Stock Price Booms and Expected Capital Gains?

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Abstract

The booms and busts in U.S. stock prices over the post-war period can to a large extent be explained by fluctuations in investors’ subjective capital gains expectations. Survey measures of these expectations display excessive optimism at market peaks and excessive pessimism at market troughs. Formally incorporating subjective price beliefs into an otherwise standard asset pricing model with utility maximizing investors, we show how subjective belief dynamics can temporarily delink stock prices from their fundamental value and give rise to asset price booms that ultimately result in a price bust. The model quantitatively replicates (1) the volatility of stock prices and (2) the positive correlation between the price dividend ratio and expected returns observed in survey data. We show that models imposing objective or ‘rational’ price expectations cannot simultaneously account for both facts. Our findings imply that large parts of U.S. stock price fluctuations are not due to standard fundamental forces, instead result from self-reinforcing belief dynamics triggered by these fundamentals.

JEL Class. No.: G12, D84
‘Bull-markets are born on pessimism, grow on skepticism, mature on optimism and die on euphoria’
Sir John Templeton, Founder of Templeton Mutual Funds

1 Motivation

Following the recent boom and bust cycles in a number of asset markets around the globe, there exists renewed interest in understanding better the forces contributing to the emergence of such drastic asset price movements. This paper argues that movements in investor optimism and pessimism, as measured by the movements in investors’ subjective expectations about future capital gains, are a crucial ingredient for understanding these fluctuations.

We present an asset pricing model that incorporates endogenous belief dynamics about expected capital gains. The model gives rise to sustained stock price booms and busts and is consistent with the behavior of investors’ capital gains expectations, as measured by survey data. The model suggests that more than half of the variance of the price dividend ratio in U.S. post-WWII data is due to movements in subjective expectations.

The standard approach in the consumption-based asset pricing literature consists of assuming that stock price fluctuations are fully efficient. Campbell and Cochrane (1999) and Bansal and Yaron (2004), for example, present models in which stock price fluctuations reflect the interaction of investor preferences and stochastic driving forces in a setting with optimizing investors who hold rational expectations.

The empirical evidence we present casts considerable doubt on the prevailing view that stock price fluctuations are efficient. Specifically, we show that the rational expectations (RE) hypothesis gives rise to an important counterfactual prediction for the behavior of investors’ expectations. This counterfactual prediction is a model-independent implication of the RE hypothesis, but - as we explain below - key for understanding stock price volatility and its efficiency properties.

As previously noted by Fama and French (1988), the empirical behavior of asset prices implies that rational return expectations correlate negatively with the price dividend (PD) ratio. Somewhat counter-intuitively, the RE hypothesis thus predicts that investors have been particularly pessimistic about future stock returns in the early part of the year 2000, when the tech stock boom and the PD ratio of the S&P500 reached its all-time max-

\[1\] The RE hypothesis implies also a negative correlation between the PD ratio and expected capital gains. Since most variation in returns is due to variation in capital gains, we tend to use both terms interchangeably.
mum. As we document, the available survey evidence implies precisely the opposite: all quantitative survey measures of investors’ return (or capital gain) expectations available for the U.S. economy, unambiguously and unanimously correlate positively with the PD ratio; and perhaps not surprisingly, return expectations reached a temporary maximum rather than a minimum in the early part of the year 2000, i.e., precisely at the peak of the tech stock boom, a fact previously shown in Vissing-Jorgensen (2003). We present a formal econometric test of the null hypothesis that the survey evidence is consistent with RE. Using this test we confirm that the survey data is at odds with the RE hypothesis at any conventional significance level. Our test is immune to the presence of differential information on the part of agents and to the presence of orthogonal measurement error in survey data. Moreover, our test allows determining which variables drive the departure from RE. In particular, we show that the RE hypothesis fails because survey expectations and RE covary differently with the PD ratio, a finding that is useful for guiding the search for alternative and empirically more plausible expectations models.

The positive comovement of stock prices and survey expectations suggests that price fluctuations are amplified by overly optimistic beliefs at market peaks and by overly pessimistic beliefs at market troughs. Furthermore, it suggests that investors’ capital gains expectations are influenced - at least partly - by the capital gains observed in the past, in line with evidence presented by Malmendier and Nagel (2011). Indeed, a simple adaptive updating equation captures the time series behavior of the survey data and its correlation with the PD ratio very well.

Taken together, these observations motivate the construction of an asset pricing model in which investors hold subjective beliefs about the capital gains from stock market investments. We incorporate such beliefs into a Lucas (1978) asset pricing model, assuming that agents are uncertain about the capital gains process they invest optimally given their beliefs. Optimal behavior dictates that agents update beliefs according to Bayes’ law.

With this modification, the Lucas model becomes quantitatively consistent with the observed volatility of stock prices and the positive correlation between the PD ratio and subjective return expectations. This is obtained even though we use the simplest version of Lucas model with time separable preferences and standard stochastic driving processes. Considering the same model under RE produces - amongst other things - too little price volatility and the wrong sign for the correlation between the PD ratio and expected returns.

As is explained in Adam and Marcet (2011), the presence of subjective price beliefs is consistent with optimizing behavior and a lack of common knowledge about agents’ beliefs and preferences.
The strong improvement in the model’s empirical performance arises because agents’ attempts to improve their knowledge about price behavior can temporarily delink asset prices from their fundamental (RE) value and give rise to belief-driven boom and bust cycles in stock prices. This occurs because with imperfect information about the price process, optimal behavior dictates that agents use past capital gains observations to learn about the stochastic process governing the behavior of capital gains; this generates a feedback between capital gain expectations and realized capital gains which drives the booms and busts in stock prices.

