Optimal Repeated Purchases When Sellers Are Learning about Costs

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A buyer repeatedly purchases some good. Suppliers privately learn their cost only upon producing at least once. Efficiency would imply sampling sellers until one is found with cost lower than a (increasing with time) reservation value. Then the good would be permanently purchased from the seller with lowest cost. A sequence of second-price auctions with participation premia and entry fees is shown to be both efficient and optimal for the buyer. An alternative calls for price offers by informed sellers, with compensations that increase in the offer. In both cases, the buyer subsidizes competition through participation premia or compensations. Journal of Economic Literature Classification Numbers: D44, D83, L14.

1. INTRODUCTION

One of the important concerns of large manufacturers, like auto makers, is their relationship with suppliers. Many aspects are involved, for example, reliability, specific investments, incentives for cost reduction and R & D, or risk sharing. Here, I consider one of these important aspects, learning about suppliers' efficiency. Suppose a new part is to be produced according to the manufacturer's specifications. From the set of suppliers, one has to be selected to do the job. However, precise information about a supplier's cost of producing the part is obtained only after this supplier has actually been involved in production, and therefore a sampling process has to be designed. On the other hand, part of the information learned in this process is private to the supplier, and therefore payments and compensation should be designed to make the sampling process incentive compatible. The

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question we study in this paper is what types of contract attain these, possibly conflicting, goals.¹

To answer this question, we model the problem faced by a buyer who has to buy one unit of a good in each period from one of a set of potential sellers, and has the ability to set and commit to the buying mechanism. We assume that these potential sellers will (privately) learn their cost only after producing the good once. This cost can be different for different sellers.

We first show that the (first best) efficient sampling rule is a search process with reservation values that increase over time. The sampling process stops at the first period in which some (experienced) seller’s cost is below the reservation value for that period. From then on, the good is permanently bought from the experienced seller with lowest cost. Then, we show that this sampling rule can be implemented and that the buyer extracts all the rents. We present two mechanisms that attain this goal. The first is a sequence of auctions. In each period, unless it has been allocated already, the right to permanently sell the good is auctioned in a second-price auction. There is a maximum acceptable bid equal to the reservation value that corresponds to that period in the efficient sampling rule. If no (acceptable) bid is submitted, the buyer buys from a new seller, and the process repeats in the next period.² If there are bids, the winner becomes the permanent seller and those who submit a bid receive a fixed premium independent of the outcome.

The second, alternative mechanism works as follows: the buyer tries a seller and, at the end of the period, asks this seller to make a price offer. Low offers (which are below both the reservation value of the period and all previous offers) are accepted and the buyer sticks to that seller forever. Higher offers (below previous offers, but above the reservation value of the period) are kept for possible acceptance in the future, and then a new seller is tried in the next period. The owner subsidizes these offers (the lower the offer, the higher the subsidy). Finally, if the offer is high (above some past offer), then it is immediately turned down. In this last case, a new seller is tried too, unless the best of the past offers is accepted, i.e., unless the (higher) reservation value of this period is already above that past offer.

The “double sourcing” system, as used in the automobile industry, could be an example of this type of mechanism. There, a main supplier is in charge of a large proportion of the production, while secondary (less

¹ One can think of other examples in which these two goals have to be reconciled: the allocation of franchise contracts, the assignment of jobs to internal workers, etc.

² We assume that the buyer can commit to the mechanism to be employed. In the “changing principal framework” (see Laffont and Tirole [2]), we need only assume that the principal must refund any amount received if it reneges. Indeed, the sequence of auctions implements the (interim) efficient sampling rule, so that reneging and refunding in this case will not be in the buyer’s interest.
efficient, ex ante) suppliers produce the rest. Indeed, assume a certain percentage of the total production of the part (say 10%) is needed for learning a supplier's efficiency (cost). Then this 10% could be thought of as the good to be bought. In every period the main supplier would be the most efficient of the suppliers sampled up to date. However, that 10% of the total production would be assigned to new suppliers, as long as the expected benefit from this sampling is higher than the loss represented by the higher cost incurred.3

The salient feature of these mechanisms is the use of subsidies to foster competition. Once a supplier has produced the part and learned its own cost, this supplier enjoys an information advantage vis-à-vis the manufacturer. Suppliers then have an incentive to understate their efficiency in the hope that bad realizations of future suppliers returns the job to them under better price conditions. Therefore, in order to avoid this strategic behavior, the manufacturer (buyer) subsidizes good offers or bids by all suppliers, even those who are not eventually selected. These subsidies can best be accomplished as part of a long-term relationship (for instance, by conceding less important contracts to losers).

