sum_{i=1}^{j-1} \beta^i}, \frac{\bar{M}_N}{\sum_{j=1}^{N} \beta^j + \kappa \sum_{j=1}^{N} \sum_{k=1}^{j} \beta^k} \right] \equiv [\tilde{M}_N, \bar{M}_N]$$

and $[\tilde{M}_1, \bar{M}_1] \equiv [\frac{M_1}{\beta}, \frac{M_1}{\beta}]$ for short bonds.\(^{47}\) The planning problem and the FOC with respect to consumption are given in the online appendix. Off corners the first order conditions for $b^t_t$ and $b^N_t$

\(^{47}\)In the US coupon payments are usually six monthly. However, since our model’s horizon is one year we model one year debt as zero coupon.
are:

\[\lambda_t E_t(\kappa \sum_{j=1}^{N} \beta^j u_{c,t+j} + \beta^N u_{c,t+N}) = E_t(\kappa \sum_{j=1}^{N} \beta^j \lambda_{t+j} + \beta^N u_{c,t+N} \lambda_{t+N}).\]

Equation (32) is the analogue of equation (15) (when the debt constraints are loose). It reveals that the multiplier follows a complicated pattern which equates it with the sum of all expected future terms \(u_{c,t+j} \lambda_{t+j}\) for \(j = 1, 2, \ldots, N\) weighted by the payments that the bond promises. Intuitively a zero-coupon \(N\) maturity bond produced a \(N\) cycle in \(\lambda_t\), if we think of coupons as themselves zero coupon bonds then we are adding additional smaller cycles in \(\lambda_t\) at \(1, 2, \ldots, N - 1\).

In Faraglia et al (2016) we study the properties of the single bond coupon model analyzing the effects of fiscal shocks. We show that the optimal policy is characterized by an important \(N\) cycle component and the main features of optimal fiscal policy under no buyback persist, since basically under coupons the bulk of interest rate payments is still concentrated every \(N\) periods. For brevity we refer the reader to Faraglia et al. (2016) for details.

In the fifth row of Table 4 we show the simulation results of the no buyback, no lending, coupon-paying model. The results show relatively minor changes from the zero coupon case - a small increase in the portfolio share of short bonds, a less volatile, less persistent share and a slightly higher correlation between short and long bond issuance. Introducing coupons to long bonds does not alter our conclusions.

### 7.2 Three Bonds

A relevant robustness exercise is to consider a government that issues a third bond. In particular, we assume that in addition to 1- and 10- year bonds now the government can issue a 5-year bond. This also serves as a test for the ability of the algorithm to deal with larger models, as the state vector now has five more variables and an additional decision variable.

We keep the zero-coupon assumption for comparability and we only study the no buyback case. We do not write the model and optimality conditions as they are similar to those in Section 3.3.

As can be seen from the corresponding row in Table 4, now we find that the average share of short debt is lower than NBB with two bonds and lower than the data. There is a similar phenomenon with the serial correlation of \(S_t\). Other statistics remain the same.

The intuition for these results is clear: the five-year bond attenuates the \(N\)–period cycles that we have discussed. By issuing 5-year bonds the height of the spikes of taxes described in Section 3.3.2 is half of the spike with a 10-year bond, hence the government is less reluctant to issuing long bonds and reduces issuance of 1-year bonds. This example demonstrates one feature of the model at hand: small changes in the model may have considerable impact on \(E(S_t)\) hence this value is not closely determined by the theory. But there are many features of optimal DM that are well determined and robust in the models we have considered, namely: \(S_t\) is never close to zero, it is stable and short bonds are highly correlated with long bonds. The main focus of our paper is to establish a role for short term bonds in optimal debt management which is clearly preserved in this case.
7.3 Callable Bonds

In our section on the stylised facts of US government debt we described how in the first half of our sample period considerable use was made of callable bonds. Just as coupons can be thought of as shortening the life of a bond so too can a callable bond. By preannouncing a future date at which the government can buy back a bond at par the government lessens the duration. If, as is the case in practice, governments tend to repurchase at the first call date, then cash flow is affected and bonds are not bought back at maturity. This raises another way in which security design may lessen the problems with long bonds and reduce the need for short bonds. We investigate this (results in an online appendix) by considering the case where the government issues bonds of maturity N but repurchases them after m years where m < N e.g we assume that governments always repurchases at the call date. We find that our main result still holds, if callable bonds are redeemed after m years they offer less fiscal insurance and introduce greater tax volatility at lower frequencies opening up a positive role for issuing short term debt to support optimal tax smoothing. If governments cannot repurchase debt each and every period then long bonds provide additional volatility and short bonds can help tax smoothing.

7.4 Accuracy

We have performed thorough accuracy tests of all the models for which we report numerical simulations. To check the accuracy of the solution we run Euler Equation Error (EEE) tests (see for example Aruoba et al (2006) for an exhaustive description of the methodology). Essentially this methodology checks that first order conditions hold with an acceptable degree of precision at many points in the state vector. For example, in the model with buyback one needs to check that Euler equations (11) hold approximately. To interpret the size of the error, the approximation is expressed in terms of the change in consumption units that would be required for the Euler equation to hold exactly.

The tests require to numerically calculate each of the conditional expectations in the Euler equations. Ours is not a routine application of this accuracy test because we have expectations involving up to N leads, so that exact integration would be too costly. For this reason we use Monte-Carlo integration to compute these expectations, using many simulations starting at each point considered of the state space. Details on how we perform the test are in the online appendix.

Since we have two Euler equations (or three in the case of the optimal repurchases model) we check separately each of them by calculating the value of the multiplier and for consumption in period t implied by the expectations Ξ_t generated by our approximation, given the portfolio b_{1T}, b_{NT}. Table 6 summarises the test for the main models presented in the paper. 48 As in Aruoba et al. (2006) we report the absolute errors using base 10 logarithms to make our findings comparable with the rest of the literature. A value of -3 means a 1$ mistake per 1000$, a value of -4 a mistake of $1 per $10000 and so on.

Table 6 shows that the average of the errors are between -3 and -4 and that the maximum errors are not large. Moreover, we found that is quite unlikely that the region of the state space where the maximum error occurs is visited in simulations. We have calculated also the percentage of positive

48An additional table in the online appendix provides all the details for each model and each Euler equation in this work.
and negative errors. A good approximation should deliver evenly distributed errors between the two signs. In the online appendix we shows that the distribution is fairly even in all the models presented with 41% to 58% of the errors being of positive sign. These results are well within the range accepted by other authors (e.g. Aruoba et al (2006)) suggesting that the model solutions are accurate.

8 Conclusion

We have studied optimal debt management under incomplete markets. The literature has to date isolated a powerful influence of fiscal insurance whereby governments can exploit the negative covariance between long bond prices and fiscal shocks in order to stabilise debt and minimise tax volatility. The implications of that channel for debt management under incomplete markets leads to a policy relying heavily on issuing long term debt. The recommendations from this canonical Ramsey approach are in stark contrast to the actual behaviour of observed US debt management. One might infer from this literature that governments should issue more long bonds, pursue very active portfolio management and actively repurchase previously issued long bonds.

Our approach has been to consider whether there are other market frictions that introduce additional considerations into debt management. If the implications of fiscal insurance are robust across a range of market frictions then the existing findings in the literature increase in their relevance. If however the implications from the standard Ramsey model are non-robust then it becomes important to isolate the exact market frictions that need to be considered in order to better understand the implications for actual debt management.

Once we introduce incomplete markets, non-negativity constraints on bonds issued and small transaction costs, the picture changes considerably. Optimal policy now involves repurchasing very rarely and a sizable and stable proportion of short bonds is key in achieving tax smoothing. Short bonds play a more important role in supporting optimal debt management than in previous papers, as short bonds provide a flexibility and optionality that helps reduce tax volatility in the face of fiscal shocks. Portfolio shares should be more stable and persistent and governments should issue positive quantities of both short and long debt in response to a shock to deficit. All of these conclusions are diametrically opposite to the standard recommendations from the canonical Ramsey model mentioned earlier, making observed US debt management look much closer to the recommendations of optimal debt management.

These main features of optimal debt management under no buyback are very robust, they stand the introduction of coupons, more bonds, and a callable bond. Clearly the recommendations around optimal debt management for governments can benefit from a deeper understanding of the reasons for market incompleteness and their practical relevance. It is of course entirely feasible that other microfounded features of market incompleteness e.g the clientele effects, asymmetric information, liquidity provision, reduced rollover risk, etc. restore a preference for long bonds. Critically however they would do so for reasons other than fiscal insurance and merely reiterate the importance under incomplete markets of specifying the reasons for market incompleteness.
References


A Database and Construction of Figures

Data on US Treasuries was obtained from the CRSP US Treasury Database and is comprised of all types of marketable treasury securities (bills, notes, bonds and inflation protected securities (TIPS)). Observations are available at monthly frequency. As discussed in text to compute key moments such as the share of short maturity debt or the ratio of issuances over total debt, we focus on data from the years 1955-2015. We extended this sample to include observations from the 1920s and 1930s and 1940s when we report moments on the frequency of repurchases or the timing of buybacks of callable bonds (e.g. Tables 2 and 3).

As discussed, not all bonds are recorded by the CRSP in the 1920s-1940s. The missing amounts outstanding are as high as 60 percent in the mid 30s and 40 percent in the early 40s. This constrains our empirical analysis to include observations from the 1950s onwards when all bond records on marketable US government debt are included in our dataset. We report the properties of share of short maturity debt over total debt in the CRSP from 1955 onwards, so that our estimates exclude the build up of public debt during the Korean war. Our results however would not be affected if we included the early 1950s in our sample.

Date variables of particular interest for this study include the quote date, the date of the first coupon and the maturity date. Amounts outstanding of the bonds are usually available in the CRSP although missing for certain observations. Gaps in amounts outstanding were filled with preceding observations for the same bond when possible or future observations if no preceding data points existed. Coupons are typically paid every six months from the date the bond is issued and until maturity.

To construct the share of short term debt (e.g. in Figure 1) we stripped the coupons. The strips, were given distinct maturity dates, face values and market values. For a ten year bond paying coupons every six months the first coupon is counted as six month maturity debt (at the issuance date), the second coupon as one year debt and so on. Market values for the strips were imputed using the yield-maturity data.

Our sample includes both nominal and inflation protected debt (TIPS). TIPS typically represent long maturity claims (five, ten or thirty year debt) and therefore contribute towards reducing the average value of the share of short term debt. The first inflation protected security in the US government bond market was issued in 1997 (before then there was no indexed government debt).

Finally, to construct the time series of the issuances (Figure 2) we adopt the following approach. Because our definition of short term debt in the data includes maturities which are less than one year (one month to six months in the CRSP data) we define the total issuance in short maturity debt as:

\[ I_{S,t} = I_{m,1/12} + I_{m,1/4} + I_{m,1/2} + I_{m,1/2} \]

where \( I_{xm,t} \) denote claims of maturity \( x \) months in the data and the quantities \( I_{xm,t} \) are sums of issuances over year \( t \). Note that differently from the case of the stocks displayed in Figure 1 the issuances share is formed by a flow variable in the numerator and a stock variable in the denominator. This approach enables us therefore to not bias the share upwards.

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49 Our approach to strip the coupons and assign an appropriate maturity to each strip is compatible with the notion that the benefits from fiscal insurance are proportional to the amount of long term government debt outstanding. If a long term bond is issued paying coupons every six months then this bond provides less hedging to the governments budget than a zero coupon long term bond, as is claimed in text. When we construct the model counterparts for the series plotted in Figure 1 we follow essentially the same approach.

50 This definition includes the first coupons of long term notes and bonds.
by counting very short term issuances (monthly debt) many times over the year.
### Table 1: Remaining Term at Time of Buyback

<table>
<thead>
<tr>
<th>Year</th>
<th>Amount Issued (in millions)</th>
<th>Share Called</th>
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</thead>
<tbody>
<tr>
<td>1931</td>
<td>755</td>
<td>100%</td>
</tr>
<tr>
<td>1934</td>
<td>491</td>
<td>100%</td>
</tr>
<tr>
<td>1935</td>
<td>2611</td>
<td>100%</td>
</tr>
<tr>
<td>1936</td>
<td>5616</td>
<td>100%</td>
</tr>
<tr>
<td>1938</td>
<td>3588</td>
<td>100%</td>
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<td>1939</td>
<td>1689</td>
<td>100%</td>
</tr>
<tr>
<td>1940</td>
<td>1404</td>
<td>100%</td>
</tr>
<tr>
<td>1941</td>
<td>9326</td>
<td>55.35%</td>
</tr>
<tr>
<td>1942</td>
<td>14061</td>
<td>38.06%</td>
</tr>
<tr>
<td>1943</td>
<td>16763</td>
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<td>1974</td>
<td>587</td>
<td>100%</td>
</tr>
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<td>1975</td>
<td>3616</td>
<td>100%</td>
</tr>
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<td>1574</td>
<td>100%</td>
</tr>
<tr>
<td>1977</td>
<td>2638</td>
<td>100%</td>
</tr>
<tr>
<td>1978</td>
<td>4516</td>
<td>100%</td>
</tr>
<tr>
<td>1979</td>
<td>4523</td>
<td>100%</td>
</tr>
<tr>
<td>1980</td>
<td>7794</td>
<td>100%</td>
</tr>
<tr>
<td>1981</td>
<td>4626</td>
<td>100%</td>
</tr>
<tr>
<td>1982</td>
<td>3163</td>
<td>100%</td>
</tr>
<tr>
<td>1983</td>
<td>4921</td>
<td>100%</td>
</tr>
<tr>
<td>1984</td>
<td>16142</td>
<td>100%</td>
</tr>
</tbody>
</table>

Notes: The table lists (by year of issuance) the total amounts of callable bonds which have been called prior to maturity. The data are extracted from the CRSP.