Suppose, in line with the empirical evidence, that agents become more optimistic about future capital gains whenever they are positively surprised by past capital gains.\footnote{Such positive surprises may be triggered by fundamental shocks, e.g., a high value for realized dividend growth.} A positive surprise for the capital gains observed in the previous period then increases optimism about the capital gains associated with investing in the asset today. If such increased optimism leads to an increase in investors’ asset demand and if this demand effect is sufficiently strong, then positive past surprises trigger further positive surprises today, and thus further increases in optimism tomorrow. As we show analytically, stock prices in our model do increase with capital gain optimism whenever the substitution effect of increased optimism dominates the wealth effect of such belief changes. Asset prices in the model then display sustained price booms, similar to those observed in the data.

After a sequence of sustained increases, countervailing forces come into play that endogenously dampen the upward price momentum, eventually halt it and cause a reversal. Specifically, in a situation where increased optimism about capital gains has led to a stock price boom, stock prices make up for a larger share of agents’ total wealth.\footnote{This occurs because stock prices are high, but also because agents discount other income streams, e.g., wage income, at a higher rate.} As we show analytically, this causes the wealth effect to become as strong as (or even stronger than) the substitution effect when expectations about stock price appreciation are sufficiently high.\footnote{With CRRA utility, this happens whenever the coefficient of relative risk aversion is larger than one.} Increases in optimism then cease to cause further increases in stock demand and thus stock prices, so that investors’ capital gains expectations turn out to be too optimistic relative to the realized outcomes. This induces downward revision in beliefs, which gives rise to negative price momentum and an asset price bust.

The previous arguments show how belief dynamics can temporarily delink asset prices from their fundamental value. Clearly, these price dynamics are inefficient as they are...
not justified by innovations to preferences or other fundamentals.

We obtain these results even though we depart from the standard paradigm in a minimal way. Specifically, we assume that investors are internally rational (IR) in the sense of Adam and Marcet (2011). This implies that all investors hold an internally consistent system of beliefs about variables that are exogenous to their decision problem and they choose investment and consumption optimally. Although agents’ beliefs do not fully capture the actual behavior of prices in equilibrium, in line with the survey evidence, agents’ beliefs are broadly plausible given the behavior of equilibrium prices and the behavior of prices in the data. In particular, agents believe the average growth rate of stock prices to slowly drift over time, which is consistent with the presence of prolonged periods of price booms followed by price busts.\(^6\)

The current paper shows how the framework of internal rationality allows studying learning about market behavior in a model of intertemporal decision making while avoiding some of the pitfalls of the adaptive learning literature, where agents’ belief updating equations and choices are often not derived from individual maximization. We thus show how explicit microfoundations can guide modelling choices in settings featuring subjective beliefs about market outcomes, as is the case in settings imposing RE.

The remainder of the paper is structured as follows. Section 3 documents that there is a strong positive correlation between the PD ratio and survey measures of investors’ return and capital gain expectations and that this is incompatible with the RE hypothesis. It then documents that from a purely statistical standpoint approximately two thirds of the variation in the PD ratio of S&P500 can potentially be accounted for by variations in expected capital gains. Section 4 presents our asset pricing model with subjective beliefs. For benchmark purposes, section 5 determines the RE equilibrium. Section 6 introduces a specific model for subjective price beliefs, which relaxes agents’ prior beliefs about price behavior relative to the RE equilibrium beliefs. This section also derives the resulting Bayesian updating equations characterizing belief dynamics over time, involving learning about the permanent component of stock price growth. After imposing market clearing in section 7, we present closed form solutions for the PD ratio in section 8 in the special case of vanishing uncertainty. Using the vanishing noise setup, we explain how the interaction between belief updating dynamics and price outcomes can endogenously generate boom and bust dynamics in asset prices. In section 9 we estimate the fully stochastic version

\(^6\)Section 6.2 in Adam Marcet and Nicolini (2014) characterizes the second order implications of agents’ beliefs in a related model, showing that these are validated by the data and - to a large extent - also by the model behavior.
of the model using a mix of calibration and simulated method of moments estimation. This section shows that the subjective belief model successfully replicates a number of important asset pricing moments, including the positive correlation between expected returns and the PD ratio. Section 10 shows that the estimated model can replicate the low frequency movements in the time series of the US postwar PD ratio, as well as the available time series of survey data. Section 11 presents a number of robustness checks and extensions of the basic model. A conclusion briefly summarizes and discusses potential avenues for future research. Technical material and proofs can be found in the appendix.

2 Related Literature

Following Bob Shiller’s (1981) seminal observation that stock price volatility cannot be explained by the volatility of rational dividend expectations, the asset pricing literature made considerable progress in explaining stock price behavior. Bansal and Yaron (2004) and Campbell and Cochrane (1999), for example, developed consumption based RE models in which price fluctuations result from large and persistent swings in investors’ stochastic discount factor. Section 3 shows, however, that RE models fail to capture the behavior of investors’ return expectations. This strongly suggests that RE models fall short of providing a complete explanation of the sources of stock price volatility.

Attributing stock price fluctuations to ‘sentiment’ fluctuations or issues of learning has long had an intuitive appeal. A substantial part of the asset pricing literature introduces subjective beliefs to model investor ‘sentiment’. The standard approach resorts to Bayesian RE modeling, which allows for subjective beliefs about fundamentals, while keeping the assumption that investors know the equilibrium pricing function linking stock prices to fundamentals. Following early work by Timmermann (1993), a substantial literature follows this approach. It finds that the additional stock price volatility generated from learning is overall small compared to the gap that exists relative to the data. Recent work by Barberis et al. (2014), for example, considers a time-separable utility framework where some investors have rational dividend beliefs while others extrapolate from past observations. In line with the Bayesian RE approach, they maintain the assumption that agents know the equilibrium pricing function and show that their model successfully replicates survey behavior. At the same time, the volatility of the PD ratio falls significantly short of that observed in the data, so that there are no significant stock price boom
and bust episodes. In ongoing work, Hirshleifer and Yu (2012) and Choi and Mertens (2013) consider Bayesian RE models with non-time separable preferences and investors who extrapolate past fundamentals. They show how extrapolation of fundamentals endogenously generates long-run consumption risk. Even with non-separable preferences, it remains challenging to generate sufficient volatility for the PD ratio.