It is worth noting that the (interim) information asymmetry does not impose efficiency costs in our case4: the buyer can control the sellers' access to the information, although not the information itself, and therefore extracting the rents is not a problem. What is important is to set the sampling process in an incentive compatible way. McAfee and McMillan [5] study a model with that same feature. In their model an exogenous, fixed search cost drives the sampling process: in looking for the seller, the buyer has to weigh this cost of communicating with a candidate against the benefit of possibly having a good realization of the production cost. The solution is a standard search problem with constant reservation value.5 In

3 The implication is a lot of turnover in the position of secondary suppliers and some possible substitution in the post of main supplier.

4 Luton and McAfee [3] consider a similar problem where, in contrast to our case, sellers know their (first-period) cost before the contract is signed. Also, a seller can have different costs in different periods, and therefore sampling is not the driving force. Then, the principal, at a cost in efficiency, discriminates against the incumbent in order to reduce his informational rents.

Our paper is also related to Bergemann and Valimaki [1]. In their model, both buyer and sellers symmetrically learn about the (heterogeneous) quality of the good by experiencing it, although the (Bayesian) learning takes more than just one period. Sellers set the prices of their good and the buyer chooses which of the sellers to sample in each period. They show that in equilibrium sellers set prices in a way that first-best sampling is attained and sellers subsidize the sampling of the buyer by setting prices below marginal cost, in the hope that good realizations make the buyer continue buying their good in the future.

5 If all sellers have been sampled without finding one with a cost below the reservation value, then the seller is decided in an auction with a reserve price.
our model, the cost of sampling one more seller is also constant and is represented by the need for having a new seller produce the good for a period. However, in the finite horizon case that we consider, the benefit from one more period of sampling declines with time. Sampling means the possibility of finding a better realization, which will then be realized in all future periods. As time passes, the number of these future periods decreases and therefore the benefit of sampling declines too. Thus, contrary to the solution proposed by McAfee and McMillan, in our mechanisms the reservation values increase with time, and so the buyer could end up (permanently) buying from a seller even if after its trial period some more sellers (not necessarily all) were sampled. In the infinite horizon case, on the other hand, our sampling process is equivalent to that of McAfee and McMillan.

The paper is organized as follows. In Section 2 we present the model and the efficient sampling rule under symmetric information. Section 3 analyses the two mechanisms described above, a sequence of second-price auctions, and a sequence of offers by the sellers. Section 4 concludes the paper.

2. Efficient Sampling of Sellers under Symmetric Information

Let \( T \) be the number of periods in which the good is to be purchased, and let \( I = \{1, 2, \ldots, n\} \) be the set of potential sellers. In general, let \( c_i \) be the (constant) per-period cost of producing the good for seller \( i \), and \( c = (c_1, c_2, \ldots, c_n) \). At time zero, all the \( c_i \) are commonly known to be realizations of i.i.d. random variables with distribution mean \( \mu \) and a c.d.f. \( F \), which we assume to be continuous and strictly increasing in some real interval \( C = [\underline{c}, \overline{c}] \), with density \( f \). Seller \( i \) privately observes \( c_i \) during the first period he produces the good. In particular, the buyer does not observe \( c_i \). We assume risk neutrality on the part of all agents. Also, we assume a common discount factor \( \delta \leq 1 \). Both time and sellers will be indexed by subscripts.

Before investigating what is the best purchasing policy for the buyer in this setting, let us start by studying what would be efficient in an environment with no asymmetry of information. That is, assume for now that the buyer can also observe a seller’s cost when the seller does and let us characterize the efficient purchase policy in this symmetric information case. Thus we start by defining a sampling rule as the mapping that determines who produces (sells) the good each period as a function of all the information available at that period. Formally, a sampling rule \( \gamma \) is a sequence \( \{\gamma_t\} \), defined inductively as:

(a) \( \gamma_1 = 1 \), let \( n_1 = 1 \).

(b) For \( t > 1 \), let \( n_t = \max_{t \leq \tau < t} \{\gamma_\tau(c_1, \ldots, c_{n_\tau})\} \); then \( \gamma_t : C^{n_t} \rightarrow \{1, \ldots, n_t + 1\} \), where \( C \) is the \( n_t \)-fold Cartesian product of \( C \).
The value of $\gamma_t$ tells us the seller who sells the good in period $t$ and $n_t$ tells us how many and which sellers have been sampled (i.e., have produced the good) before time $t$. Since all sellers are equal ex ante, we assume that every time the good is to be bought from a new seller, it will be bought from the next new seller in the integer order. $\gamma_t$ depends on the costs already sampled before $t$, i.e., the costs observed by seller 1 through $n_t$. This is all the information available at time $t$.