### Table 2: Share of Redeemed Callable Treasuries by Year of Issuance

<table>
<thead>
<tr>
<th>Remaining Term (in quarters)</th>
<th>Normalized Count</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>98.86%</td>
</tr>
<tr>
<td>1</td>
<td>0.70%</td>
</tr>
<tr>
<td>2-4</td>
<td>0.24%</td>
</tr>
<tr>
<td>5-9</td>
<td>0.05%</td>
</tr>
<tr>
<td>10-14</td>
<td>0.02%</td>
</tr>
<tr>
<td>≥15</td>
<td>0.13%</td>
</tr>
</tbody>
</table>

Notes: The table provides information on the redemption profiles of non callable bonds. The data are all non callable debt issued by the US Treasury since the 1920s. 'Remaining Term' counts the number of quarters remaining until maturity when debt is bought back. When 0 this signifies that debt is bought at maturity. The data are extracted from the CRSP.
<table>
<thead>
<tr>
<th>Bond Term (in years)</th>
<th>Call Window* (in years)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
</tr>
<tr>
<td>9</td>
<td>1</td>
</tr>
<tr>
<td>10</td>
<td>8</td>
</tr>
<tr>
<td>11</td>
<td>1</td>
</tr>
<tr>
<td>12</td>
<td>3</td>
</tr>
<tr>
<td>13</td>
<td>1</td>
</tr>
<tr>
<td>14</td>
<td>3</td>
</tr>
<tr>
<td>15</td>
<td>1</td>
</tr>
<tr>
<td>16</td>
<td>1</td>
</tr>
<tr>
<td>17</td>
<td>1</td>
</tr>
<tr>
<td>18</td>
<td>0</td>
</tr>
<tr>
<td>20</td>
<td>0</td>
</tr>
<tr>
<td>23</td>
<td>0</td>
</tr>
<tr>
<td>24</td>
<td>0</td>
</tr>
<tr>
<td>25</td>
<td>0</td>
</tr>
<tr>
<td>26</td>
<td>0</td>
</tr>
<tr>
<td>27</td>
<td>0</td>
</tr>
<tr>
<td>29</td>
<td>0</td>
</tr>
<tr>
<td>30</td>
<td>0</td>
</tr>
</tbody>
</table>

Notes: The table shows the call windows (maturity minus first possible call date) for callable bonds in the US. The sample used is the same as the sample used for Table 2.

Table 3: Bond Terms and Call Windows
Table 4: Moments: Data and Models

<table>
<thead>
<tr>
<th></th>
<th>$S_t$ (%)</th>
<th>$\sigma_{S_t}$ (%)</th>
<th>$\rho(S_t,S_{t-1})$</th>
<th>$\rho(\tilde{b}_S^{GDP}, \tilde{b}_N^{GDP})$</th>
<th>$%S_t = 0$</th>
<th>$%S_t \leq 0.1$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>US DATA</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>BuyBack</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>'Lend.'</td>
<td>4·10³</td>
<td>3·10⁵</td>
<td>0.47</td>
<td>-0.01</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>'No Lend.'</td>
<td>43</td>
<td>7.8</td>
<td>0.94</td>
<td>0.86</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td><strong>No Buyback</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>'Lend.'</td>
<td>76</td>
<td>3·10³</td>
<td>0.42</td>
<td>0.87</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>'No Lend.'</td>
<td>48</td>
<td>8.1</td>
<td>0.92</td>
<td>0.92</td>
<td>0.01%</td>
<td>0.02%</td>
</tr>
<tr>
<td>'No Lend.+Coupons'</td>
<td>51</td>
<td>4.9</td>
<td>0.90</td>
<td>0.94</td>
<td>0.01%</td>
<td>0.02%</td>
</tr>
<tr>
<td>'3 Bonds'</td>
<td>31</td>
<td>5.5</td>
<td>0.81</td>
<td>0.93</td>
<td>0.11%</td>
<td>0.64%</td>
</tr>
<tr>
<td><strong>Repurchases+</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>T Costs</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>'No Lend.'</td>
<td>45</td>
<td>9.0</td>
<td>0.92</td>
<td>0.93</td>
<td>0.01%</td>
<td>0.01%</td>
</tr>
</tbody>
</table>

Notes: $S_t$ denotes the share of debt of maturity less than or equal to one year over the total (market value) of debt. $\bar{S}_t$ represents the average share and $\sigma_{S_t}$ denotes the standard deviation. The statistic $\rho(\tilde{b}_S^{GDP}, \tilde{b}_N^{GDP})$ is the correlation between the market value of short debt and the value of long debt both divided by GDP. The exact definition of the market values, varies depending on the model specification. For example under buyback it holds that $\rho(\tilde{b}_S^{GDP}, \tilde{b}_N^{GDP}) \equiv \rho(\frac{p_S b_S^{GDP}}{GDP}, \frac{p_N b_N^{GDP}}{GDP})$. Under no buyback and no coupons $\tilde{b}_N^{GDP} \equiv \frac{\sum_{i=S+1}^{N} p_i b_{t-i}^{GDP}}{GDP}$. Therefore, when $S = 1$ the value of long debt outstanding is the value of all debt which in $t$ is of maturity greater than one year and $b_S^{GDP}$ is the market value of all outstanding debt less than one year maturity, divided by GDP. The data counterpart is constructed applying this logic (see text).

$\%S_t \leq x$ denotes the percentage of times that $S_t$ is less than or equal to $x$. Finally $T$ denotes the transaction cost function specified in Section 6. See text for details.

Table 5: t stats: Data and Model Moments

<table>
<thead>
<tr>
<th></th>
<th>$S_t$ (%)</th>
<th>$\sigma_{S_t}$ (%)</th>
<th>$\rho(S_t,S_{t-1})$</th>
<th>$\rho(\tilde{b}_S^{GDP}, \tilde{b}_N^{GDP})$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>std. of sample moment</strong></td>
<td>0.0314</td>
<td>0.0139</td>
<td>0.0209</td>
<td>0.0354</td>
</tr>
<tr>
<td><strong>Buyback</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>'Lend.'</td>
<td>-1355</td>
<td>247540</td>
<td>22.44</td>
<td>24.65</td>
</tr>
<tr>
<td>'No Lend.'</td>
<td>9.87</td>
<td>-3.76</td>
<td>3.79</td>
<td>17.32</td>
</tr>
<tr>
<td><strong>No Buyback</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>'Lend.'</td>
<td>-10.52</td>
<td>-2744</td>
<td>24.83</td>
<td>-0.17</td>
</tr>
<tr>
<td>'No Lend.'</td>
<td>-1.60</td>
<td>-0.24</td>
<td>0.92</td>
<td>-1.59</td>
</tr>
<tr>
<td>'No Lend.+Coupons'</td>
<td>-2.55</td>
<td>2.05</td>
<td>1.88</td>
<td>-2.15</td>
</tr>
<tr>
<td>'3 Bonds'</td>
<td>3.81</td>
<td>1.61</td>
<td>6.18</td>
<td>-1.87</td>
</tr>
<tr>
<td><strong>Repurchases</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>'No Lend.'</td>
<td>-0.65</td>
<td>-0.89</td>
<td>0.92</td>
<td>-1.87</td>
</tr>
</tbody>
</table>

Notes: The Table presents t statistics testing the hypothesis that the data moments are equal to the model generated moments summarized in Table 4.
<table>
<thead>
<tr>
<th></th>
<th>BB</th>
<th>NBB</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>lending</td>
<td>no lending</td>
</tr>
<tr>
<td>EEE(^A)</td>
<td>ave: -3.97</td>
<td>-3.72</td>
</tr>
<tr>
<td></td>
<td>max: -2.30</td>
<td>-2.28</td>
</tr>
<tr>
<td>EEE(^N)</td>
<td>ave: -3.18</td>
<td>-3.06</td>
</tr>
<tr>
<td></td>
<td>max: -1.81</td>
<td>-1.93</td>
</tr>
<tr>
<td>EEE(^N)</td>
<td>ave: -3.23</td>
<td></td>
</tr>
<tr>
<td></td>
<td>max: -1.94</td>
<td></td>
</tr>
</tbody>
</table>

Notes: The Table reports average and maximum Euler equation errors (EEE) for the benchmark models (buyback/ no buyback, lending / no lending, and optimal repurchases). Additional moments and errors for other models considered in this paper can be found in the online appendix.

Table 6: Accuracy Tests

Figure 1: **Share of Short Term Debt in the US**

Notes: The Figure plots the share of short maturity government debt (less than or equal to one year) in the US over the period 1955-2015. The data are annual observations (time aggregated from monthly data extracted from the CRSP). Details on the data construction are contained in the Appendix.
Figure 2: **Total Issuance as a Fraction of the Market Value of Outstanding Debt**

Notes: The Figure plots the issuance of new government debt by year and in market value, as a fraction of the total market value of debt outstanding in the United States. The data are from the CRSP and refer to the period 1955-2015.
Figure 3: Callable Bonds over Long Bonds in the US data

Notes: The Figure plots the fraction of long term callable debt over total long term debt outstanding in the CRSP sample.
Figure 4: Callable Bonds: Timing of Buybacks and Call Windows

Notes: The plots show the timing of redemptions of callable debt in the US. The top left shows this timing for bonds whose first call date is 2 years before maturity. The y-axis is in percentage points. Therefore, roughly 75 percent of the bonds are redeemed 2 years before maturity, 15 percent 1 year and 10 percent at the maturity date. The top right panel represents bonds with 3 year call windows, and the bottom panels 4 and 5 years, left and right respectively.
Figure 5: **Response of the Tax Schedule - No Buyback Model**

Notes: The Figure plots the tax rate in the single bond model without buyback presented in Section 3.3.2. The solid line corresponds to a bond of one year maturity. The dashed line sets the maturity to three years and the crossed line to 10 years.
Figure 6: Model Simulations: Buyback and Lending

Notes: The Figure plots a typical sample of the optimal portfolio under buyback. The bound on each maturity is equal to 100% of steady state GDP. The lower bound constraint equals -100% of GDP.
Figure 7: Model Simulations: Buyback and No Lending

Notes: The Figure plots a typical sample of the optimal portfolio under buyback and no lending. The upper bounds on short and long maturities equal 100 percent of steady state GDP. The lower bounds equal 0.
Figure 8: Model Simulations: No Buyback and Lending

Notes: The Figure plots a typical sample of the optimal portfolio under no buyback and lending. The upper bounds imposed on the market value of short and long debt equal 100% of steady state GDP. The lower bounds are equal to -100% of GDP.
Figure 9: Model Simulations: No Buyback and No Lending

Notes: The Figure plots a typical sample of the optimal portfolio under no buyback and lending. The upper bounds imposed on the market value of short and long debt equal 100% of steady state GDP. The lower bounds are equal to 0.
Figure 10: Models vs Data

Notes: The solid line shows the average share of short term debt which was also shown in Figure 1. The dashed line shows the fit of the no buyback model. As described in text we constructed this figure through the following steps: First, we used model simulations to estimate the ‘short debt share’ policy function, regressing the share of short debt on its first and second lags, two lags of the market value of total debt over GDP and the current value and first lag of the spending to GDP series. We also included higher order terms of these variables to produce a ‘good fit’ of the model’s policy function. Second, we fed the ‘data state variables’ to the model’s estimated policy rule, and obtained the dashed line shown. For the buyback model (crossed line) we applied the same procedure. This gave us the crossed line. Because of the inclusion of two lags of the share in the regressions, the years covered in the figure are 1957-2015.
Figure 11: Model Simulations: Debt to GDP ratio and Repurchases

Notes: The Figure plots the debt to GDP ratio (solid line) and the absolute level of repurchases (dashed line) in model of Section 6.3. We used the sample of spending as in Figures 6 to 9. The upper bounds imposed on the market value of short and long debt equal 100% of steady state GDP. Hence the market value of total government debt can be as high as 200% of steady state GDP. The lower bounds of short and long bonds are equal to 0.
Online Appendix to Government Debt Management: The Long and the Short of It

Faraglia, Elisa† Marcet, Albert‡ Oikonomou, Rigas§ Scott, Andrew¶

August 6, 2017
This appendix contains two main sections. Section A contains some analytic results. We first study the optimal portfolio under no buyback and complete markets to show that the usual prediction of ABN models (issue long debt and finance short) holds also in this case. Second, we setup the Lagrangeans that have not been included in the main text. In particular, the Lagrangean for the model with coupons of section 7.1, the ‘callable bonds’ model of section 7.3, and the optimal repurchases with transaction costs model of section 6.3. We derive the state vectors and first order optimality conditions for these models.