The modeling approach pursued in the present paper differs fundamentally from the one discussed in the previous paragraph. The Bayesian RE literature assumes that agents find it difficult to forecast fundamental shocks (e.g., agents hold subjective dividend beliefs), but that agents can predict perfectly price outcomes conditional on the history of observed fundamentals (agents know the equilibrium pricing function mapping the history of dividends into price outcomes). Assuming that agents know the pricing function provides agents with a substantial amount of information about market behavior, suggesting that it is of interest to study the effects of relaxing this informational assumption.

The present paper assumes that agents find it easy to predict fundamentals (e.g., agents hold RE about dividends), but difficult to predict price behavior (agents do not know the equilibrium pricing function). We show that a simple asset pricing model can then replicate survey data and generate sufficient volatility for the PD ratio, including occasional boom and bust episodes. This is achieved in a setting with standard time-separable preferences and obtained because there is a much stronger propagation of economic disturbances when agents learn about the equilibrium pricing function: belief changes then affect stock price behavior and stock prices feed back into belief changes; this allows movements in prices and beliefs to mutually reinforce each other during price boom and bust phases, thereby increasing price volatility. The feedback from market outcomes into beliefs is absent in a Bayesian RE setting.

The literature on adaptive learning previously considered deviations from rational price expectations using asset pricing models where investors learn about price behavior. Marcet and Sargent (1992), for example, study convergence to RE when agents estimate an incorrect model of stock prices by least squares learning. A range of papers in the

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7 Due to the CARA utility setup the volatility of the PD ratio also asymptotically converges to zero.  
8 Hirshleifer and Yu (2012) and Choi and Mertens (2013) do not evaluate the match with survey return expectations.  
9 Sometimes Bayesian RE models can be written as if there is feedback from market outcomes to investors beliefs, see Barberis et al. (2014) for an example. The learning setup is, however, then isomorphic to one where agents beliefs are affected by fundamentals only (dividend behavior). This is the case because in a Bayesian RE equilibrium there is a tight and stable functional relationship between fundamentals (dividends) and equilibrium prices. In contrast, when agent learn about the equilibrium pricing function, this relationship is time-varying.
adaptive learning literature argues that learning generates additional stock price volatility. Bullard and Duffy (2001) and Brock and Hommes (1998), for example, show that learning dynamics can converge to complicated attractors that increase asset return volatility, if the RE equilibrium is unstable.\textsuperscript{10} Lansing (2010) shows how near-rational bubbles can arise in a model with learning about price behavior. Branch and Evans (2011) present a model where agents learn about risk and return and they show how it gives rise to bubbles and crashes. Boswijk, Hommes and Manzan (2007) estimate a model with fundamentalist and chartist traders whose relative shares evolve according to an evolutionary performance criterion, showing that the model can generate a run-up in asset prices and subsequent mean-reversion to fundamental values. DeLong et al. (1990) show how the pricing effects of positive feedback trading survives or even get amplified by the introduction of rational speculators.

The approach used in the present paper differs along several dimensions from the contributions mentioned in the previous paragraph. First, we compare quantitatively the implications of our model with the data, i.e., we match a standard set of asset pricing moments capturing stock price volatility and use formal asymptotic distribution to evaluate the goodness of fit. Second, we compare the model to evidence obtained from survey data. Third, we present a model that derives investors’ consumption and stockholding plans from properly specified microfoundations. In particular, we consider agents that solve an infinite horizon decision problem and hold a consistent set of beliefs. By contrast, the adaptive learning literature often employs shortcuts that amount to introducing additional behavioral elements into decision making and postulates beliefs that become well specified only in the limit when there is convergence to RE.\textsuperscript{11} Finally, within our setup we discuss conditions for existence and uniqueness of optimal plans, as well as conditions insuring that the optimal plan has a recursive representation.

In prior work, Adam Marcet and Nicolini (2014) present a model in which investors learn about risk-adjusted price growth and show how such a model can quantitatively replicate a set of standard asset pricing moments describing stock price volatility. While replicating stock price volatility and postulating beliefs that are hard to reject in the light of the existing asset price data, as well as the outcomes generated by the model, their setup falls short of explaining survey evidence. Specifically, it counterfactually implies that stock return expectations are constant over time. Adam Marcet and Nicolini (2014) also solve for equilibrium prices under the assumption that dividend and trading income

\textsuperscript{10}Stability under learning dynamics is defined in Marcet and Sargent (1989).

\textsuperscript{11}See section 2 in Adam and Marcet (2011) for a detailed discussion.
are a negligible part of total income. We solve the model without this assumption and show that it can play an important role for the model solution, for example, it gives rise to an endogenous upper bound for equilibrium prices.\textsuperscript{12}

The experimental and behavioral literature provides further evidence supporting the presence of subjective price beliefs. Hirota and Sunder (2007) and Asparouhova, Bossaerts, Roy and Zame (2013), for example, implement the Lucas asset pricing model in the experimental laboratory and document that there is excess volatility in prices that is unaccounted for by the rational expectations equilibrium and that likely arises from participants’ expectations about future prices. Furthermore, the type of learning employed in the present model is in line with evidence presented in Malmendier and Nagel (2011) who show that experienced returns affect beliefs about future asset returns.\textsuperscript{13}

3 Stock Prices & Stock Price Expectations: Facts

This section explains how two important and widely accepted asset pricing facts imply a counterfactual behavior for stock price expectations, whenever one imposes that agents hold rational price expectations. We present the evidence informally in section 3.1 and derive a formal statistical test in section 3.2 The test shows that the RE hypothesis is inconsistent with the behavior of the survey data due to the way survey expectations covary with the PD ratio. Section 3.3 illustrates how simple adaptive prediction of prices, in line with Malmendier and Nagel (2011, 2013), quantitatively captures the relationship between survey expectations and the PD ratio. It also shows how, in a purely statistical sense, variations in expected capital gains can potentially account for up to two thirds of the variation of the U.S. PD ratio over the postwar period.