Given a realization of the cost vector $c$ and a sampling rule, the sequence of sellers and the sequence of costs actually realized from period 1 to period $T$ are determined. A sampling rule is (ex ante) efficient if it minimizes the sum of this sequence (in expected terms). We call this expected sum the total cost induced by the sampling rule, and represent it as $S(\gamma)$. That is,

$$S(\gamma) = E\left[\sum_{t=1}^{T} \delta^{t-1}c_{\gamma_t}\right],$$

where the expectation is taken with respect to the c.d.f. $F$ and $\gamma_t = \gamma(c_1, ..., c_n), \text{ for } t > 1$. The following proposition characterizes the sampling rule that minimizes $S(\gamma)$.

**Proposition 1.** The efficient sampling rule $\gamma^*$ is characterized by a monotone increasing sequence of real numbers $z_1, z_2, ..., z_{T-1}$ in $C$, such that for all $t \leq T$,

(a) if $c_1, ..., c_{n_t} > z_{t-1}$, then $\gamma_t(c_1, ..., c_m) = t,$

(b) if $c_1, ..., c_{n_t-1} > z_{t-2}$, but $\min_{i \leq n_t} \{c_i\} \leq z_{t-1}$, then

$$\gamma_t(c_1, ..., c_m) = \arg \min_{i \leq n_t} \{c_i\}.$$  

**Proof.** See Appendix.

In words, there exists a sequence of increasing values $z_t$ such that the production of the good is assigned to new sellers until one is found with a cost lower than the $z_t$ corresponding to the period and/or this $z_t$ is higher than the cost of the best among the experienced sellers. When this happens, the good is produced in the remaining periods by the experienced seller with the lowest cost. Thus, $\{z_t\}$ is a sequence of stopping points for the sampling process, one for each period. Therefore, if the buyer has observed a cost equal to $z_1$ in period 1, she should be indifferent about whether or not she samples once more. But, since $z_2 \geq z_1$, optimal sampling will

If $T > n$, then we define $\gamma_t : C^n \rightarrow \{1, ..., n\}$, whenever $n_t = n$ (i.e., after all potential sellers have produced the good, no new seller remains and, therefore, from then on the seller has to be an experienced one).
involve stopping after one more observation and buying from seller 1 forever, or buying from seller 2 forever if seller 2 has a lower cost than seller 1. The same can be said with respect to any $z_t$; the buyer must be indifferent about whether or not to sample for exactly one more period. Therefore, $z_t$ solves

$$
(1 - \delta^{T-t}) \left( 1 - \delta \right) z_t = \mu + \delta (1 - \delta^{-(t+1)}) \int_c^\infty \min\{z_t, x\} f(x) \, dx.
$$

There is also a straightforward relationship among the sequences $\{z_t\}$ that define the efficient sampling rule for different values of $T$. Indeed, let us use superscripts to refer to the number of periods. If $T < n$, notice that $z_{T-1}^T$ solves the same equation as $z_T^T$, and therefore $z_T^T = z_{T-1}^{T-1}$.

When $T > n$, of course, the sampling process cannot go beyond period $n$ (and so, $z_t = \hat{c}$ for $t \geq n$). Apart from that, $z_t \leq \mu$ in all cases.

In the next section, we show that, for any $T$, this optimal sampling rule can be obtained as the outcome of the two mechanisms described in Section 1 even when only the seller observes his cost after producing the good for one period.

3. Sequential Auctions and Sequential Offers under Asymmetric Information

In this section we present two mechanisms whose outcome is the efficient sampling rule $\gamma^*$. We assume no restriction on the buyer’s use of entry fees. Therefore, since there is no ex ante information asymmetry and the buyer can set entry fees as part of the contract offered to prospective sellers (thereby extracting all the rents), these mechanisms will be optimal also from the buyer’s viewpoint. That is, the ability to set entry fees reduces the search for optimal mechanisms (from the buyer’s viewpoint) to a search for mechanisms that obtain efficient outcomes independently of the division of the surplus, which is then extracted via these entry fees.

We start by studying the following auction mechanism: Let $\{P_t, A_t, B_t\}$ for $t = 1, \ldots, T$ be a sequence of lump sum payments. The buyer begins by charging seller 1 $P_1$ for the right to sell to the buyer at a price $\mu$ in the first period then participate in any future auction that the buyer might hold. At the end of the period, the buyer offers the seller a payment $A_1$ if he is willing to supply the good for the remaining $T-1$ periods at a price $z_1$. If the first seller refuses, the buyer charges the second seller $P_2$ for a deal similar to the one offered to the first seller. At the end of the second period the buyer offers seller 1 $A_2$ and seller 2 $B_2$ to submit prices below $z_2$, at which they are willing to supply the good for the remaining $T-2$ periods.
If either seller bids a price below $z_2$, the buyer agrees to purchase from the low bidder and to pay a price equal to $z_2$ or the second lowest bid, if more than one seller bids. If neither seller submits a bid below $z_2$, then the seller repeats the process with seller 3, and so on. At the end of any future period $t$ in which a new seller has provided the good this new seller is offered $B_t$ to submit bids, whereas the offer to old sellers is $A_t$.