Section B contains the Numerical Appendix. It discusses in detail the implementation of the ‘Forward States’ and ‘Condensed PEA’ algorithms and several practical issues on solving portfolio models with incomplete markets with the PEA.

A  Extensions and Additional Theoretical Results

A.1  Complete Markets and No Buyback

We describe in this section the debt management strategy under a complete financial market assuming no buyback. We show that under no buyback it also holds that the government wants to issue long bonds and save in short bonds.

As in Chari and Kehoe (1999) we formulate the planner’s program as a maximization of the household’s utility subject to the following implementability constraint at date 0:

$$E_0 \sum_{t=0}^{\infty} \beta^t [(c_t + g_t) - u_{ct} g_t] = E_0 \sum_{i \in \{1,N\}} \sum_{t=0}^{i-1} \beta^t u_{ct} b_{i-t}$$

where for simplicity we assume that the government may issue debt in one year and $N$ year maturities.

The optimal allocation derives as a solution to the following equation:

$$L_{CM} = E_0 \left\{ \sum_{t=0}^{\infty} \left[ \beta^t u(c_t) + v(T - c_t - g_t) \right] + \Lambda \left[ \sum_{t=0}^{\infty} \beta^t ((u_{ct} - v_{xt})(c_t + g_t) - u_{ct} g_t) - \sum_{i \in \{1,N\}} \sum_{t=0}^{i-1} \beta^t u_{ct} b_{i-t} \right] \right\}$$

The first order conditions for the optimum are given by:

$$\Gamma_t - \Lambda [u_{cc,t} c_t + u_{ct} - v_{xt} + v_{xx,t}(c_t + g_t)] = 0 \quad \text{for} \quad t > N - 1$$

$$\Gamma_t + \Lambda [u_{cc,t} c_t + u_{ct} - v_{xt} + v_{xx,t}(c_t + g_t)] - \Lambda u_{cc,0} \sum_{i \in \{1,N\}} b_{i-t} = 0 \quad \text{for} \quad t \leq N - 1$$

where $\Gamma_t = u_{ct} - v_x (T - x_t)$.

A.1.1  Optimal Debt Management

Let $g^{t-1} = (g_0, g_1, ..., g_{t-1})$ be the history of government spending shocks up to date $t - 1$. As ABN and Faraglia et al. (2010) we define $z$ as the present discounted value of the government surplus
contingent on \( g^{t-1} \) and the current realization of spending \( g_t \).

\[
(4) \quad z_t(g^{t-1}, g_t) = E_t \sum_{i=0}^{\infty} \beta^i \frac{1}{u_{c,t}} [(u_{c,t+i} - v_{x,t+i})(c_{t+i} + g_{t+i}) - g_{t+i}u_{c,t+i}]
\]

Assuming that debt is not state contingent, and that the asset returns are not perfectly correlated, one of the key results of Buera and Nicolini (2004) is that \( z \) is the only variable (along with the asset prices) that is needed to pin down the optimal issuance of the government. Let’s recall the simple example of Section 2 of Faraglia et al. (2010) which makes use of this result. We assume for simplicity that government expenditure follows a two step Markov process taking values \( g_H > g_L \) with probabilities, \( \pi_{HH} \) and \( \pi_{LL} \), of remaining in the same state. We need only two risk free bonds to complete the markets. If initial debt equals 0 we have \( z_t(g^{t-1}, g_t) = z_i \) and \( p_i^{N-1}(g^{t-1}, g_t) = p_i^N \) for \( i = H, L \) and for all \( t \). If we further assume that \( g_0 = g_H \) then \( z_H = 0 < z_L \).

The optimal portfolio must be such that the outstanding value of the government’s liability is equal to 0 whenever government spending is high and equal to \( z_L \) when spending is low. Recall that under buyback the optimal portfolio needs to satisfy:

\[
(5) \quad b_{t-1}^i (g^{t-1}) + p_i^{N-1} (g^{t-1}) b_{t-1}^N (g^{t-1}) = z_i \quad \text{for} \quad i = H, L \quad \forall t
\]

and if \( p_H^{N-1} \neq p_L^{N-1} \) we have

\[
(6) \quad \begin{pmatrix} 1 & p_H^{N-1} \\ 1 & p_L^{N-1} \end{pmatrix} \begin{pmatrix} b_{t-1}^H (g^{t-1}) \\ b_{t-1}^N (g^{t-1}) \end{pmatrix} = \begin{pmatrix} 0 \\ z_L \end{pmatrix}
\]

yielding

\[
(7) \quad \begin{pmatrix} b_{t-1}^H (g^{t-1}) \\ b_{t-1}^N (g^{t-1}) \end{pmatrix} = \begin{pmatrix} \frac{p_H^{N-1} z_L}{p_H^{N-1} - p_L^{N-1}} \\ \frac{p_L^{N-1} z_L}{p_H^{N-1} - p_L^{N-1}} \end{pmatrix} = \begin{pmatrix} B^1 \\ B^N \end{pmatrix}
\]

The key properties of the solution are that first the issuance of each security is constant over time and second that if \( p_H^{N-1} < p_L^{N-1} \) then \( B^1 < 0 \) and \( B^N > 0 \).

Consider now the case where the government has to hold all the debt to maturity. The analogous of equation (5) assuming no buyback is:

\[
(8) \quad b_{t-1}^i (g^{t-1}) + \sum_{j=1}^{N} p_i^{N-j} (g^{t-1}) b_{t-1}^{N-j} (g^{t-1}) = z_i \quad \text{for} \quad i = H, L \quad \forall t
\]

which states that in order to complete the market the government will issue bonds ensuring that the total value of debt (include unmatured debt) each period equals the discounted sum of surpluses \( z_i \).

Given initial conditions \( b^1_{t-1} \) and \( b^N_{t-1} \) for \( i = 1, ..., N - 1 \) this equation describes the whole evolution of \( b^1_{t} \) and \( b^N_{t} \). To evaluate its implications we focus on the steady state.

In our two state example a generalization of (6) gives:

\[
(9) \quad \begin{pmatrix} 1 & \sum_{i=1}^{N-1} p_i^H \\ 1 & \sum_{i=1}^{N-1} p_i^L \end{pmatrix} \begin{pmatrix} b_{t-1}^H (g^{t-1}) \\ b_{t-1}^N (g^{t-1}) \end{pmatrix} = \begin{pmatrix} 0 \\ z_L \end{pmatrix}
\]
yielding

\[
\begin{pmatrix}
\frac{b_{t-1}^1}{b_{t-1}^N}
\end{pmatrix}
\begin{pmatrix}
(g^{t-1})
\end{pmatrix}
= \begin{pmatrix}
\frac{\sum_{i=1}^{N-1} p_H^{-i} z_t}{\sum_{i=1}^{N-1} (p_H^{-i} - p_L^{-i})}
\end{pmatrix}
= \begin{pmatrix}
B^1_{ss} \\
B^N_{ss}
\end{pmatrix}
\]

For standard utility functions it holds that \( p_i^H < p_i^L \) for all \( i = 1, \ldots, N \) so that, as before, \( B^N_{ss} > 0 \), showing that in this economy the government should issue long bonds in order to hedge against variations in \( z_t \). The government saves in short bonds, e.g. \( B^1_{ss} < 0 \).

### A.1.2 Relative Debt Positions

Let us now compare these steady state positions with the quantities given by (7). In the case of \( N > 2 \) we can expect \( \left| \sum_{i=1}^{N-1} (p_H^{N-1} - p_L^{N-1}) \right| > |p_H^{N-1} - p_L^{N-1}| \) implying \( B^N_{ss} < B^N \). In other words, in steady state the government will issue lower amounts of long term bonds every period if it holds the bonds until maturity. This is not surprising, any long bond now constitutes debt for the following \( N \) periods, so that less debt needs to be issued every period.

In order to compare the debt positions when buyback is ruled out we shall focus on the ratio of the **value** of total long debt (RVLD):

\[
RVLD^j \equiv \frac{p_j^{N-1} B^N}{\left( \sum_{i=1}^{N-1} p_j^i \right) B^N_{ss}}
\]

that compares the value of total government long term outstanding debt in the model with buyback (in the numerator) with the value of total long term outstanding debt in the model without buyback. Even though bonds are constant at steady state, this ratio depends on the realization \( j = H, L \) due to the fact that prices change with the current realization \( j \).

To gain some insight on likely values of this ratio, we assume that the process for government spending shocks \( g \) i.i.d. and that \( \beta \) close to 1. Under these assumptions we can show that

\[
E(RVLD) \approx 1 - \frac{1}{N}
\]

Therefore, according to (11), ruling out buyback may increases the value of total long debt held by the government, since \( 1 - \frac{1}{N} < 1 \). Moreover, for the short term bond we have that:

\[
\begin{align*}
\frac{B^1}{B^1_{ss}} &= RVLD^H \\
\end{align*}
\]

where the equality uses the first equations in (10) and in (7) and the definition of \( RVLD^H \). Therefore, except for the discrepancy between \( RVLD^H \) and \( E \ (RVLD) \), which is likely to be small, we can claim that for the case of i.i.d. expenditure shocks and \( \beta \) close to 1 we have that:

\[
\frac{B^1}{B^1_{ss}} \approx 1 - \frac{1}{N}.
\]

Therefore ruling out buyback leads also to larger positions in short bonds. As \( N \) grows the difference becomes smaller.
A.2 Coupon Bonds and No Buyback: the Ramsey Program

We solve the optimal policy problem under no buyback and coupons assuming that the government issues debt in one year and $N$ year bonds. As in Aiyagari et al. (2002) we introduce debt limits, these are parameterized as:

$$\sum_{j=1}^{N} \beta^j + \kappa \sum_{j=1}^{N} \sum_{l=1}^{j} \beta^l \equiv \left[ M_N, \bar{M}_N \right]$$

Letting $[\tilde{M}_1, \tilde{M}_N] \equiv [M_1, \bar{M}_1]$ be the analogous constraint set for one year debt the planning problem is given by:

$$\mathcal{L} = E_0 \sum \beta^t \left\{ u(c_t) + v(T - c_t - g_t) + \lambda_t \left\{ b_{t-1}^t \beta u_{c,t+1} + b_t^N \left( \beta^N u_{c,t+N} + \sum_{j=1}^{N} \beta^j u_{c,t+j} \right) - b_{t-1}^t u_{c,t} - b_{t-N}^t u_{c,t} - \kappa \sum_{j=1}^{N} b_{t-j}^j u_{c,t} - g_t u_{c,t} + (u_{c,t} - v_{x,t})(g_t + c_t) \right\} + \sum_{i \in \{1,N\}} \xi_{i,t}^t (\tilde{M}_i - b_i^t) + \sum_{i \in \{1,N\}} \xi_{i,t}^t (b_i^t - \tilde{M}_i) \right\}$$

The first order condition for consumption is:

$$u_{c,t} - v_{x,t} + \lambda_t(u_{cc,t} c_t + u_{c,t} + v_{xx,t}(c_t + g_t)) = 0$$

and off corners the analogous conditions for $b_1^t$ and $b_N^t$ are:

$$\lambda_t E_t (u_{c,t+1}) = E_t (\lambda_{t+1} u_{c,t+1})$$

$$\lambda_t E_t (\kappa \sum_{j=1}^{N} \beta^j u_{c,t+j} + \beta^N u_{c,t+N}) = E_t (\kappa \sum_{j=1}^{N} \beta^j u_{c,t+j} \lambda_{t+j} + \beta^N u_{c,t+N} \lambda_{t+N})$$

For brevity we summarized the properties of this model in Table 4 in text. In Figure 1 of this appendix we show a typical sample of long and short debt (analogous to Figures 5-9 in the main text). As was claimed in text, when bonds pay positive coupons the properties of the model remain very close to the no buyback and zero coupons case.

A.3 Callable Bonds: the Ramsey Program

As explained in Section 2 of the paper the US government issued callable bonds in the past. These types of securities give to the issuer the option to buy them back after $m$ years, at every coupon date until the bond matures. Their price at buyback is at par. We showed that historically the US government has repurchased callable bonds at the start of the call window.