3.1 Survey Expectations and the PD Ratio

This section explains how the presence of boom and bust dynamics in stock prices, together with the unpredictability of dividend growth, imply that rational stock return forecasts should correlate \textit{negatively} with the PD ratio. It then documents that survey measures of investors’ return expectations correlate instead \textit{positively} with the PD ratio.

\textsuperscript{12}In line with the approach in the Bayesian RE literature, Adam Marcet and Nicolini (2014) impose an exogenous upper bound on agents’ beliefs, a so-called ‘projection facility’, so as to insure existence of finite equilibrium prices.

\textsuperscript{13}Nagel and Greenwood (2009) show that - in line with this hypothesis - young mutual fund managers displayed trend chasing behavior over the tech stock boom and bust around the year 2000.
The discrepancy in terms of correlations with the PD ratio is in line with recent independent findings by Greenwood and Shleifer (2014). The positive comovement between survey return expectations and the PD ratio has also been noted before by Vissing-Jorgensen (2003) and Bacchetta, Mertens, and Wincoop (2009). While generally insightful, one must emphasize that - in econometric terms - the presence of such a discrepancy is only suggestive. In particular, if investors possess private information that is not observed by the econometrician or if survey expectations are measured with error, as one can reasonably expect to be the case, then the correlation between fully rational return forecasts and the PD ratio will differ from the correlation between realized returns and the PD ratio. Simply comparing correlations is thus not sufficient to reject the hypothesis that survey expectations are driven by RE. Furthermore, formal tests must always take into account the joint distribution of the correlation estimates in order to make statistically valid statements. The informal discussion below abstracts from these aspects, but the next section takes them fully into account.

As is well known, stock prices experience substantial price booms and price busts. Figure 1 illustrates this behavior for the post-WWII period for the United States, using the quarterly price dividend ratio (PD) of the S&P 500 index.\textsuperscript{14} The PD ratio displays persistent run-ups and reversals, with the largest one occurring around the year 2000. This shows that price growth can persistently outstrip dividend growth over a number of periods, but that the situation eventually reverses. In fact, the quarterly autocorrelation of the PD ratio equals 0.98. Similar run-ups and reversals can be documented for other mature stock markets, e.g., for the European or Japanese markets.

Equally well-known is the fact that the growth rate of dividends is largely unpredictable, e.g., Campbell (2003). It is especially hard to predict using the PD ratio. The $R^2$ values of an in-sample predictive regression of cumulative dividend growth 1, 5 or 10 years ahead on a constant and the log PD ratio are rather small and amount to 0.03, 0.04, and 0.07, respectively, for the U.S. post-war data.\textsuperscript{15}

Taken together the previous two facts imply that under RE one would expect that the PD ratio negatively predicts future stock market returns. To see this, let the stock

\begin{footnotesize}
\begin{itemize}
  \item \textsuperscript{14}Quarterly dividend payments have been deseasonalized in a standard way by averaging them across the current and preceding 3 quarters. See appendix A.1 for details about the data used in this section.
  \item \textsuperscript{15}We use logPD as a regressor, in line with Campbell (2003). The $R^2$ values are unchanged when using the level of the PD ratio instead.
\end{itemize}
\end{footnotesize}
return $R_{t+1}$ be defined as

$$R_{t+1} = \frac{P_{t+1} + D_{t+1}}{P_t} = \frac{R_{t+1}}{D_{t+1}} + \frac{1}{D_t} D_{t+1},$$

where $P$ denotes the stock price and $D$ dividends. Given a high value of $P_t/D_t$, we have - due to the mean reverting behavior of the PD ratio - that $P_{t+1}/D_{t+1} < P_t/D_t$ on average. Since $D_{t+1}/D_t$ is unpredictable, it follows that a high PD ratio negatively predicts future returns.\(^\text{16}\) A symmetric argument holds if $P_t/D_t$ is low.

In the setup just described, expectations about future stock returns should covary negatively with the PD ratio if investors hold RE. In particular, rational expectations about stock returns should be very low at the height of the tech stock boom in the year 2000 when the PD ratio reached its historical peak.

Survey evidence on investors’ return expectations displays instead a strong positive correlation between investors’ expected returns and the PD ratio. Figure 2 depicts this for our preferred survey, the UBS Gallup Survey, which is based on a representative sample of approximately 1,000 U.S. investors that own at least 10,000 US$ in financial wealth.\(^\text{17}\) Figure 2 graphs the US PD ratio (the black line) together with measures of the cross-sectional average of investors’ one-year ahead expected real return.\(^\text{18}\) Return expectations are expressed in terms of quarterly real growth rates and the figure depicts two expectations measures: investors’ expectations about the one year ahead stock market return, as well as their expectations about the one year ahead returns on their own stock portfolio. These measures behave very similarly over the period for which both series are available, but the latter is reported for a longer time period, so that we focus on it as our baseline. Figure 2 reveals that there is a strong positive correlation between the PD ratio and expected returns. The correlation between the expected own portfolio returns and the PD ratio is +0.70 and even higher for expected stock market returns (+0.82). Moreover, investors’ return expectations were highest at the beginning of the year 2000, which is precisely the year the PD ratio reached its peak during the tech stock boom.

\(^\text{16}\)There may exist, of course, other predictors of future returns which correlate negatively with the PD ratio and that overturn the negative relationship between PD ratio and expected stock returns emerging from the forces described above. We take these formally into account in our statistical test in section 3.2.

\(^\text{17}\)About 40% of respondents own more than 100,000 US$ in financial wealth. As documented below, this subgroup does not behave differently.

\(^\text{18}\)To be consistent with the asset pricing model presented in later sections we report expectations of real returns. The nominal return expectations from the survey have been transformed into real returns using inflation forecasts from the Survey of Professional Forecasters. Results are robust to using other approaches, see the subsequent discussion.
boom. At that time, investors expected annualized real returns of around 13% from stock investments. Conversely, investors were most pessimistic in the year 2003 when the PD ratio reached its bottom, expecting then annualized real returns of below 4%.