Again, notice that this mechanism implements $\gamma^*$, provided that all accept the corresponding deals and all use a truthful bidding strategy (bid at $t$ when their value is below $z_t$ and bid an amount equal to their valuation). The first result in this section shows that truthful bidding is, indeed, a perfect Bayesian equilibrium when $A_t$ and $B_t$ are appropriately defined.

**Proposition 2.** There is a sequence of lump-sum payments $\{P_t, A_t, B_t\}$ such that a perfect Bayesian equilibrium for this mechanism generates an allocation that coincides with $\gamma^*$, and such that the buyer's expected surplus is the same as the total expected surplus under the efficient sampling rule.

**Proof.** See the Appendix.

What are these participation premia $A_t$ and $B_t$, offered to participants in the auctions? To answer this question, consider the situation faced by an informed (past) seller $i$. Its first decision at time $t$ is whether to participate in the auction held at $t$ or to wait and participate in the auction at $t+1$, if it ever takes place. We would like seller $i$ to participate at $t$ if and only if $c_i \leq z_t$. However, if $c_i$ is very close to $z_t$, this seller would expect (virtually) zero surplus from participating. On the other hand, this seller $i$ expects positive surplus if it submits no bid: if no one bids today and seller $i$ wins tomorrow, it would sell the good forever at a price higher than $c_i$. Therefore, without any compensation, there is a disincentive to submitting a bid. To avoid this problem, the mechanism introduces the participation premia $A_t$, set in a way such that a seller $i < t$ with cost $c_i = z_t$ (a marginal bidder at $t$) is just indifferent between:

(i) participating at $t$ with a bid equal to $z_t$, and so obtaining an expected payoff $A_t$ (if $i$ wins the auction, he gets a price $z_t = c_i$, the maximum bid in this auction, so that his payoff is only the participation premium); and

(ii) not participating, then, if there is an auction at $t+1$ (i.e., if nobody participates at auction $t$), bidding $z_t$, thereby $A_{t+1}$ is obtained plus some surplus associated with the higher expected price per period. This surplus is calculated in the Appendix and is denoted by $a_{t+1}$. The probability (if $i$ does not bid) that no bid is submitted at $t$ is $[1 - F(z_t)]^{t-1}/[1 - F(z_{t-1})]^{t-2}$ (the probability that the other $t - 2$ old
sellers have costs above $z_t$ conditional on the fact that each has a cost of at least $z_{t-1}$, all times the unconditional probability that the new seller $t$ also has a cost above $z_t$.

Therefore, the participation premia $A_t$ for sellers other than $t$ are set so that

$$A_t = \delta \left[ \frac{1 - F(z_t)}{1 - F(z_{t-1})} \right]^{t-1} [a_{t+1} + A_{t+1}].$$

That is, the premia set the incentives so that competition (participation) in the auctions occurs thereby assuring efficiency.

As indicated above, there is an alternative mechanism whose outcome is $\gamma^*$ (and allows the buyer to extract all the surplus), which is based on offers made sequentially by the sellers, instead of auctions conducted sequentially. We will discuss this scheme in a less formal way. Indeed, consider the following, alternative scheme: at every period $t$ in which a new seller provides the good (following $\gamma^*$), the seller is asked to make a (standing) price offer for the good. Then, if the offer is between $z_{t+k-1}$ and $z_{t+k}$ for some $k \in \{1, \ldots, T-t\}$, the offer is accepted at time $t+k$ provided that this is the lowest offer received up to that point. Moreover, the seller receives today (period $t$), in any case, a certain compensation payment which depends only on his offer.

In this scheme, if we define the compensation payments appropriately, making a price offer equal to the observed cost constitutes an (perfect Bayesian) equilibrium for which the outcome is, again, $\gamma^*$. Indeed, assume all offers are publicly observed (which is not necessary), and assume at time $t$ a sequence $s=(m_1, m_2, \ldots, m_{t-1})$ of offers by previous sellers. Then define the following function $N^S_t: C \to R$:

\begin{enumerate}
  \item $N^S_t(x) = \delta^{j-t+1} [(1 - \delta^{T-j})/(1 - \delta)] [1 - F(x)]^{j-t-1}$, if $x \leq \min(m_1, m_2, \ldots, m_{t-1})$ and either $\exists j > t$ ($j \leq T$) such that $z_{j-1} < x \leq z_j$ or $x \leq z_j$;
  \item $N^S_t(x) = 0$, if $x > \min(m_1, m_2, \ldots, m_{t-1})$ or $x > z_j$ for all $j \leq T$.
\end{enumerate}