Proposing this model, our intention is not to motivate the empirical observations of why the Treasury chooses to buyback at the first call date. Rather we seek to establish that removing debt
Notes: The Figure plots a typical sample path from the no buyback model with positive coupons. As explained in text the value of the coupon \( \kappa \) is calibrated so that bonds trade on average at par. The upper bound on \( b_t^N \) is given in (12). The lower bound is zero. The value of short term debt in a given period \( t \) in the Figure, is constructed by adding the coupon payments and principals which are to mature in \( t + 1 \) to the market value of one year bonds issued in \( t \).

from the market before, but close to, the maturity date is akin to the model of no buyback and that our findings about the importance of short term debt and the positive comovement between short and long debt still hold.

We solved our model assuming that the buyback of the \( N \)-year bond occurs \( m \) years after issuance. To make our analysis as conservative as possible we calibrated \( N = 10 \) and \( m = 5 \). Notice that a lower \( m \) makes the model closer to the buyback section, since buyback is equivalent with \( m = 1 \). If we were to find that even for a low \( m \) the model behaves similar to no buyback, higher \( m \) are likely to be even closer to no-buyback. The call window in the data for 10 year bonds starts 2 years before maturity, suggesting a buyback period of \( m = 8 \). In this sense, our model choice of \( m = 5 \) is conservative.

The budget constraint of the government is:

\[
\sum_{i \in \{1, N\}} p_i b_i^t = b_{t-1}^1 + b_{t-m}^N + g_t - \tau_t (T - x_t)
\]
The ad hoc debt constraints for the $N$ year bond are

\[ b^N_t \in \left[ \frac{M_N}{\sum_{j=0}^{m-1} \beta^{N-j}}, \frac{\bar{M}_N}{\sum_{j=0}^{m-1} \beta^{N-j}} \right] = [\tilde{M}_N, \bar{M}_N] \]

Letting $[\tilde{M}_1, \bar{M}_1] = [\frac{M_1}{\beta}, \frac{\bar{M}_1}{\beta}]$ be the analogous constraints for one year debt and substituting the equilibrium expressions for the tax rate and the bond prices we represent the planning problem as follows:

\[
\mathcal{L} = E_0 \sum_t \beta^t \left\{ u(c_t) + v(T - c_t - g_t) + \lambda_t \left[ \sum_{i \in \{1,N\}} b^i_t \beta u_{c,t+i} - b^{i}_{t-1} u_{c,t} - b^N_t \beta^{N-m} u_{c,t+N-m} \right. \right.
\]

\[
- g_t u_{c,t} + (u_{c,t} - v_{x,t})(g_t + c_t) \right.
\]

\[
+ \sum_{i \in \{1,N\}} \xi^i_{U,t}(\tilde{M}_i - b^i_t) + \sum_{i \in \{1,N\}} \xi^i_{U,t}(\bar{M}_i - \tilde{M}_i) \left\} \right. \}
\]

The first order conditions for the optimum are given by:

\[
u_{c,t} - v_{x,t} + \lambda_t \left( u_{cc,t} c_t + u_{cx,t} c_t + v_{xx,t} (c_t + g_t) - v_{x,t} \right) + u_{ct} \left[ (\lambda_{t-1} - \lambda_t) b^1_{t-1} + (\lambda_{t-N} - \lambda_{t-N+m}) b^N_{t-N} \right] \]

\[
\beta E_t(u_{c,t+1} \lambda_t - u_{c,t+1} \lambda_{t+1}) + \xi^1_{L,t} - \xi^1_{U,t} = 0
\]

\[
\beta^N E_t(u_{c,t+N} \lambda_t - u_{c,t+N} \lambda_{t+m}) + \xi^N_{L,t} - \xi^N_{U,t} = 0
\]

We assume $\tilde{M}_1 = \bar{M}_1 = 0$. In Figure 2 we plot a typical sample of the market value of short and long debt. Notice that indeed assuming that the government repurchases debt from the market $m = 5$ years after issuance, reduces the share of short bonds in the portfolio compared to a model with no buyback. The portfolio is somewhere between the ‘full buyback model’ studied in the paper (i.e. when $m = 1$) and the no buyback model where $m = N$. In terms of the moments reported in Table 4 in the paper the model of this section gives us the following: $\bar{S}_t = 19\%$, $\sigma_{S_t} = 11.6\%$, $\rho_{S_t S_{t-1}} = 0.84$ $\hat{\rho}_{S_t^N, \hat{S}_t^N} = 0.62$ $\%_{S_t=0} = 0.65\%$.

We view these results as encouraging because they confirm the hypothesis that even if the government buys back some of the debt before maturity there is still a role for short bonds. First, because the share of short debt is very rarely zero in simulations (e.g. $%_{S_t=0} = 0.65\%$ versus the analogous figure in the buyback no lending model in the paper of 13%). As we mentioned, the choice of $m = 5$ is quite conservative. The data suggest that $m = 8$ would be more appropriate. With $m = 8$ we expect the model to generate results very close to the no buyback ones.

This is only a partial study of callable bonds. Clearly, the modelling of callable bonds can be made closer to the data by introducing that they can be repurchased at par or the transaction costs that are involved in the recall. A model taking all these features into account is beyond the scope of this paper. However the message that there is still a role for short bonds comes out clearly from the analysis.
Notes: The Figure plots a typical sample path from the model of Section A.3. We assume that government buybacks of 10 year bonds occur 5 years after issuance.

A.4 Optimal Repurchases: the Ramsey Program

In the optimal repurchase (OR) model of Section 6 the government maximizes the utility of the household subject to the following constraints

\[
\sum_{i \in \{S,N\}} p_i^t b_i^t (1 - T^i(b_i^t)) = b_{t-S}^S + b_{t-N}^N - R_{t-N+1} + p_t^{N-1} R_t (1 + T^R(R_t)) + g_t - \tau_t (T - x_t)
\]

(15)

\[
T - x_t = c_t + g_t + TC_t
\]

(16)

\[
0 \leq b_{t_i}^t \leq \frac{M_i}{\sum_{j=1}^{\beta_i}}, \quad 0 \leq R_t \leq b_{l-1}^N
\]

(17)

where \(TC_t = \sum_{i \in \{S,N\}} p_i^t b_i^t T^i(b_i^t) + p_t^{N-1} R_t (1 + T^R(R_t))\).

To simplify the solution of this model we assume that the government treats as exogenous the function \(TC_t\), in other words it does not take derivatives of \(TC_t\) with respect to consumption and
the 'out'-vector, \( X \)

discuss some practical features of our numerical procedure that can help the algorithm convergence.

In section 4 of the paper we described the ‘Condensed PEA’ that deals with the high dimensionality of the state vector, and the ‘Forward States PEA’ that deals with the indeterminacy of the portfolio generated by the use of the PEA. This numerical appendix outlines in greater detail the two methods, their implementation and the steps we followed to approximate the conditional expectations in the generated by the use of the PEA. This numerical appendix outlines in greater detail the two methods, of the state vector, and the ‘Forward States PEA’ that deals with the indeterminacy of the portfolio

\[ \mathcal{L} = E_0 \sum_t \beta^t \left[ u(c_t) + v(T - c_t - g_t - TC_t) + \lambda_t \sum_{i \in \{S,N\}} b_i^t \beta^t u_{c,t+1}(1 - T(b_i^t)^t) 
- \beta^{N-1} u_{c,t+N-1} R_t (1 + T^R(R_t)) - (b^S_{i-N} + b^N_{i-N} - R_{t-N+1}) u_{c,t} 
- g_t u_{c,t} + (u_{c,t} - v_{x,t})(g_t + c_t) \right] + \sum_{i \in \{S,N\}} \xi_{i,t}^L \left( \sum_{j=1}^{M_i} \beta^j - b_i^t \right) + \sum_{i \in \{S,N\}} \xi_{i,t}^R (b_i^t + R_t) \]

The FONC are given by:

\[ u_{c,t} - v_{x,t} + \lambda_t (-u_{cc,t} g_t + u_{c,t} + u_{cc,t} (T - x_t) + v_{xx,t} (T - x_t) - v_{x,t}) - u_{cc,t} [B_t - \lambda_t - B \lambda_t] = 0 \]

\[ E_t \beta^t (-u_{cc,t+1} \lambda_t + u_{cc,t+1} \lambda_t (1 - T_t^{1} - T_t^{1} b_i^{1})) + \xi_{i,t}^L - \xi_{i,t}^U = 0 \quad \text{for} \quad i = S \]

\[ E_t \beta^t (-u_{cc,t+N} \lambda_t + u_{cc,t+N} \lambda_t (1 - T_t^{N} - b_i^{N})) + E_t \beta (\xi_{U,t+1}^R) + \xi_{L,t}^N - \xi_{U,t}^R = 0 \quad \text{for} \quad i = N \]

\[ E_t \beta^t (u_{c,t+N-1} \lambda_t - u_{c,t+N-1} \lambda_t (1 + T_t^{R} + T_t^{R} R_t)) + \xi_{L,t}^R - \xi_{U,t}^R = 0 \]

where

\[ B_t \equiv b_t^S + B_{t-N+1+S} \]

\[ B_{t}^{net} \equiv b_{t-1} - R_t \]

\[ B \lambda_t \equiv \lambda_t (1 - T_t^{S}) b_t^S + B \lambda_{t-N+1+S} \]

\[ B \lambda_{t}^{net} \equiv \lambda_{t-1} (1 - T_{t-1}^{N}) b_{t-1} - \lambda_t (1 + T_t^{R}) R_t \]

## B Numerical Appendix

In section 4 of the paper we described the ‘Condensed PEA’ that deals with the high dimensionality of the state vector, and the ‘Forward States PEA’ that deals with the indeterminacy of the portfolio generated by the use of the PEA. This numerical appendix outlines in greater detail the two methods, their implementation and the steps we followed to approximate the conditional expectations in the different models. In particular, we report how we selected the state variables of the core vector, \( X_{t}^{core} \), the ‘out’-vector, \( X_{t}^{out} \) and the order of the polynomials of the states that were used. Moreover, we report how many linear combinations of state variables were added to the approximations, and also discuss some practical features of our numerical procedure that can help the algorithm convergence.

\(^1\)Without this assumption we would need to keep track of the fact that there is a conditional expectation in the determination of \( TC_t \), therefore the solution to the model would feature both current and lagged values of the multiplier on this constraint. This would add yet more state variables in the model but with minimal quantitative effects.

Note that another way to simplify the planner’s program (avoid having to keep track of the resource constraint as a separate object in the Lagrangean) is to assume \( TC_t \) do not enter the feasibility constraint. In this case transaction costs do not impact the overall resources of the economy, this would correspond to a situation where a financial firm can charge transaction costs on bond issuances without actually spending labor resources in it. When we run the model under this assumption (which appears sometimes in the literature) we found virtually no effect on our results.
B.1 Implementation of ”Condensed PEA” and ”Forward states PEA”

B.1.1 Selection of variables in the approximation

Recall that ‘Condensed PEA’ divides the state vector \( X \) in two subvectors: the core vector, \( X_{t+1}^{\text{core}} \), which includes variables that (we believe a priori) are of primary importance in the approximation, and the \( X_{t+1}^{\text{out}} \) vector, which includes the remaining state variables and possibly higher order polynomial terms. ‘Forward States PEA’ resolved the portfolio indeterminacy issue through approximating \( E_t u_{c,t+1} \) with \( E_t (\Phi (X_{t+1}, \gamma^i)) \) and \( E_t \lambda_{t+1} u_{c,t+1} \) with \( E_t (\Psi (X_{t+1}, \delta^i)) \). To clearly show how we implemented these two methods, we bring them now together and in what follows we outline the ‘Condensed PEA’ using since the first PEA iteration (i.e. with core state variables only) ‘Forward States’ to solve the portfolio choice.

The \( X^{\text{core}} \) vector:

In all the models presented in the paper, but the 3 bond model, we used the core vector

\[
X_{t+1}^{\text{core}} = \left\{ 1, g_{t+1}, \{ b_i^t \}_{i=1,N}, \lambda_t, \{ (b_i^t)^2 \}_{i=1,N}, \{ g_{t+1} b_i^t \}_{i=1,N} \right\}
\]

i.e. \( X_{t+1}^{\text{core}} \) is composed of a constant, the level of government spending in \( t + 1 \), the levels of date \( t \) variables (the bonds and the multipliers), the square of the bonds and the interaction term between the bonds in \( t \) and \( g_{t+1} \).

We solve the system of FONCs after integrating out the term \( g_{t+1} \) as discussed in footnote 25 of the main text. We use the analytical formula for the conditional expectation of \( g_{t+1} \) at time \( t \) given by:

\[
\int g_{t+1} f_{g_{t+1}|g_t} dg_{t+1} = \omega_t + (\bar{g} - \omega_t) \Phi \left( \frac{\bar{g} - \omega_t}{\sigma_e} \right) + (\bar{g} - \omega_t) \left[ 1 - \Phi \left( \frac{\bar{g} - \omega_t}{\sigma_e} \right) \right] - \sigma \left( \Phi \left( \frac{\bar{g} - \omega_t}{\sigma_e} \right) - \phi \left( \frac{\bar{g} - \omega_t}{\sigma_e} \right) \right)
\]

where \( \omega_t \equiv \rho_g g_t + (1 - \rho_g) g_{t-1} \), \( \Phi(\phi) \) is the standard normal cdf (pdf), \( \bar{g} \) and \( g \) are upper and lower bounds on government spending. \( \sigma_e \) is the standard deviation of the spending shock.