Table 1 shows that the strong positive correlation evident from figure 2 is robust to a number of alternative approaches for extracting expectations from the UBS survey, such as using the median instead of the mean expectation, when using inflation expectations from the Michigan survey to obtain real return expectations, when considering plain nominal returns instead of real returns, or when restricting attention to investors with more than 100,000 US$ in financial wealth. The numbers reported in brackets in table 1 (and in subsequent tables) are autocorrelation robust p-values for the hypothesis that the correlation is smaller or equal to zero. The p-values for this hypothesis are all below the 5% significance level and in many cases below the 1% level.

A positive and statistically significant correlation is equally obtained when considering other survey data. Table 2 reports the correlations between the PD ratio and the stock price growth expectations from Bob Shiller’s Individual Investors’ Survey. The table shows that price growth expectations are also strongly positively correlated with the PD ratio, suggesting that the variation in expected returns observed in the UBS survey is due to variations in expected capital gains. Table 2 also shows that correlations seem to become stronger for longer prediction horizons.

Table 3 reports the correlations for the stock return expectations reported in the Chief Financial Officer (CFO) survey which surveys chief financial officers from large U.S. corporations. Again, one finds a strong positive correlation; it is significant at the 1% level in all cases.

Table 4 reports the correlations between the PD ratio and the realized real returns (or capital gains) in the data, using the same sample periods as are available for the surveys considered in tables 1 to 3, respectively. The point estimate for the correlation is negative in all cases, although the correlations fall short of being significant the 5% level due to the short sample length for which the survey data is available.

These tables suggest that investors’ expectations are most likely incompatible with RE. The next section investigates this issue more formally.

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19 The sampling width is four quarters, as is standard for quarterly data, and the test allows for contemporaneous correlation, as well as for cross-correlations at leads and lags. The p-values are computed using the result in Roy (1989).

20 Shiller’s price growth data refers to the Dow Jones Index. The table thus reports the correlation of the survey measure with the PD ratio of the Dow Jones.
3.2 Survey Expectations versus Rational Expectations

Using a formal econometric test, this section shows that the RE assumption is indeed incompatible with the behavior of survey expectations. As suggested by the informal arguments in the previous section, the failure is due to the fact that RE and survey expectations covary differently with the PD ratio. The testing approach presented below is immune to the presence of measurement error in surveys, allows for unobserved information on the side of investors and properly takes into account the joint distribution of estimates.

Let $E_P$ denote agents’ subjective (and potentially less-than-fully-rational) expectations operator based on information up to time $t$, and $R_{t,t+N}$ the cumulative stock returns between period $t$ and $t+N$. Furthermore, let $E_t^N = E_t^PR_{t,t+N} + \mu_t^N$ denote the (potentially noisy) measurement of expected returns, as obtained - for example - from survey data. Since we explore the rationality of return and excess return expectations, we slightly abuse notation and let $R_{t,t+N}$ denote both real return expectations and real excess return expectations.\(^{21}\) We construct excess return expectations following Bacchetta et al. (2009), i.e., we assume that the $N$ period ahead risk-free interest rate is part of agents’ information set and subtract it from the expected stock return.\(^{22}\)

Let us linearly project the random variable $E_t^P R_{t,t+N}$ on $\frac{P_t}{D_t}$ to define

$$E_t^P R_{t,t+N} = a^N + c^N \frac{P_t}{D_t} + u_t^N,$$

where

$$E(x_t u_t^N) = 0$$

for $x_t = (1,P_t/D_t)$, and where the operator $E$ denotes the objective expectation for the true data generating process, whatever is the process for agents’ expectations. The projection residual $u_t^N$ captures the variation in agents’ observed expectations that cannot be linearly attributed to the price-dividend ratio.\(^{23}\) It summarizes all other information that agents believe to be useful in predicting $R_{t,t+N}$.\(^{24}\)

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\(^{21}\)Since the Shiller survey reports expectations about capital gains instead of returns, $R_{t,t+N}$ denotes the real (excess) growth rate of stock prices between periods $t$ and $t+N$ when using the Shiller survey.

\(^{22}\)Following Bacchetta et al. (2009), we use the constant maturity interest rates available from the FRED database at the St. Louis Federal Reserve Bank.

\(^{23}\)The residual $u_t^N$ is likely to be correlated with current and past observables (other than the PD ratio) and thus serially correlated.

\(^{24}\)The projection and the error are well-defined as long as agents’ expectations $E_t^P R_{t,t+N}$ and $P_t/D_t$ are stationary and have bounded second moments.
Due to the potential presence of measurement error, one cannot directly estimate equation (1). Yet, given the observed (excess) return expectations $E^N_t$, one can write the following regression equation

$$E^N_t = a^N + c^N P^t D^t + u^N_t + \mu^N_t. \quad (2)$$

Assuming that the measurement error $\mu^N_t$ is orthogonal to the current PD ratio$^{25}$, we have the orthogonality condition

$$E [x_t(u^N_t + \mu^N_t)] = 0. \quad (3)$$

Finally, we let $\hat{c}^N$ denote the OLS estimator of $c^N$ in equation (2)$^{26}$.

Under the null hypothesis of rational expectations ($E^P_t = E_t$) equation (1) implies

$$R_{t,t+N} = a^N + c^N P^t D^t + u^N_t + \varepsilon^N_t, \quad (4)$$

where $\varepsilon^N_t$ is equal to the prediction error $R_{t,t+N} - E_t R_{t,t+N}$ from the true data-generating process. Importantly, $\varepsilon^N_t$ is orthogonal to all past observations dated $t$ or earlier and we have that under RE

$$E [x_t(\varepsilon^N_t)] = 0. \quad (5)$$

Therefore, applying OLS to (4) delivers another estimate of $c^N$. We let $\hat{c}^N$ denote this estimate.