The expression in (i) is the probability that $j - t$ potential sellers find their cost above $x$ (in which case the good is bought indefinitely from seller $t$ from period $j+1$ on, if this user $t$ bids $x$) times the “discounted” (at time $t$) number of remaining periods. In all cases included in (ii)—the complement

\footnote{For seller $t$, the probability of a new auction if he does not bid is $[1 - F(z_t)]^{-1} [1 - F(z_{t-1})]^{-1}$, since he knows that all $t - 1$ possible competitors have valuations below $z_{t-1}$. Also the surplus associated to the higher expected price tomorrow is higher for seller $t$ (equal to $a_t[1 - F(z_{t-1})]$). Therefore $B_t$ is slightly higher than $A_t$.}
to $C$ of (i)—the seller will never again provide the good. Notice that $N^S_t(\cdot)$ is a monotone (decreasing) function. Now, define $\Phi^S_t$ as

$$\Phi^S_t(x) = \left[ - dN^S_t(\zeta) \right].$$

(Remember that $-dN^S_t(\zeta)$ is the increase in the number of discounted periods that $i$ expects to sell resulting from decreasing a little below $\zeta$ the reported cost.)

Then, with compensation payments $P_t(s, c_t)$ (the payment that seller $t \leq T$ receives when offering to sell at a price $c_t$ per period) defined as

$$P_t(s, c_t) = \Phi^S_t(c_t) - c_t N^S_t(c_t),$$

it is straightforward to prove that making a standing offer equal to the observed cost constitutes an (perfect Bayesian) equilibrium in the offers scheme delineated above. Again, full extraction of the surplus could then be obtained through entry fees before the first period the seller provides the good (and thus learns his cost).

Notice that $P_t$ is decreasing in $c_t$ (its derivative is $-N^S_t(c_t) < 0$). That is, the compensation paid to the seller is decreasing in its offered price.

As one would expect, there is a close connection between both mechanisms considered. Indeed, assume we change the auction mechanism in the following way: at every period that a new seller is chosen, that seller is asked to declare in which auction (as defined by the auctioning mechanism) it wants to participate and what bid it would like to make. Setting the participation premia as in the original bidding mechanism, it is still true that every seller $i$ would decide to participate in the auction at time $t$ such that $z_{t-1} < c_t \leq z_t$. Then, these bids could be understood as the offers in the alternative mechanism. To get this latter mechanism, we only have to substitute the certain payment at $t$ for the participation premia of the uncertain future.$^8$

4. Concluding Remarks

We have considered the problem faced by a buyer with repeated unit purchases when deciding the cost-minimizing (from its viewpoint), buying mechanism. We have assumed that there were several potential sellers

$^8$ In the auction mechanism, the winner gets a price per period higher than its cost (bid), whereas in the offers mechanism the winner gets a price equal to its cost (offer). Therefore, the fixed payments exceed the present value of the participation premia by an amount equal to the (expected) extra surplus obtained in the auction scheme.
for the good with possibly different costs. Each potential seller (privately) learned its own cost only after producing the good for one period.

An efficient sampling rule for this buyer is defined by a sequence of increasing cutoff points, one for each period that a unit is to be purchased. The good should be bought from new sellers until one of them observes a cost lower than the cutoff point corresponding to the period, and/or the cutoff point increases above the cost observed by a previous seller. From that period on, the good should be bought on a permanent basis from the experienced seller with lowest cost, with no further sampling.

We have shown that a certain sequence of second-price auctions is an optimal buying mechanism for this buyer. In this auction scheme, at any point in time, if a permanent seller has not been selected, the good is bought from a new seller. Then, at the end of the period a second-price auction is conducted with participation premia and a maximum bid, where all experienced sellers can bid. If at least one bid is made, the winner sells the good forever at a price per period equal to the second lowest bid or the maximum bid, if no one else bids. If no bid is submitted, the good is bought from a new seller, and the process repeats in the next period with new premia and a (higher) maximum bid. The scheme is defined in a way such that truthful bidding is an equilibrium, the good is bought efficiently, and the buyer gets all the rents. An alternative, equivalent mechanism would ask for a price offer from every new seller. These offers would be understood as standing offers to sell the good at the stated (per period) price. The buyer would make a payment to sellers that submit an offer, which is decreasing with the offer, whether the offer is ever accepted or not. New sellers would be tried until one of these standing offers falls below the reservation value of the period.

One important feature of these mechanisms is the subsidization of competition. Indeed, in the auction scheme the buyer pays participation premia to induce former sellers to bid and forego opportunities to sell at a higher price in the future. In the offers scheme, the buyer subsidizes low offers, so that the opportunity to sell the good at higher prices in the hope of bad realizations for competitors is less attractive.