We have chosen to introduce higher order terms in the core vector for three reasons. First, approximating the conditional expectations \( E_t u_{c,t+1} \) means that we are approximating the bond

---

2 As discussed in the calibration section we assume that spending fluctuates between 15 and 35 percent of steady state GDP.

3 The expression is reached as follows:

\[
\int g_{t+1} f_{g_{t+1}|g_t} dg_{t+1} = \int_{-\infty}^{\bar{g} - \omega_t} g dF(\epsilon_{t+1}) + \int_{\bar{g} - \omega_t}^{\infty} g dF(\epsilon_{t+1}) + \int_{\rho_g g_t + (1 - \rho_g) g_{t-1}}^{\bar{g} - \omega_t} (g - \omega_t) dF(\epsilon_{t+1}) + \int_{\rho_g g_t + (1 - \rho_g) g_{t-1}}^{\infty} (\bar{g} - \omega_t) dF(\epsilon_{t+1}) + \int_{\rho_g g_t + (1 - \rho_g) g_{t-1}}^{\bar{g} - \omega_t} \epsilon_{t+1} dF(\epsilon_{t+1})
\]

where \( F \) denotes the cdf of \( \epsilon \). Standard results give:

\[
\int_{-\infty}^{\bar{g} - \omega_t} (g - \omega_t) dF(\epsilon_{t+1}) = \int_{-\infty}^{\bar{g} - \omega_t} (g - \omega_t) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} \frac{(g - \omega_t)^2}{\sigma_e^2}} dz_{t+1} = (g - \omega_t) \Phi \left( \frac{\bar{g} - \omega_t}{\sigma_e} \right)
\]

where \( z \) is a standard normal variable and \( \Phi \) is the cdf. Analogously:

\[
\int_{\bar{g} - \omega_t}^{\infty} (\bar{g} - \omega_t) dF(\epsilon_{t+1}) = (\bar{g} - \omega_t) (1 - \Phi \left( \frac{\bar{g} - \omega_t}{\sigma_e} \right))
\]
prices. If we include only the levels of the bonds, \( \{ b_i^t \}_{i=1,N} \), we are imposing that bonds are close substitutes in terms of their influence on prices.\(^4\). This is not a property that we are sure to find in equilibrium and non-linear terms may be potentially important. Secondly, if the ad hoc debt limits are occasionally binding this is known to introduce non-linearities so that higher order terms may be important. Finally the ’Condensed PEA’ inclusion criterion suggested that these non linear terms are important for the approximations. Indeed when we solved the models without higher order terms and tested whether they should be included in the linear combinations, the percentage gains in \( R^2 \) were substantial. Higher order terms therefore should be included in the polynomials, either in \( X_{t+1}^{\text{core}} \) or in \( X_{t+1}^{\text{out}} \). We ultimately chose to introduce some higher order terms in the core vector and left others for the linear combinations (see below) finding this helpful for the stability of the convergence.

The \( X_{t+1}^{\text{out}} \) vector:

Notice that if we think of introducing higher-order terms \( X_{t+1}^{\text{out}} \) is potentially infinite-dimensional, so we need to specify what we actually included in \( X_{t+1}^{\text{out}} \) in our simulations, as the linear combinations of this vector are the ones that eventually can play a role in the solution.

Assuming \( S = 1 \), the ‘out’ vector is different for each model presented in the paper. To identify the elements of \( X_{t+1}^{\text{out}} \) we use as guidance the FONC of each model.

Consider first the buyback model. The first order conditions are

\[
\begin{align*}
(21) & \quad u_{c,t} - v_{x,t} + \lambda_t \left( u_{c,t+1} + u_{c,t} + v_{x,t}(c_t + g_t) - v_{x,t} \right) + u_{c,t} \sum_{i \in \{1,N\}} \left( \lambda_{t-i} - \lambda_{t-i+1} \right) b_{i-t-i}^t = 0 \\
(22) & \quad \beta^t E_t \left( u_{c,t+i} \lambda_t - u_{c,t+i} \lambda_{t+1} \right) + \xi_{L,t}^i - \xi_{U,t}^i = 0 \quad \text{for } i = 1, N
\end{align*}
\]

When markets are incomplete, the term \( \sum_{i \in \{1,N\}} \left( \lambda_{t-i} - \lambda_{t-i+1} \right) b_{i-t-i}^t \) summarises interest rate manipulation under commitment (see FMOS (2016)). Suppose that a positive spending shock arrives in period \( t \) and that \( b_t^N > 0 \). Since \( \left( \lambda_{t-i} - \lambda_i \right) b_t^N \) becomes negative, the government finds optimal to promise a tax cut in \( t + N - 1 \) and lower the marginal utility of consumption in that period. It is then evident that the terms \( \lambda_{t-1} b_t^N \) and \( \lambda_t b_t^N \) are important determinants of \( u_{c,t+N-1} \) and \( u_{c,t+N} \) and hence they should be accounted for when we approximate the conditional expectations.\(^5\)

Finally,

\[
\int_{\bar{g} - \omega_i}^{\bar{g} - \omega_i} \frac{1}{\sqrt{2\pi}} \exp \left( \frac{-z_i^2}{2} \right) dz_i + \int_{\bar{g} - \omega_i}^{\bar{g} - \omega_i} \frac{1}{\sqrt{2\pi}} \exp \left( \frac{-z_i^2}{2} \right) dz_i = -\sigma \left( \phi \left( \frac{\bar{g} - \omega_i}{\sigma} \right) - \phi \left( \frac{\bar{g} - \omega_i}{\sigma} \right) \right)
\]

Putting everything together we get (20)

\(^4\)To see this consider the approximation of \( E_t u_{c,t+N} \approx \gamma_0^N + \gamma_1^N \int g_{t+1} f(g_{t+1} | g_t) dg_{t+1} + \gamma_2^N b_t^1 + \gamma_3^N b_t^N + \gamma_4^N \lambda_t \) under linear polynomials. Clearly there are (infinitely) many pairs \( (b_t^1, b_t^N) \) that give the same bond price (holding \( L_t \) fixed).

Notice that the optimal portfolio is nonetheless identified under linear polynomials since \( \{ b_i^t \}_{i=1,N} \) influence all conditional expectations and enter in a nonlinear fashion in the system of FONCs (for example in the budget constraint of the government).

\(^5\)In the text the implementation of ‘Forward States’ to the buyback model was summarized in the following equations

\[
(23) & \quad \lambda_t = \frac{E_t \left( \Psi \left( X_{t+1}, \delta_t^i \right) \right)}{E_t \left( \Phi \left( X_{t+1}, \gamma_t^i \right) \right)} \quad \text{for } i = S, N
\\
(24) & \quad \sum_{i \in \{S,N\}} b_i^t \beta^t E_t \left( \Phi \left( X_{t+1}, \gamma_t^i \right) \right) = \sum_{i \in \{S,N\}} b_i^{t-1} \beta^{t-1} \Phi \left( X_t, \gamma_t^i \right) + g_t u_{c,t} - (u_{c,t} - v_{x,t})(g_t + c_t)
\]
Applying the above argument to determine which states potentially exert a significant influence to the expectations of date \(t+1, t+N-1\) and \(t+N\) variables in the buyback model, we include in \(X_{t+1}^{out}\) the following terms: \(\lambda b_t^N, \lambda b_t^{1}, \lambda b_{t-1}^N, \lambda_{t-1} b_{t-1}^N, \lambda_{t-N+1} b_{t-N+1}^N\) and \(\lambda_{t-N+2} b_{t-N+1}^N\).

Two more comments about this choice are necessary. Firstly, despite the fact that each of the above terms is potentially important for (some of) the conditional expectations we wish to approximate, it is unlikely that each term bears the same importance to each conditional expectation. For example, the term \(\lambda b_t^N\) clearly exerts an influence on \(u_{c,t+N}\) (through the FONC) but it is less likely to exert a significant influence on \(u_{c,t+N-1}\). In this case the ‘Condensed PEA’ will assign a coefficient close to zero to \(\lambda b_t^N\) in the approximation of \(E_t u_{c,t+N-1}\) and a coefficient different from zero in the approximation of \(E_t u_{c,t+N}\). This shows how convenient it is to include these terms in \(X^{out}\) where having coefficients close to zero for some state variables is not an issue, as opposed to including them in \(X^{core}\), in which case variables with close to zero coefficients may cause convergence problems.

Secondly, as explained before, (21) suggests that the cross terms between \(\lambda\) and \(b\) are potentially important for the solution. However, one may wonder whether the levels of these variables should also be included in the state vector. The FONCs show that the influence of \(\lambda_{t-N+1}\) on the optimal allocation in \(t+1\) is close to zero if \(b_{t-N+1}^N\) is close to zero. The effect of changes in the value of the multiplier is felt more when government debt is high. This nonlinear influence seems to be (sufficiently) well captured in our specification by the cross terms and not by the levels since, as we verify in section 7.4 of the main text, we pass accuracy tests.\(^6\)

We apply the above selection criterion to the other models. Consider the no buyback model and its first order conditions and budget constraint:

\[
\begin{align*}
  u_{c,t} - v_{x,t} + \lambda_t \left( u_{cc,t} c_t + u_{ct} + v_{xx,t} (c_t + g_t) - v_{x,t} \right) + u_{cc,t} \sum_{i \in \{1,N\}} \left( \lambda_{t-i} - \lambda_t \right) b_{t-i}^1 = 0 \\
  \beta^i E_t \left( u_{c,t+i} \lambda_t - u_{c,t+i} \lambda_{t+i} \right) + \xi^i_{t,t} - \xi^i_{t,t-1} = 0 \quad \text{for} \quad i = 1, N
\end{align*}
\]

\[
\sum_{i \in \{1,N\}} b_{t-i}^1 \beta^i E_t u_{c,t+i} = g_t u_{c,t} + u_{ct} \sum_{i \in \{1,N\}} b_{t-i}^1 - (u_{c,t} - v_{x,t}) (c_t + g_t).
\]

We include in the \(X^{out}\) vector: \(\lambda_t b_t^N, \lambda_t b_t^{1}, \lambda_{t-N+1} b_{t-N+1}^N, \) and \(b_{t-N+1}^N,\) as these appear directly on the FONC.

Next, consider the no buyback model with coupons. To solve the coupon model we need to approximate the term \(\sum_{j=1}^N \beta^j E_t u_{c,t+j} \lambda_t + \beta^N u_{c,t+N}\) and the term \(\sum_{j=1}^N \beta^j E_t u_{c,t+j} \lambda_{t+j} + \beta^N u_{c,t+N} \lambda_{t+N}\).

Notice that in (24) we parameterize the term \(E_t u_{c,t+N-1}\) as \(\Phi^N(X_t, \gamma^N)\). In other words we apply the standard PEA to this term. An alternative is to define \(E_t u_{c,t+N-2}\) as \(\Phi^N(X_t, \gamma^{N-1})\) and then use Forward States to get: \(E_t u_{c,t+N-1} = E_t \Phi^N(X_{t+1}, \gamma^{N-1})\). We follow the latter route in the numerical implementation. We therefore write (24) as follows:

\[
b_t^i \beta^i E_t (\Phi^i (X_{t+1}, \gamma^i)) + b_{t-N}^i \beta^N E_t (\Phi^N (X_{t+1}, \gamma^N)) = b_{t-1}^{i-1} u_{c,t} + b_{t-1}^{i-1} \beta^{N-1} E_t (\Phi^{N-1} (X_{t+1}, \gamma^{N-1})) + g_t u_{c,t} - (u_{c,t} - v_{x,t}) (g_t + c_t)
\]

i.e. when \(S = 1\) and realizing that \(E_t u_{c,t} = u_{c,t} = \Phi^1(X_t, \gamma^1)\).

The two ways of solving the model are obviously conceptually equivalent.

\(^6\)Recall that \(X^{core}\) includes the variables \(\lambda_t, b_t^N\) and \(b_t^{1}\) in levels. These first order terms, help us to identify the portfolio, but combined with their squares, cubes and so on can (practically speaking), explain part of the variability of some of the cross terms in \(X^{out}\). To avoid having residuals close to zero from the regressions of \(X^{out}\) on \(X^{core}\) when we compute linear combinations, we use an additional selection criterion that we describe in the next subsection.
From the FONC of consumption and the government budget constraint (omitted for brevity), it is easy to show that all the lags of \(b_{t-j}^N\) and \(\lambda_{t-j}b_{t-j}^N\), for \(j = 1, 2, ..., N - 1\) should be introduced in the out vector. The \(X^{out}\) vector is therefore composed by: \(\{\lambda_{t-j}b_{t-j}^N\}_{j=0}^{N-1}, \lambda_t b_t, \{b_{t-j}^N\}_{j=1}^{N-1}\).