The correlations reported in tables 1-4 imply - by construction - that $\hat{c}^N > 0$ and $\hat{c}^N < 0$. The regression estimates are useful here because under the hypothesis of RE the estimates $\hat{c}^N$ and $\hat{c}^N$ are both consistent estimates of the same parameter $c^N$. Therefore, under the null hypothesis of RE we can test $H_0 : \hat{c}^N = \hat{c}^N$.$^{27,28}$ Clearly, if the asset

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$^{25}$We allow $\mu^N_t$ to be serially correlated and correlated with equilibrium variables other than $PD_t$.

$^{26}$Although the residuals $u^N_t$ and the measurement errors $\mu^N_t$ are likely to be serially correlated, the OLS estimate is consistent. The presence of serial correlation influences, however, the way t-tests are computed, see appendix A.2.

$^{27}$Under the hypothesis of RE, the correlations in tables 1-3 are not equal to the corresponding correlations reported in table 4, albeit both should have the same sign. That both correlations have the same sign is a weaker test than ours. Also, constructing a formal test for the hypothesis that the correlations have jointly the same sign is a non-trivial task that, as far as we know, has not been carried out in the literature. Furthermore, the test we present is valid in the presence of measurement error, while tests involving correlations are sensitive to the presence of measurement error. There is a literature that essentially tests RE by regressing realized variables on observed expectations about these variables, see for example section 3 in Greenwood and Schleifer (2014). As is well known, such results are also inconsistent in the presence of measurement error.

$^{28}$To obtain p-values for $H_0 : \hat{c}^N = \hat{c}^N$, we stack up equations (2) and (4), so as to create a seemingly
price and survey data were generated by a rational expectations model, say the models of Campbell and Cochrane (1999) or Bansal and Yaron (2004), this test would be accepted.

Test outcomes are reported in table 5a using stock returns and table 5b using excess returns. Both tables report the point estimates $\hat{c}$ and $\hat{c}'$, as well as the p-values for $H_0: \hat{c}' = \hat{c}$. Using the survey data sources considered in the previous section. The point estimates satisfy in all but two cases $\hat{c} > 0$ and always satisfy $\hat{c}' < 0$. The difference between the two estimates is statistically significant at the 1% level in all cases, except for the survey median from the CFO survey. Given the relatively short sample lengths, this is a remarkable outcome. Tables 5a and 5b thus provide overwhelming evidence against the notion that survey expectations are rational.

### 3.3 How Models of Learning May Help

This section illustrates that a simple ‘adaptive’ approach to forecasting stock prices is a promising alternative to explain the joint behavior of survey expectations and stock price data.

Figure 2 shows that the peaks and troughs of the PD ratio are located very closely to the peaks and troughs of investors’ return expectations. This suggests that agents become optimistic about future capital gains whenever they have observed capital gains in the past. Such behavior can be captured by models where agents expectations are influenced by past experience, prompting us to temporarily explore the assumption that agents’ subjective conditional capital gain expectations $E_t[P_{t+1}/P_t]$ evolve according to the following adaptive prediction model

$$E_t[P_{t+1}/P_t] = E_{t-1}[P_t/P_{t-1}] + g \left( \frac{P_t}{P_{t-1}} - E_{t-1}[P_t/P_{t-1}] \right),$$

where $g > 0$ indicates how strongly capital gain expectations are updated in the direction of the forecast error. While equation (6) may appear ad-hoc, we show in section 6 how a very similar equation can be derived from Bayesian belief updating in a setting where agents estimate the persistent component of price growth from the data.

unrelated regression (SUR) system of equations to find the joint distribution of $\hat{c}'$ and $\hat{c}$ and use it to build an asymptotically valid t-test for $H_0$. We use serial-correlation and heteroskedasticity robust asymptotic covariance matrix of the estimators using 4 lags. We checked that results are robust to increasing the lag length up to 12 lags. For each considered survey, we use data on actual (excess) returns (or price growth) for the same time period for which survey data is available when computing the p-values. Further details of the test are described in appendix A.2.

Tests for ‘own portfolio’ expectations are not shown because we do not observe agents’ returns on their own portfolio.

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29Tests for ‘own portfolio’ expectations are not shown because we do not observe agents’ returns on their own portfolio.
One can feed into equation (6) the historical price growth data of the S&P 500 over the postwar period. Together with an assumption about capital gain expectations at the start of the sample this will deliver a time series of implied capital gain expectations \( E_t [P_{t+1}/P_t] \) that can be compared to the expectations from the UBS survey.\(^{30}\) Figure 3 reports the outcome of this procedure when assuming initial beliefs in Q1:1946 to be equal to \(-1.11\%\) per quarter and \( g = 0.02515 \), which minimizes the sum of squared deviations from the survey evidence.\(^{31}\) Figure 3 shows that the adaptive model captures the behavior of UBS expectations extremely well: the correlation between the two series is equal to \(+0.89\).

A similarly strong positive relationship between the PD ratio and the capital gains expectations implied by equation (6) exists over the entire postwar period, as figure 4 documents. The figure plots the joint distribution of the capital gains expectations (as implied by equation (6)) and the PD ratio in the data. When regressing the PD ratio on a constant and the expectations of the adaptive prediction model, one obtains an \( R^2 \) coefficient of 0.55; using also the square of the expectations, the \( R^2 \) rises further to 0.67. Variations in expected capital gains \( E_t [P_{t+1}/P_t] \) can thus account - in a purely statistical sense - for up to two thirds of the variability in the postwar PD ratio.\(^{32}\)

The previous findings suggest that an asset pricing model consistent with equation (6), which additionally predicts a positive relationship between the PD ratio and subjective expectations about future capital gains, has a good chance of replicating the observed positive comovement between price growth expectations and the PD ratio. The next sections spell out the microfoundations of such a model. As we show, the model can simultaneously replicate the behavior of stock prices and stock price expectations.