All the results were derived for the case in which the costs for different potential sellers were independent and identically distributed. With correlated costs, as long as the first-best sampling rule is monotone (in the sense that if a seller is not chosen as a permanent seller when its cost is equal to $x$, then it is not chosen when its cost is higher than $x$), on top of being incentive compatible, a version of the offers scheme should still be efficient and optimal for the buyer. The terms of the contract as well as the parameters defining the first-best sampling rule would be more complex (the sequence of cutoff points would not be a sequence of real values, but a sequence of functions), but even then, the basic tradeoff between
sampling for better realizations and exploiting the best realization to date would be the determinant of these parameters. Even then, subsidization of informed sellers would be needed to induce efficiency.

Appendix

Proof of Proposition 1. First we prove the result for $T \leq n$. Again, we use superscript to indicate the number of periods that a purchase is to be made. We proceed by induction on $T$, from $T = 2$ up to $T = n$. When $T = 2$, the result is trivial, and $z_1^n = \mu$.

Now, for $T > 2$, assume that the optimal sampling rule $\gamma^{*T-1}$ for the $T-1$ periods case is characterized by the sequence $z_1^{T-1}$, $z_2^{T-1}$, ..., $z_T^{T-1}$. The $T$ periods case starts by buying the good from a seller, say seller 1. Let $x$ denote the cost realized by this seller. Now the good could be bought from seller 1 forever, in which case the realized total cost would be $x(1 - \delta^T)/(1 - \delta)$, or else, it could be bought from a new seller at some time $t$ for the first time. Consider this second case and denote by $c_i$ the cost realized by this new seller. Notice that, for efficiency, from $t$ on we should be following $\gamma^{*T-t+1}$, except that after $t$ we substitute seller 1 for the new seller at $t$ whenever $x < c_i$. Denote by $S[\gamma^{*T-t+1}/x]$ the total cost realized from $t$ on following this rule. Notice that $S(\gamma^{*1}/z_1^1) = z_1^1 = \mu$, and as another induction hypothesis, assume that for all $t < T$,

$$S(\gamma^{*t-1}/z_1^t) = \frac{(1 - \delta^t)}{(1 - \delta)} z_1^t.$$

Now, for efficiency,

$$x \frac{(1 - \delta^{T-t+1})}{(1 - \delta)} \geq S[\gamma^{*T-t+1}/x],$$

since otherwise buying the good from user 1 forever would realize a lower total cost.

We now show that, also for efficiency, $t = 2$. Indeed, assume $t > 2$. Then consider buying the good from a new seller in period $t-1$, and follow $\gamma^{*T-t+2}$ afterwards, substituting seller 1 for the new seller whenever $x < c_i$, as before. Denote the cost from $t-1$ by $S[\gamma^{*T-t+2}/x]$. We know that

$$S[\gamma^{*T-t+2}/x] \leq S[\gamma^{*T-t+1}/x] + \lambda \delta^{T-t+1},$$

since otherwise following the rule $\gamma^{*T-t+1}/x$ from period $t-1$ and then buying the good from seller 1 at period $T$ would reduce the cost (realizing the right hand side). Then, from A1 and A2,

$$S[\gamma^{*T-t+2}/x] \leq x + \delta S[\gamma^{*T-t+1}/x],$$

or
which means that savings could be obtained by buying the good from the new seller in period $t-1$. Applying this reasoning recursively, we conclude that, if the good is not bought from seller 1 forever, it should be bought from a new seller already in period 2, as we wanted to show.

Now, we should compare these two alternatives, i.e.,

(a) $x + \delta S[y^*T^{-1}/x]$,  
(b) $x(1-\delta^T)/(1-\delta)$.

First notice that (b) < (a) when $x = \hat{c}$. Also, from the induction hypothesis and A2, (a) < (b) when $x = z_{1}^{T-1}$. Finally,

$$\frac{(1-\delta^T)}{(1-\delta)} > \frac{\partial S[y^*T^{-1}/x]}{\partial x} \geq 0.$$  

Indeed, the total cost cannot decrease when $x$ increases, and at least in period 1 the good is not bought from the outsider. Therefore, there exists a unique $z_{1}^{T}$ in the interval $(\hat{c}, z_{1}^{T-1})$ such that

$$z_{1}^{T} \frac{(1-\delta^T)}{(1-\delta)} = S[y^*T^{-1}/z_{1}^{T}].$$

For values of $x$ lower than $z_{1}^{T}$, (b) is smaller than (a), and the opposite is true for values of $x$ higher than $z_{1}^{T}$. Also, the sequence of stopping values for $t > 1$ are $z_{2}^{T} = z_{1}^{T-1}$, $z_{3}^{T} = z_{2}^{T-1}$, ... This recovers the induction hypothesis and concludes the proof for $T \leq n$.