Similarly, when we consider the *callable bond* model using the same criteria discussed above we know that the repurchase date, \(m\), will exert an influence on the solution and then we have chosen the \(X^{out}\) vector to be: \(\{b_t^\lambda\}_{t=1,N}, \lambda_{t-N+m}b_{t-N+m}, \lambda_{t-N+1}b_{t-N+1}, \lambda_{t-N+m}b_{t-N+1}^N\) and \(b_{t-N+m+1}^N\).

Moreover for each model discussed we include in \(X^{out}\) other higher order terms of date \(t\) variables that have not been included in \(X^{core}\). In each approximation we add in \(X^{out}\) the following terms: \(\lambda_t^2\), \((b_t^\lambda)^3\), \((b_t^\lambda)^3\), \(b_t^N\).

We now consider the *optimal repurchases* of section 6.3 in the main text, which FONCs shown in a previous subsection. The following expectations need to be approximated with PEA in this case:

\[
E_t \xi^R_{U,t+1} \quad \text{and} \quad E_t u_{c,t+i}, \quad E_t \lambda_{t+i} u_{c,t+i}, \quad i = 1, N, N - 1
\]

where \(\xi^R_{U,t}\) is the Langrange multiplier on the constraint \(R_t \leq b_{t-1}^N\).

As discussed in the text, one way to rewrite the state vector and reduce the total number of state variables is to redefine the state vector as:

\[
X_{t+1} = \left[ \begin{array}{c} g_{t+1}, B_t, B\lambda_t, (B_{t+1-i}^{net}, B\lambda_{t+1-i}^{net})_{i=1}^N, \lambda_{t-N}, b_{t-N}^N \end{array} \right]
\]

where

\[
B_t^{net} = b_{t-1}^N - R_t,
B_t \equiv b_t^S + B_{t-N+2}^R,
B\lambda_t^{net} = \lambda_{t-1}(1 - T^N)b_{t-1}^N + \lambda_t(1 + T^R)R_t,
B\lambda_t \equiv \lambda_{t-1}(1 - T^1)b_t^N + B\lambda_{t-N+2}^{net}.
\]

As for the other models we chose \(X^{core}\) and \(X^{out}\) to include the state variables which appear in the FONC of the model and which therefore exert a direct influence on the conditional expectations. We specified \(X^{core}\) as in \((19)\) and \(X^{out}\) as follows:

\[
X^{out}_{t+1} = \left\{ \{b_t^\lambda\}_{t=1,N}, \{b_{t-N+1}^N - R_{t-N+2}, (b_{t-N+1}^N - R_{t-N+2})\lambda_{t-N+1}, R_{t-N+2}\lambda_{t-N+2} \} \right\}.
\]

It is worth noting that \(B_t, B\lambda_t, (B_{t+1-i}^{net}, B\lambda_{t+1-i}^{net})_{i=1}^N\) are combinations of the variables included in this specification of \(X^{out}_{t+1}\). For example \(B_t = b_t^1 + b_{t-N+1}^N - R_{t-N+2}\) and \(B_{t+1}^{net} = b_t^N, \lambda_t - R_{t-N+2}\) are part of the state vector. Therefore we are using the state variables that are sufficient. We have chosen to separate the terms \(b_t^1\) and \(b_{t-N+1}^N - R_{t-N+2}\) in the approximations assigning \(b_t^1\) to the core and \(b_{t-N+1}^N - R_{t-N+2}\) to the out vector. We did this for convenience and most importantly to be able to use as initial guess for our approximation the solution of the no buyback model.

Moreover, notice that though in principle we could introduce \(R_t\) as a variable in \(X^{core}\) \(^7\), this is not necessary to identify the optimal path of \(R_t\). Since this is a model where the government can

\(^7\)From \((26)\) we know that \(b_{t-1}^N - R_t\) is a state variable. However, this will not appear in the FONC in periods \(t+1, t+N, t+N-1\) and for this reason we dropped it from the core state vector and from the out vector \((27)\).
repurchase only after one period and we assume positive transaction costs, we do not need $R_t$ in the core states to determine the portfolio.\footnote{In other words $R_t$ can still be identified through the budget constraint or through the nonlinear transaction costs. Had we allowed the government to repurchase more than once and if the transaction costs were assumed independent of (the vector in the case of many repurchases) $R$ we would need the control variables $R$ to be in $X^{\text{core}}$ in order to solve the model.}

Finally notice that in $X^{\text{core}}$ and $X^{\text{out}}$, the bond and repurchases variables are not multiplied by transaction costs. Since these variables (mostly) enter separately in the approximations and since the costs $T$ are small, this does not influence the properties of the solution.

Let’s now turn to the model with three bonds. When the government issues debt in three maturities ($1 < M < N$) under no buyback the FONC are given by:

$$u_{c,t} - v_{x,t} + \lambda_t \left( u_{c,t} c_t + u_{c,t} + v_{xx,t} (c_t + g_t) - v_{x,t} \right) + u_{cc,t} \sum_{i \in \{1,M,N\}} \left( \lambda_{t-i} - \lambda_t \right) b_{t-i}^i = 0$$

$$\beta^i E_t \left( u_{c,t+i} \lambda_t - u_{c,t+i} \lambda_{t+i} \right) + \xi_{t}^{i,}\xi_{t+i}^{i} = 0 \quad \text{for} \quad i = 1, M, N$$

$$\sum_{i \in \{1,M,N\}} b_{t}^{i,} \beta^i E_t u_{c,t+i} = g_t u_{c,t} + u_{c,t} \sum_{1, M, N} b_{t-i}^{i} - (u_{c,t} - v_{x,t})(c_t + g_t).$$

We need to approximate now 6 conditional expectations. We specify the ‘core’ and ‘out’ vectors as follows:

$$X_{t+1}^{\text{core}} = \left\{ 1, g_{t+1}, \lambda_t, \{b_i^1\}_{i=1, M, N}, \{(b_i^2)^2\}_{i=1, M, N}, \{g_{t+1}b_i^1\}_{i=1, M, N} \right\}$$

$$X_{t+1}^{\text{out}} = \left\{ \{b_i^1\lambda_t\}_{i=1, M, N}, \{(b_i^2)^3\}_{i=1, M, N}, \{b_i^2b_i^k\}_{i, k \in \{1, M, N\}, k \neq i}, \lambda_t^2, \lambda_{t-i}^{1,}, \lambda_{t-i+1}^{i,} \}_{i=M, N}, \{b_i^{i-1+1}\}_{i=M, N} \right\}$$

Therefore we have 12 variables in the core vector and 14 variables in the out vector.

Table 1 summarises the previous discussion on our choices for $X^{\text{core}}$ and $X^{\text{out}}$.

**B.1.2 An $R^2$ selection criterion for the elements of $X^{\text{out}}$**

Once we have chosen the composition of $X^{\text{core}}$ and $X^{\text{out}}$, we apply the following procedure:

1. In implementing the ‘Condensed PEA’ and calculating the linear combinations, we regress each variable $X_{t+1}^{\text{out}}$ on $X_{t+1}^{\text{out}}$ and $X^{\text{core}}$ and compute the R-square of the regression, $R^2_{j}$.\footnote{Moreover, since $R_t$ is always close to zero introducing it as an independent variable in the core vector leads to convergence problems. We discuss this further below.}

2. We find the variable $k$ with the highest $R^2_k$, that is $k = \arg \max_{j \in \{1, 2, \ldots, \text{length}(X^{\text{out}})\}} \{R^2_{j} \}$. If $R^2_{k} > 0.995$ we set the coefficient $\alpha_k = 0$. In other words, we set the coefficient of this variable in the first linear combination (and in all approximations) equal to zero.

3. We repeat Steps 1 and 2 removing the excluded variables from $X^{\text{out}}$ until $R^2_{k} < 0.995$.

4. We apply the ‘Condensed PEA’ to find the coefficients $\overline{\psi}_{i,j}^{m,f}$ and $\overline{\delta}_{i,j}^{m,f}$, i.e. the new fixed point in the model, with the first linear combination of the elements of $X^{\text{out}}$ which ‘survive’ Steps 1-3.
In order to decide whether a new linear combination is needed we study how much of the variation of the expression inside the conditional expectation is explained by the new linear combination. In this subsection, $R^2$ denotes this coefficient of variation. So, for example, for the Euler equation involving equation (20) in the main paper, $R^2$ now denotes how much of the variance of $u_{t+1}$ is explained by a certain linear combination of the $X^{out}$.

The rows in the table summarise the gains in $R^2$ in percentage terms for each linear combination.

5. When we recover $\gamma_{ij}$ and $\delta_{ij}$, we repeat steps 1-4 to determine which of the variables in $X^{out}$ have a non-zero coefficient in the second linear combination. We apply this procedure to all linear combinations we include to the model.

To understand why the above criterion is useful notice that when $R^2 > 0.995$, most of the variability of $X^{out}_j$ is either explained by the core state variables and/or $X^{out}_i$ is highly correlated with other variables in $X^{out}$. In the first case the residuals of the regression of $X^{out}_j$ on $X^{core}$ (required to estimate the linear combination) will be close to zero so that the variable does not add almost anything to the approximation. In the second case, the residuals will be highly correlated with the residuals of other $X^{out}$ variables. In both cases estimating the coefficients $\alpha$ becomes problematic and the convergence of the model with linear combinations becomes more difficult. Since a high $R^2$ denotes that the $k$–th variable is redundant, it helps the algorithm to converge if its coefficient is set to zero beforehand.

### Number of linear combinations used in the approximation

Tables 2 to 4 summarize the number of linear combinations we add to each approximation to some of the models considered in the paper. Consider first Table 2 which reports the results for the buyback models (under ‘no lending’, top panel and under ‘lending’, bottom panel).

<table>
<thead>
<tr>
<th>$X^{core}$</th>
<th>$X^{out}$</th>
<th>common var.</th>
<th>ad hoc</th>
<th>total</th>
</tr>
</thead>
<tbody>
<tr>
<td>BB</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>coupons</td>
<td>$1, g_{t+1}, {b^i_t}<em>{t=1,N}$, $\lambda_t, {(b^i_t)^3}</em>{t=1,N} {b^i_t}_{t=1,N}$</td>
<td></td>
<td>${b^i_t\lambda_t}<em>{t=1,N}$, $\lambda_t b^i</em>{t-1}$, $\lambda_{t-N} b^i_{t-N-1}$</td>
<td>19</td>
</tr>
<tr>
<td>callables</td>
<td>$1, g_{t+1}, {b^i_t}<em>{t=1,M,N}$, $\lambda_t, {(b^i_t)^3}</em>{t=1,M,N}$</td>
<td></td>
<td>${b^i_t\lambda_t}<em>{t=1,M,N}$, $\lambda_t b^i</em>{t-1}$, $\lambda_{t-N} b^i_{t-N-1}$</td>
<td>19</td>
</tr>
<tr>
<td>repurchases</td>
<td>$1, g_{t+1}, {b^i_t}<em>{t=1,M,N}$, $\lambda_t, {(b^i_t)^3}</em>{t=1,M,N}$</td>
<td></td>
<td>${b^i_t\lambda_t}<em>{t=1,M,N}$, $\lambda_t b^i</em>{t-1}$, $\lambda_{t-N} b^i_{t-N-1}$</td>
<td>19</td>
</tr>
<tr>
<td>3 bonds</td>
<td>$1, g_{t+1}, {b^i_t}<em>{t=1,M,N}$, $\lambda_t, {(b^i_t)^3}</em>{t=1,M,N}$</td>
<td></td>
<td>${b^i_t\lambda_t}<em>{t=1,M,N}$, $\lambda_t b^i</em>{t-1}$, $\lambda_{t-N} b^i_{t-N-1}$</td>
<td>26</td>
</tr>
</tbody>
</table>

When we recover $\gamma_{ij}$ and $\delta_{ij}$, we repeat steps 1-4 to determine which of the variables in $X^{out}$ have a non-zero coefficient in the second linear combination. We apply this procedure to all linear combinations we include to the model.

To understand why the above criterion is useful notice that when $R^2 > 0.995$, most of the variability of $X^{out}_j$ is either explained by the core state variables and/or $X^{out}_i$ is highly correlated with other variables in $X^{out}$. In the first case the residuals of the regression of $X^{out}_j$ on $X^{core}$ (required to estimate the linear combination) will be close to zero so that the variable does not add almost anything to the approximation. In the second case, the residuals will be highly correlated with the residuals of other $X^{out}$ variables. In both cases estimating the coefficients $\alpha$ becomes problematic and the convergence of the model with linear combinations becomes more difficult. Since a high $R^2$ denotes that the $k$–th variable is redundant, it helps the algorithm to converge if its coefficient is set to zero beforehand.