### 4 A Simple Asset Pricing Model

Consider an endowment economy populated by a unit mass of infinitely lived agents \( i \in [0, 1] \) with time-separable preferences. Agents trade one unit of a stock in a competitive stock market. They earn each period an exogenous non-dividend income \( W_t \geq 0 \) that we

\(^{30}\)We transform the UBS survey measures of return expectations into a measure of price growth expectations using the identity \( R_{t+1} = \frac{P_{t+1}}{P_t} + \frac{D_{t+1}}{P_t} = \frac{P_{t+1}}{P_t} + \beta^D D_t \), where \( \beta^D \) denotes the expected quarterly growth rate of dividends that we set equal to the sample average of dividend growth over Q1:1946-Q1:2012, i.e., \( \beta^D = 1.0048 \). Results regarding implied price growth are very robust towards changing \( \beta^D \) to alternative empirically plausible values.

\(^{31}\)The figure reports growth expectations in terms of quarterly real growth rates.

\(^{32}\)Interestingly, the relationship between implied price growth expectations and the PD ratio seems to have shifted upwards after the year 2000, as indicated by the squared icons in figure 4. We will come back to this observation in section 10.
refer to as ‘wages’ for simplicity. Stocks deliver the exogenous dividend $D_t \geq 0$. Dividend and wage incomes take the form of perishable consumption goods.

**The Investment Problem.** Investor $i$ solves

$$\max_{\{C_t^i, S_t^i \in S\}} \sum_{t=0}^{\infty} \delta^t \ u (C_t^i)$$

s.t. 

$$S_t^i P_t + C_t^i = S_{t-1}^i (P_t + D_t) + W_t$$

for all $t \geq 0$

where $S_{t-1}^i = 1$ and $C^i$ denotes consumption, $u$ the instantaneous utility of the consumer, assumed to be continuous, differentiable, increasing and strictly concave, $S^i$ the agent’s stockholdings, chosen from some compact, non-empty and convex set $S \subset \mathbb{R}$ such that $1 \in S$, and $P \geq 0$ the (ex-dividend) price of the stock. $\mathcal{P}^i$ denotes the agent’s subjective probability measure, which may or may not satisfy the rational expectations hypothesis. Further details of $\mathcal{P}^i$ will be specified below.

**Dividend and Wage Income.** As is standard in the literature, we assume that dividends grow at a constant rate and that dividend growth innovations are unpredictable

$$\ln D_t = \ln \beta^D + \ln D_{t-1} + \ln \varepsilon_t^D,$$

where $\beta^D \geq 1$ denotes gross mean dividend growth, $\ln \varepsilon_t^D$ an i.i.d. growth innovation described further below.

We also specify an exogenous wage income process $W_t$, which is chosen such that the resulting aggregate consumption process $C_t = W_t + D_t$ is empirically plausible. First, in line with Campbell and Cochrane (1999), we set the standard deviation of consumption growth to be $1/7$ of the standard deviation of dividend growth. Second, again following these authors, we set the correlation between consumption and dividend growth equal to 0.2. Third, we choose a wage process such that the average consumption-dividend ratio in the model ($E [C_t / D_t]$) equals the average ratio of personal consumption expenditure to net dividend income, which equals approximately 22 in U.S. postwar data. All this can be parsimoniously achieved using the following wage income process

$$\ln W_t = \ln \rho + \ln D_t + \ln \varepsilon_t^W,$$

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33 See appendix A.3 for details.
where
\[
\begin{pmatrix}
\ln \varepsilon_i^D \\
\ln \varepsilon_i^W
\end{pmatrix} \sim \text{i}iN\left(-\frac{1}{2} \begin{pmatrix}
\sigma_D^2 \\
\sigma_W^2
\end{pmatrix}; \begin{pmatrix}
\sigma_D^2 & \sigma_{DW} \\
\sigma_{DW} & \sigma_W^2
\end{pmatrix}\right)
\] (9)
and \(E\varepsilon_i^D = E\varepsilon_i^W = 1\). Given the variance of dividend growth \(\sigma_D^2\), which can be estimated from dividend data, one can use \(\sigma_{DW}\) and \(\sigma_W^2\) to impose the desired volatility of consumption growth and the desired correlation with dividend growth. Furthermore, one can choose \(\rho = 22\) to obtain the targeted average consumption-dividend ratio. Appendix A.3 explains how this is achieved.

**The Underlying Probability Space.** Agents hold a set of subjective probability beliefs about all payoff-relevant variables that are beyond their control. In addition to fundamental variables (dividends and wages), agents perceive competitive stock prices as beyond their control. Therefore, the belief system also specifies probabilities about prices. Formally, letting \(\Omega\) denote the space of possible realizations for infinite sequences, a typical element \(\omega \in \Omega\) is given by \(\omega = \{P_t, D_t, W_t\}_{t=0}^{\infty}\). As usual, \(\Omega^t\) then denotes the set of all (nonnegative) price, dividend and wage histories from period zero up to period \(t\) and \(\omega^t\) its typical element. The underlying probability space for agents’ beliefs is then given by \((\Omega, \mathcal{B}, \mathcal{P}^i)\) with \(\mathcal{B}\) denoting the corresponding \(\sigma\)-Algebra of Borel subsets of \(\Omega\), and \(\mathcal{P}^i\) a probability measure over \((\Omega, \mathcal{B})\).

The agents’ plans will be contingent on the history \(\omega^t\), i.e., the agent chooses state-contingent consumption and stockholding functions
\[
C_t^i : \Omega^t \to \mathcal{R}^+ \\
S_t^i : \Omega^t \to \mathcal{S}
\] (10) (11)
The fact that \(C^i\) and \(S^i\) depend on price realizations is a consequence of optimal choice under uncertainty, given that agents consider prices to be exogenous random variables.

The previous setup is general enough to accommodate situations where agents learn about the stochastic processes governing the evolution of prices, dividends, and wages. For example, \(\mathcal{P}^i\) may arise from a stochastic process describing the evolution of these variables that contains unknown parameters about which agents hold prior beliefs. The presence of unknown parameters then implies that agents update their beliefs using the observed realizations of prices, dividends and wages. A particular example of this kind will be presented in section 6 when we discuss learning about stock price behavior.