For $T > n$, assume $n = 2$ and let $H = T - n$. Then, trivially, the optimal sampling rule is characterized by $z_{t} = \hat{c}$ for all $t \geq 2$ and $z_{1}$ equals the unique solution to

$$z_{1} \frac{(1-\delta^T)}{(1-\delta)} = \mu + \delta z_{1} \left\{ (1 - F(z_{1})) z_{1} + F(z_{1}) E[c_i | c_i \leq z_{1}] \right\}.$$

Notice that, indeed, $z_{1} < \mu$. Now, assume that the result has been proved for $n \geq 2$ and $T - n = H$ fixed. Consider the case $T + 1$ and $n + 1$ and perform an induction argument like the one above. The proposition is then proved for $T - n = H$, for arbitrary $H$.

**Proof of Proposition 2.** Define the function

$$g_{t}(c_{i}, b) = \delta(1-\delta^{T-1}) \left\{ (z_{t} - c_{i}) \left[ 1 - F(z_{t-1}) \right]^{t-1} \left[ 1 - F(z_{t-1}) \right]^{t-2} \right\}$$

and

$$+ \int_{b}^{\omega} (x-c_{i})(t-1) f(x) \left[ 1 - F(x) \right]^{t-2} \left[ 1 - F(z_{t-1}) \right] dF + H_{t}(c_{i}, b).$$

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where
\[ H_t(c, b) = 0, \quad \text{if} \quad z_{t-1} \leq b \leq z, \]
and
\[ H_t(c, b) = \int_b^{z_t} (x - c) f(x) \left[ 1 - \frac{[1 - F(x)]^{t-2}}{[1 - F(z_{t-1})]^{t-2}} \right] dx, \quad \text{if} \quad b < z_{t-1}. \]

This function \( g_t(c, b) \) represents the payoff expected at \( t \) by an experienced seller other than the new one at period \( t \), with cost \( c_i \), who participates in auction \( t \) and bids \( b \leq z_t \), given that auction \( t \) takes place (if no participation premia are paid and under the assumption that all other sellers are bidding truthfully). For the new seller in \( t \), this value would be \( g_t(c, b)/\left(1 - F(z_{t-1})\right) \). Indeed, given that the auction at \( t \) is being held, there are \( t-1 \) informed sellers other than \( i \). Out of them, \( t-2 \) have revealed themselves as having observed costs above \( z_{t-1} \), whereas seller \( i \) has not yet acted. Now, in the auction at \( t \), bidding \( b \leq z_t \) seller \( i \) will win the auction and pay \( z_t \) if all other sellers observe costs above \( z_t \). This happens now with probability
\[ \frac{[1 - F(z_t)]^{t-1}}{[1 - F(z_{t-1})]^{t-2}}. \]

It also wins if the minimum bid is lower than \( z_t \) but higher than \( b \). The density function of the minimum realization of \( t-1 \) independent random variables, \( t-2 \) of which are distributed with density \( f(x)/(1 - F(z_{t-1})) \) and the other with density \( f(x) \), is
\[ \frac{(t-1) f(x) [1 - F(x)]^{t-2}}{[1 - F(z_{t-1})]^{t-2}}. \]

If \( b < z_{t-1} \), then \( i \) will win the auction for sure except if the new seller \( t \) has observed a cost below \( z_{t-1} \). That is the reason why we introduce \( H_t \). Now, define
\[ a_i = g_t(z_{t-1}, z_{t-1}), \]
which is simply (without participation premia) the expected payoff for a seller (other than \( t-1 \)) with cost \( z_{t-1} \) who decides to wait until period \( t \) and then bid its true cost \( z_{t-1} \). Finally, define \( A_t \), the participation
premium in auction \( t \) for all experienced sellers other than \( t \) (i.e., for sellers \( 1, 2, ..., t-1 \)), recursively, as

\[
A_t = 0 \quad \text{for all} \quad t \geq \min\{T, n\},
\]

\[
A_{t-1} = \delta \left( \frac{1 - F(z_{t-1})}{1 - F(z_{t-2})} \right)^{t-2} \left[ a_t + A_t \right] \quad \text{otherwise},
\]

and the participation premium for seller \( t \) as

\[
B_{t-1} = \delta \left( \frac{1 - F(z_{t-1})}{1 - F(z_{t-2})} \right)^{t-2} \left[ A_t + \frac{a_t}{1 - F(z_{t-2})} \right],
\]