### Number of linear combinations used in the approximation

Tables 2 to 4 summarize the number of linear combinations we add to each approximation to some of the models considered in the paper. Consider first Table 2 which reports the results for the buyback models (under ‘no lending’, top panel and under ‘lending’, bottom panel).

In order to decide whether a new linear combination is needed we study how much of the variation of the expression inside the conditional expectation is explained by the new linear combination. In this subsection, $R^2$ denotes this coefficient of variation. So, for example, for the Euler equation involving equation (20) in the main paper, $R^2$ now denotes how much of the variance of $u_{t+1}$ is explained by a certain linear combination of the $X^{out}$.

The rows in the table summarise the gains in $R^2$ in percentage terms for each linear combination.
Table 2: Linear Combinations: Buyback Model

<table>
<thead>
<tr>
<th></th>
<th>BuyBack 'no Lending'</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$u_{c,t+1}$</td>
<td>$u_{c,t+N}$</td>
<td>$u_{c,t+N-1}$</td>
<td>$u_{c,t+1}\lambda_{t+1}$</td>
</tr>
<tr>
<td>$R^2_{aug} - R^2_{old}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$R^2_{old}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$LC_1$</td>
<td>0.0757</td>
<td>0.0169</td>
<td><strong>0.1677</strong></td>
<td>0.0441</td>
<td>0.0258</td>
</tr>
<tr>
<td>$LC_2$</td>
<td>0.0026</td>
<td>0.0228</td>
<td>0.0043</td>
<td><strong>0.0547</strong></td>
<td>0.0417</td>
</tr>
<tr>
<td>$LC_3$</td>
<td>0.0259</td>
<td>0.0232</td>
<td>0.0234</td>
<td>0.0060</td>
<td>0.0308</td>
</tr>
<tr>
<td>Total</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

|                    |                      |                      |                      |                      |                      |
| $R^2_{aug} - R^2_{old}$ |                      |                      |                      |                      |                      |
| $R^2_{old}$        |                      |                      |                      |                      |                      |
| $LC_1$             | 0.0081               | 0.0451               | 0.0403               | 0.0385               | 0.0322               |
| Total              | 0                    | 0                    | 0                    | 0                    | 0                    |

Note: The table shows the number of linear combinations in the buyback models (‘no lending’, top panel and ‘lending, bottom panel). The columns list the conditional expectations we approximate in these models. The rows report the percentage gains in $R^2$ from adding a further linear combination to the model. Hence row $LC_1$ shows the gains when we compare the regressions with $X^{core}$ only ($R^2_{old}$) to the regressions with $X^{core}$ and one linear combination ($R^2_{aug}$). In row $LC_2$ $R^2_{old}$ derives from a regression on $X^{core}$ and the first linear combination and $R^2_{aug}$ adds a second linear combination and so on.

We denote in bold values of $\frac{R^2_{aug} - R^2_{old}}{R^2_{old}} \times 100$ which exceed the 0.05 percent threshold (above which we introduce an additional linear combination to the model).

$R^2_{aug}$ is the value of the coefficient of variation we obtain when we include an additional linear combination to the model. $R^2_{old}$ the coefficient of variation without the additional linear combination. The row labeled $LC_1$ corresponds to the ‘Condensed PEA’ test when we solve the model only with the $X^{core}$ variables and test the inclusion of the first linear combination. $LC_2$ tests the significance of the second linear combination and so on.

We add a further linear combination to an approximation when

$$R^2_{diff} = \frac{R^2_{aug} - R^2_{old}}{R^2_{old}} \times 100 > 0.05,$$

in other words when the gain in $R^2$ is greater than 0.05 percent.

As Table 2 shows the buyback model under ‘no lending’ requires one linear combination. The approximations of $E_t u_{c,t+1}$ and $E_t u_{c,t+N-1}$ include a linear combination in the first round and the approximation of $E_t u_{c,t+1} \lambda_{t+1}$ includes one in the second round. In the buyback ‘lending’ model the importance of the $X^{out}$ variables is limited and so this model does not require any linear combinations.

Table 3 reports the analogous findings in the no buyback models and Table 4 for the case of coupons. Each of these models is solved with linear combinations.
Table 3: Linear Combinations: No-Buyback model

<table>
<thead>
<tr>
<th></th>
<th>$u_{c,t+1}$</th>
<th>$u_{c,t+N}$</th>
<th>$u_{c,t+1} \lambda_{t+1}$</th>
<th>$u_{c,t+N} \lambda_{t+N}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R^2_{aug} - R^2_{old}/R^2_{old} \times 100$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>LC1</td>
<td>0.0173</td>
<td>0.0166</td>
<td><strong>0.0561</strong></td>
<td><strong>0.0646</strong></td>
</tr>
<tr>
<td>LC2</td>
<td><strong>0.0578</strong></td>
<td>0.0112</td>
<td>0.0109</td>
<td>0.0035</td>
</tr>
<tr>
<td>LC3</td>
<td>0.0002</td>
<td>0.0194</td>
<td>0.0058</td>
<td>0.0001</td>
</tr>
<tr>
<td>Total</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Note: The table shows the number of linear combinations in the no buyback models (’no lending’, top panel and ’lending, bottom panel). See Table 3 for details.

Table 4: Linear Combinations: No-Buyback Model with Coupons

<table>
<thead>
<tr>
<th></th>
<th>$u_{c,t+1}$</th>
<th>$q_{c,t}$</th>
<th>$u_{c,t+1} \lambda_{t+1}$</th>
<th>$q_{\lambda,c,t}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R^2_{aug} - R^2_{old}/R^2_{old} \times 100$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>LC1</td>
<td>0.0067</td>
<td>0.0154</td>
<td>0.0213</td>
<td><strong>0.0654</strong></td>
</tr>
<tr>
<td>LC2</td>
<td>0.0048</td>
<td>0.0108</td>
<td>0.0077</td>
<td>0.0011</td>
</tr>
<tr>
<td>Total</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Note: The table shows the number of linear combinations in the no buyback model with coupons. See Table 3 for details. $q_{c,t} \equiv \sum_{j=1}^{N} \beta^j u_{c,t+j} + \beta^N u_{c,t+N}$ $q_{\lambda,c,t} \equiv \sum_{j=1}^{N} \beta^j u_{c,t+j} \lambda_{t+j} + \beta^N u_{c,t+N} \lambda_{t+N}$
The picture is similar when we consider the models not included in the tables: the callable bonds, three maturities and the model with optimal repurchases. The callable bond model needs one linear combination to be added to get accurate solutions. For the three maturities model we find that $X^{core}$ is sufficient and therefore we do not include any linear combinations.

The optimal repurchase model requires two linear combinations to be accurately solved. The first linear combination is introduced to the approximations of $E_t u_{c,t+i} \lambda_{t+i}$, $i = 1, N, N - 1$ and $E_t \xi_{t+1}^R$. The second linear combination is introduced to the approximation of $E_t u_{c,t+1}$.

### B.2 Some practical features of the numerical implementation

#### B.2.1 Dealing with occasionally binding constraints on debt

As explained in the main text, we impose an upper and lower bound on the issuance of short and long bonds. In a two bond model we have in total four constraints. These constraints are only occasionally binding and in theory we could use the approach explained, for example, in Marcet and Singleton (1999) to deal with them. Suppose that the government can issue only one bond, whatever the maturity. Marcet and Singleton suggest for every period $t$ first to solve the unconstrained problem and check whether one of debt constraints is violated. If it is violated, the value of the bond is set equal to the value of the debt limit and $c_t$ and $\lambda_t$ are recalculated accordingly.

Unfortunately, this cannot be easily applied in the case of more than one maturity because of the number of constraints involved. If one of the constraints is violated when solving the unconstrained problem, we need to verify that forcing the constraint is not going to generate a violation of one of the constraints on the other bond. This problem presents too many cases to be checked one by one and the computational burden increases considerably when an additional maturity is introduced to the model. For this reason we impose the following (quadratic) costs when the bonds violate the limits in the buyback model:

$$
C(b_i^t) = \begin{cases} 
\frac{\phi_1}{2} \left( b_i^t - \frac{M_i}{\beta_i} \right)^2 & \text{if } b_i^t > \frac{M_i}{\beta_i} \\
\frac{\phi_1}{2} \left( \frac{M_i}{\beta_i} - b_i^t \right)^2 & \text{if } b_i^t < \frac{M_i}{\beta_i} \\
0 & \text{otherwise}
\end{cases}
$$

for $i = 1, N$. $\phi_1$ governs the penalty from deviating from the debt limits $\frac{M_i}{\beta_i}$ and $\frac{M_i}{\beta_i}$. We choose a value of $\phi_1$ equal to unity. Analogous cost functions are used in the no buyback and coupons models, the debt limits have to be adjusted in these cases as described in text.

In the optimal repurchase model we have an additional constraint on the level of repurchases: $0 \leq R_t \leq b_{i-1}^N$. In this case we continue to impose $C(b_i^t)$ for $i = 1, N$ however we use Lorenzoni and Marcet’s approach to deal with this extra constraint. When $R$ violates a limit (either because $R_t < 0$ or $R_t > b_{i-1}^N$) we fix the value of $R_t$ to the constraint and solve the FONC to determine the optimal portfolio and the value of the multipliers, $\xi_{L,t}^R$ and $\xi_{U,t}^R$.

#### B.2.2 Initial conditions and samples’ size
In order to generate a more precise approximation of the policy functions over the debt space we use PEA with oversampling. We choose 25 different initial conditions for the debt levels $b_{1-1}$ and $b_{N-j}$, where $j = 1, 2, .. N - 1$ uniformly distributed in the interval $[M_i/\beta_i, M_i/\beta_i]$ (e.g. in the buyback model). We draw 10 samples of 500 periods for each initial condition. The total number of observations is then 125000.

Given the initial conditions for the portfolio, we also need to specify some initial values for the $\lambda$’s. For this purpose we recover initial values $\lambda_{N-1}^t = ..., \lambda_1^t$ that would be consistent with the deterministic steady state. As is well known in steady state the debt level in these models is indeterminate and so we can obtain a different $\lambda$ (consistent with a different $c$) for each bond vector. Under no buyback we obviously need to set $b_{N-1}^t = b_{N-2}^t = ... = b_{N-N+1}^t$ to be in steady state.

B.2.3 Rescaling

To improve the stability of the algorithm, we rescale the variables which enter in $X_{core}$ and $X_{out}$. For example we use $\frac{b_i^t}{\beta_i}$ and $\frac{\lambda_i^t}{\lambda_s}$ instead of $b_i^t$ and $\lambda_t$. This is applied to every lag of the independent variables used in the approximation. We also rescaled the dependent variables in the PEA regressions by their steady state values such that their means are close to one in the approximations. For example, we regress $\frac{u_{c,t+1}}{\lambda_s}$ and $\frac{u_{c,t+1}^*}{\lambda_s^*}$ on $X_{core}$ and the linear combinations, to obtain the approximations of $E_t(\frac{u_{c,t+1}}{\lambda_s})$ and $E_t(\frac{u_{c,t+1}^*}{\lambda_s^*})$ respectively. The same is done for the other expectations.

Rescaling is useful because some of the coefficients could be very small without it. For example, consider the buy back no lending model; $b_N^t$ can fluctuate in simulations between 0 and $\frac{M_i}{\beta_i} \approx 117$ and its square between 0 and $117^2$. It is obvious that the estimated coefficients of these terms may be close to zero. This makes it difficult to find a reliable convergence criterion for the model. Through rescaling the state variables fluctuate between 0 and 1. This improves significantly the stability of our algorithm (see also Judd et al (2011a)).

B.2.4 Convergence of PEA - Finding Good initial conditions of the coefficients

Den Haan and Marcet (1990) show that PEA does not guarantee convergence. Convergence is more likely if we use good initial conditions for the coefficients. This is even more necessary in the context

\[ \text{Converge if } \frac{|\gamma_{N,1} - \gamma_{N,0}|}{|\gamma_{N,0}|} < \epsilon \]

and $\gamma_{N,1}, \gamma_{N,0} \approx 0$, then the behavior of (28) will be very erratic (both very high and very low values are possible, and this does not tell us much about convergence of the model’s quantities). Analogously, if we use the convention

\[ \text{Converge if } \frac{|\gamma_{N,1} - \gamma_{N,0}|}{1 + |\gamma_{N,0}|} < \epsilon \]

for some $\epsilon$, then the algorithm may (wrongly) converge after a few iterations.

In our codes we employ the criterion (29), but since the variables are rescaled, we are sure that coefficients which are small in values, do not matter much for the optimal policy.
of the optimal portfolio under incomplete markets. If the initial coefficients constitute a very poor guess of the equilibrium of the model, then the algorithm may circle for a long time and subsequently diverge.

Good initial conditions for portfolio choice models can be obtained as follows:

1. Solve portfolio models with positive transaction costs.

   For example consider solving the Ramsey problem under buyback subject to the following government budget constraint:

   \[
   \sum_{i=1}^{N} p_i^t b_i^t = \sum_{i=1}^{N} p_i^{t-1} b_{i-1}^t + g_t - \left(1 - \frac{u_{x,t}}{u_{c,t}}\right) (c_t + g_t) + \sum_{i=1}^{N} \omega_i (b_i^t)^2
   \]

   where \(\omega_i (b_i^t)^2\) is a transaction cost paid by the government at issuance.\(^{12}\) It is obvious that in this model the optimal portfolio is determinate (even with the conventional PEA). In the limit when \(\omega_i \to 0\) we obtain the buyback model considered in this paper, if \(\omega_i \to \infty\) there is no trade in bonds. Hence, good initial conditions can be found from solving models with positive transactions costs and gradually reducing \(\omega_i\) till 0.

2. Solve models under tight debt limits and gradually loosen them.

   We found that models with tight debt constraints converge more easily than models with looser ones. Generally speaking, models with very loose debt constraints can converge to a wrong equilibrium which features for example a constant \(\lambda\) as in the case of complete markets. This holds in particular because running the models with samples of 500 observations may imply that the debt limits are rarely hit, if they are very loose.

   To illustrate this property we consider two simple examples under very poor initial conditions for the PEA.

   (a) Suppose that the initial guess for the polynomials is \(E_t u_{c,t+i} = X_{t+1}^{core} \gamma^i = \gamma_0^i + \gamma_1^i E_t g_{t+1} + \gamma_2^i b_1^t + \gamma_3^i b_N^t + \gamma_4^i \lambda_t\) and \(E_t \lambda_{t+1} u_{c,t+i} = \lambda^* (X_{t+1}^{core} \gamma^i)\). Assume that under very loose bounds (e.g. \(M = -M = \infty\)) for every \(t\) we get \(\lambda_t = \lambda^*\) as a solution to the system of FONC.

   (b) Suppose that \(E_t \lambda_{t+1} u_{c,t+N} = \alpha \lambda^* (X_{t+1}^{core} \gamma^N)\) where \(\alpha < 1\) and \(E_t \lambda_{t+1} u_{c,t+1} = \lambda^* (X_{t+1}^{core} \gamma_N)\). Suppose further that \(M = 0, \bar{M} = \infty\). For every \(t\) the optimal portfolio features \(b_N^t = 0\).\(^{13}\)

Under both (a) and (b) the algorithm will not converge. When we update the coefficients \(\gamma\), the independent variable \(\lambda_t\) \((b_N^t)\) is constant under (a) (b), so that the least squares problem cannot be solved. Assuming tight bounds can mitigate this problem. To see this note that when a bound is hit under a) the Euler equations do not hold. The value of \(\lambda_t\) is recovered from the FONC of consumption and generally it will be that \(\lambda_t \neq \lambda^*\). This introduces variability in \(\lambda_t\). Similarly under (b), when \(b_1^t\) reaches its upper bound we will have \(b_N^t > 0\), so that \(b_N^t\) is not constant and equal to zero for every \(t\).

\(^{12}\)Since this cost does not distort the household’s program, it does not show up in bond prices.

\(^{13}\)Since the Euler equations cannot hold simultaneously, there are two possible solutions. Either \(\lambda_t = \lambda^*\) or \(\lambda_t = \alpha \lambda^* < \lambda^*\). But the latter case \((which\ would\ give\ b_N^t > 0)\) can be ruled out since \(b_1^t\) must be at the upper bound which is assumed arbitrarily loose here.
Note that though the above examples are extreme, they are not completely unlikely. It is not uncommon to initiate coefficients with steady state values, or choose values so that one of the bonds is always at the lower bound. When the bounds are loose, poor initial conditions may make the PEA circle or diverge. Assuming tight bounds helps the algorithm converging. The converged coefficients can then be used as initial conditions for models with looser bounds and so on.

A similar argument, showing the importance of ‘moving bounds’ for convergence in the PEA, was made by Maliar and Maliar (2003).

B.2.5 Calculating the sample moments

As discussed in the text, to compute the moments reported in Table 4 in the paper, we simulated the model 1000 times using as initial conditions the values of $S_t$ and the market value of debt, we recovered from the data. In 1955 the share of short debt equaled 39% and the initial debt to GDP ratio was 38% in the CRSP sample.

We then computed the values $b_{1,-1}$ and $b_{N,-j}$, $j = 1, 2, ..., N$ in the deterministic steady state such that the initial share and market value of debt are consistent with these targets. For example in the buyback model we have

$$\frac{\beta b_{1,-1}}{\beta b_{1,-1} + \beta^N b_{N,-1}} = 0.39 \quad \beta b_{1,-1} + \beta^N b_{N,-1} = 0.38 * 70$$

The analogous expressions for the other models are omitted for brevity.

Given the initial conditions for the bonds, we found the initial values of $c$ and $\lambda$ to satisfy the FONC of consumption and the government budget constraint in the deterministic steady state.

We then simulated the models and computed the market value of government debt and the share of short bonds. Notice that whereas in the buyback models to construct the market values for short and long bonds it is sufficient to use the approximations of $E_t u_{c,t+1}$ and $E_t u_{c,t+N}$, in the no buyback model this is not the case. In particular we need to compute the value of non-matured debt in period $t$. This requires all the prices $p^t_j$ for $j = 2, 3, ..., N - 1$. Since these prices do not affect the equilibrium properties of optimal allocations, we computed the approximations through simple regressions of $u_{c,t+i}$ on $X_{t+i}$ once our algorithm has converged. In other words, we do not use the ‘Condensed PEA’ to calculate these bond prices since these approximations are performed when the algorithm has converged. The model’s convergence properties and the equilibrium allocation do not depend on them.

Finally, note that because the model is solved with quadratic costs if the debt limits are violated, as described in subsection B.2.1, the market value of government debt can become (slightly) negative in the no lending models, in some periods and samples. The statistics reported in Table 4 in the paper are calculated after dropping samples where the market value becomes negative in ‘no lending

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14We used all the elements of $X_t$ not just the ones included in $X_t^{out}$ above. In other words we have all bonds and cross terms $b_{N,t-j}$ and $b_{N,t-j} \lambda_{t-j}$ respectively for $j = 0, 1, ..., N - 1$, in these regressions. The logic is the same as described previously. Since all of these terms will appear in the first order conditions in periods $t + 2, t + 3, ..., t + N - 1$ we can claim that they exert a direct impact on the conditional expectations. To simplify we used the same state vector for all the additional conditional expectations we wanted to approximate. This is possible because the stability properties of our algorithm do not depend on the extra terms.
models’. For the same reason in order to avoid having a negative share of short debt in simulations (if say \( b_1^t < 0, b_N^t > 0 \) or greater than unity (i.e. when \( b_N^t < 0, b_1^t > 0 \)) we computed the moments using \( \min \{ \max \{ S_t, 0 \}, 1 \} \) in the no lending models: we forced the share to be equal to zero when it was negative and 1 when it exceeded unity. This adjustment obviously was much more frequent in the buyback ‘no lending model’ than in the no buyback model.

B.2.6 Some limitations of the PEA

We cannot claim that the numerical algorithm we propose in this paper can solve every portfolio choice problem under incomplete markets. To make this point, we describe here a few cases where the approximation of conditional expectations under ‘Forward States’ may not compute accurately equilibria with multiple assets.

The first noteworthy difficulty of our methodology is that as the number of assets increases the optimal allocation may be close to the complete markets’ one. Recall that in this case the portfolio and the multiplier \( \lambda \) are constant through time. Clearly, such equilibria cannot be approximated with polynomials of the form \( E_{t+1} u_{t+1} = \gamma_1 + \gamma_1 E_{t+1} g_{t+1} + \gamma_2 b_{1, t} + \gamma_3 b_{N, t} + \gamma_3 \lambda_t + \ldots \); if the RHS variables are roughly constant, the estimation of the polynomial coefficients will not be reliable. Our algorithms are designed to deal with cases where markets are incomplete, this involves either few assets, or tight debt constraints or both.

Second, even under incomplete markets if the government can trade three or more maturities we cannot rule out equilibria with multiple assets and realistic frictions (e.g. imperfect substitutability among the assets). Small transaction costs, bond clienteles and preferences for short term (safe) assets, will give well defined demand curves for each maturity and these are realistic features of government debt markets. Our methodology is therefore

15Recall that one of the main findings of the paper is that under no buyback long and short debt levels comove strongly. This property also holds for the issuances \( b_1^t \) and \( b_N^t \). It is therefore rare that \( b_1^t \) is slightly negative and \( b_N^t > 0 \). If this occurs in our simulations it is likely that the overall market value is slightly negative in which case the sample is dropped as described previously.

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However, under buyback and no lending, we frequently have \( b_N^t >> 0 \) and \( b_1^t \approx 0 \) so that small negative values of short debt can occur. In these cases we set the short term share to 0.

16If the debt limits on \( b_M^t \) are sufficiently loose, this strategy is feasible.
broadly applicable to solve models with many assets, the limitations described in this subsection arise because our model is a simplistic one and abstracts from several realistic frictions.

**B.3 Accuracy of the solutions**

To check the accuracy of the solution we run Euler Equation Error (EEE) tests (see for example Arouba et al (2006) for an exhaustive description of the methodology).

The tests require to numerically calculate each of the conditional expectations in the Euler equations, using the approximated policy functions. Since we have expectations up to \( N \) leads, we use Montecarlo integration to approximate the integrals. We simulate our model for 450 periods and for each of the 250 samples (25 initial conditions times 10 samples per initial condition) using our approximation. We discard the first 200 periods of each sample, and for each subsequent period, we draw \( k = 10000 \) different shock paths of length \( N \). We solve our model for each shock path separately using our approximated policy functions. For example for the buy back and no buy back models we then compute the numerical expectations as:

\[
\Xi_{t,1} = E_t \left( u_{c,t+1} \right) = \frac{\sum_{i=1}^{k} u_{c,t+1}}{k} \\
\Xi_{t,2} = E_t \left( u_{c,t+N} \right) = \frac{\sum_{i=1}^{k} u_{c,t+N}}{k} \\
\Xi_{t,3} = E_t \left( u_{c,t+1} \lambda_{t+1} \right) = \frac{\sum_{i=1}^{k} u_{c,t+1} \lambda_{t+1}}{k} \\
\Xi_{t,4}^{NBB} = E_t \left( u_{c,t+N} \lambda_{t+N} \right) = \frac{\sum_{i=1}^{k} u_{c,t+N} \lambda_{t+N}}{k} \text{ or } \Xi_{t,4}^{BB} = E_t \left( u_{c,t+1} \lambda_{t+1} \right) = \frac{\sum_{i=1}^{k} u_{c,t+1} \lambda_{t+1}}{k}
\]

Since we have two Euler equations, we checked separately each of them, calculating the value of the multiplier in period \( \tilde{t} \) implied by the expectations \( \Xi_{t} \) and given the portfolio \( b_{1,t}, b_{N,t} \). We then calculate the implied consumption error using the FONC of \( c_t \). The results from the accuracy tests are stated (following the relevant literature) in terms of consumption deviations, this enables us to give an economic interpretation to the errors.

In particular we compute the following quantities:

\[
EEE_{t}^{1} = \frac{\tilde{c}_{t} - c_{t}}{\tilde{c}_{t}} \\
EEE_{t}^{N} = \frac{\tilde{c}_{t}^{N} - c_{t}}{\tilde{c}_{t}^{N}}
\]

where \( \tilde{c}_{t} \) is the consumption implied by the new approximation of the expectations and \( c_{t} \) the one implied by our approximation. We compute the average EEE across all samples and initial conditions, the maximum error and the percentage of positive and negative errors. We average over 62500 errors.

As in Aruoba et al. (2006) we report the *absolute* errors using base 10 logarithms to make our

\[^{17}\text{For the 3 bond model we check the errors every 5 periods. This choice was made for computational purposes because in this model we have 125 initial conditions and 10 samples per initial condition.}
\[^{18}\text{Since the optimal portfolio is determined through 'Forward States' it is not possible to use objects } \Xi_{t} \text{ to determined new values of } b_{1,t}, b_{N,t}.\]
findings comparable with the rest of the literature. A value of -3 means a 1$ mistake per 1000$, a value of -4 a mistake of $1 per $10000 and so on. Table 5 reports the results.

Table 5 shows that the average of the errors are between -3 and -4, that the percentage of positive errors is close to 50% and that the maximum errors are not large. Moreover, we found that it is quite unlikely that region of the state space where the maximum error occurs is visited in simulations. These results and well within the range accepted by other authors (e.g. Aruoba et al (2006)). This suggests that the model solutions are accurate.

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