for all \( t \leq \min\{T, n\} \), and \( B_t = 0 \) otherwise. The first term in the right-hand side is the (discounted) seller’s probability assessment of having an auction at \( t \) conditional on having an auction at \( t-1 \). With this definition, it is easy to check that truthful bidding is an equilibrium strategy vector. Indeed, given that it decides to participate in an auction \( t-1 \), a seller’s best bid is its true cost (in case this is below the maximum bid, otherwise this maximum bid is the best bid). The choice is therefore between participating and not participating. We will show that the best decision is to participate if and only if \( c_t \leq z_{t-1} \). To see this, notice that at time \( t-1 \) the payoff for an experienced seller (other than \( t-1 \)) with cost \( c_t \) is

\[
A_{t-1} + g_{t-1}(c_t, \min\{c_t, z_{t-1}\})
\]

if it participates and then bids optimally (truth, if that is a valid bid, or the maximum bid \( z_{t-1} \) otherwise), whereas waiting to (hopefully) participate in auction \( t \) it expects

\[
\delta \left( \frac{1 - F(z_{t-1})}{1 - F(z_{t-2})} \right)^{t-2} \left[ A_t + g_t(c_t, \min\{c_t, z_t\}) \right].
\]

The difference is

\[
g_{t-1}(c_t, \min\{c_t, z_{t-1}\}) + \delta \left( \frac{1 - F(z_{t-1})}{1 - F(z_{t-2})} \right)^{t-2} \left[ a_t + g_t(c_t, \min\{c_t, z_t\}) \right]
\]

\[
= g_{t-1}(c_t, \min\{c_t, z_{t-1}\}) - \delta \left( \frac{1 - F(z_{t-1})}{1 - F(z_{t-2})} \right)^{t-2} \times \left\{ g_t(c_t, \min\{c_t, z_t\}) - g_t(z_{t-1}, z_{t-1}) \right\}. \quad (A3)
\]
For the moment, assume that $c_i \leq z_t$ (although it could be higher or lower than $z_{t-1}$). Then,

$$g_i(c_i, \min\{c_i, z_{t-1}\}) - g_i(z_{t-1}, z_{t-1}) = g_i(c_i, c_i) - g_i(z_{t-1}, z_{t-1})$$

$$= \delta \frac{1 - \delta^{T-i}}{1 - \delta} \times \left\{ z_{t-1} - c_i \right\} \left[ \frac{1 - F(z_t)}{1 - F(z_{t-1})} \right]^{t-2} \times \left( t - 1 \right) f(x) \left[ \frac{1 - F(z_{t-1})}{1 - F(z_{t-2})} \right]^{t-1} dx \right\} + \left\{ x - c_i \right\} \left( t - 1 \right) f(x) \left[ \frac{1 - F(z_{t-1})}{1 - F(z_{t-2})} \right]^{t-1} dx \right\}. \quad (A4)$$

Now, if $c_i \leq z_{t-1}$, this is (positive and) smaller than

$$\frac{\delta}{(1 - \delta)} \left( z_{t-1} - c_i \right)[1 - F(c_i)],$$

and then (A3) is positive (i.e., the result follows) since

$$g_{t-1}(c_i, \min\{c_i, z_{t-1}\}) = \frac{\delta}{(1 - \delta)} \left( z_{t-1} - c_i \right)[1 - F(z_{t-1})],$$

Now, assume $z_{t-1} < c_i \leq z_t$. Then (A4) is negative, since $(z_{t-1} - c_i) < 0$ (the third term in (A4) should be understood, as usual, as the negative of the same term exchanging the upper and lower limits of integration), and its absolute value is smaller than

$$\frac{\delta}{(1 - \delta)} \left( z_{t-1} - c_i \right)[1 - F(z_{t-1})],$$

whereas $g_{t-1}(c_i, \min\{c_i, z_{t-1}\}) = g_{t-1}(c_i, z_{t-1})$ is negative and its absolute value equals

$$\frac{\delta}{(1 - \delta)} \left( z_{t-1} - c_i \right)[1 - F(z_{t-1})].$$

Therefore (A3) is negative and the result follows for $z_{t-1} \leq c_i \leq z_t$. In a similar fashion we can show that, for $z_{t+j} \leq c_i \leq z_{t+j+1}$, for some $j \geq 0$, seller $i$ is better off bidding in period $t+j+1$ than bidding in period $t-1$. Finally, the same procedure can be used to prove that truthful bidding is also the best response for seller $t-1$, (where $B_t$ would be substituted
for $A_t$). This holds for all $t$, and then truthful bidding is a (perfect Bayesian) equilibrium.

The last thing to show is that the buyer can extract all the rents. That can be easily obtained by setting the fixed payments (due upon accepting to sell the good for the first time) equal to the expected profits from having the right to be a bidder from then on. This concludes the proof of Proposition 2.

REFERENCES