The probability space defined above is more general than that specified in a RE
analysis of the model, where $\Omega$ contains usually only the variables that are exogenous to the model (in this case $D_t$ and $W_t$), but not variables that are endogenous to the model and exogenous to the agent only (in this case $P_t$). Under the RE hypothesis, agents are assumed to know the pricing function $P_t((D,W)^t)$ mapping histories of dividends and wages into a market price. Prices then carry only redundant information and can be excluded from the probability space without loss of generality. The more general formulation we entertain here allows us to consider agents who do not know exactly which price materializes given a particular history of dividends and wages; our agents do have a view about the distribution of $P_t$ conditional on $(D,W)^t$, but in their minds this is a proper distribution, not a point mass as in the RE case. Much akin to academic economists, investors in our model have not converged on a single asset pricing model that associates one market price with a given history of exogenous fundamentals.

**Parametric Utility Function.** To obtain closed-form solutions, we consider in the remaining part of the paper the utility function

$$u(C_t) = \frac{C_t^{1-\gamma}}{1-\gamma} \quad \text{with } \gamma > 1, \quad (12)$$

and also consider agents who hold rational expectations about dividends and wages ($P^i$ incorporates knowledge of the process (9)), so as to be able to isolate the pricing effects arising from subjective capital gains beliefs. We furthermore assume that

$$\delta \beta^{RE} < 1, \quad (13)$$

where $\beta^{RE} \equiv (\beta^D)^{1-\gamma} e^{(\gamma-1)\sigma^2_D/2}$. This insures existence of an equilibrium under rational price expectations. Since solving the optimization problem (7) for general (potentially non-rational) price beliefs is non-standard, appendix A.4 discusses conditions guaranteeing existence of an optimum, sufficiency of first order conditions and the existence of a recursive solution. These conditions are all satisfied for the preference specification (12) and the subjective price beliefs introduced in the remaining part of the paper and guarantee that the optimal solution to (7) takes the form

$$S^i_t = S^i_t \left( S^i_{t-1}, \frac{P_t}{D_t}, \frac{W_t}{D_t}, m^i_t \right), \quad (14)$$

where $m^i_t$ is a sufficient statistic characterizing the subjective distributions about future values of $\left( \frac{D_{t+j}}{D_{t+j-1}}, \frac{P_{t+j}}{D_{t+j}}, \frac{W_{t+j}}{D_{t+j}} \right)$ for $j > 0$. 

18
5 Rational Expectations (RE) Equilibrium

As a point of reference, we determine the equilibrium stock price implied by the RE hypothesis. Appendix A.5 derives the following result:

**Proposition 1** If agents hold rational expectations and if price expectations satisfy the usual transversality condition (stated explicitly in appendix A.5), then the RE equilibrium price is

\[
P_{RE}^t = \frac{(1 + \rho \varepsilon_t^W)\gamma b}{1 - \delta \beta_{RE}}
\]

where \( b = E[(1 + \rho \varepsilon_t^W)^{-\gamma} (\varepsilon_t^D)^{1-\gamma}]\gamma(1-\gamma)^{\sigma_D^2/2} \) and \( \beta_{RE} = (\beta_D)^{1-\gamma} e^{\gamma(\gamma-1)\sigma_D^2/2} \).

The PD ratio is an iid process under RE, thus fails to match the persistence of the PD ratio observed in the data. Moreover, since the volatility of \( \varepsilon_t^W \) tends to be small, it fails to match the large variability of stock prices. Furthermore, the RE equilibrium implies a negative correlation between the PD ratio and expected returns, contrary to what is evidenced by survey data. To see this note that (15) implies

\[
\ln P_{t+1}^{RE} - \ln P_t^{RE} = \ln \beta^D + \ln \varepsilon_{t+1}^P;
\]

where \( \varepsilon_{t+1}^P \equiv \varepsilon_{t+1}^D (1 + \rho \varepsilon_{t+1}^W)/(1 + \rho \varepsilon_t^W) \), so that one-step-ahead price growth expectations covary negatively with the current price dividend ratio.\(^{34}\) Since the dividend component of returns also covaries negatively with the current price, the same holds true for expected returns.

In the interest of deriving analytical solutions, we consider below the limiting case with vanishing uncertainty \( (\sigma_D^2, \sigma_W^2 \to 0) \). The RE solution then simplifies to the perfect foresight outcome

\[
\frac{P_t^{RE}}{D_t} = \frac{\delta \beta_{RE}}{1 - \delta \beta_{RE}};
\]

which has prices and dividends growing at the common rate \( \beta_D \).

---

\(^{34}\)The PD ratio under RE is proportional to \( 1 + \rho \varepsilon_t^W \), see equation (15), while \( \varepsilon_{t+1}^P \) depends inversely on \( 1 + \rho \varepsilon_t^W \).
6 Learning about Capital Gains and Internal Rationality

Price growth in the RE equilibrium displays only short-lived deviations from dividend growth, with any such deviation being undone in the subsequent period, see equation (16). Price growth in the data, however, can persistently outstrip dividend growth, thereby giving rise to a persistent increase in the PD ratio and an asset price boom; conversely it can fall persistently short of dividend growth and give rise to a price bust, see figure 1. This behavior of actual asset prices suggests that it is of interest to relax the RE beliefs about price behavior. Indeed, in view of the behavior of actual asset prices in the data, agents may entertain a more general model of price behavior, incorporating the possibility that the growth rate of prices persistently exceeds/falls short of the growth rate of dividends. To the extent that the equilibrium asset prices implied by these beliefs display such data-like behavior, agents’ beliefs will be generically validated.

Generalized Price Beliefs. In line with the discussion in the previous paragraph, we assume agents perceive prices evolving according to the process

\[ \ln P_{t+1} - \ln P_t = \ln \beta_{t+1} + \ln \varepsilon_{t+1}, \]  

(18)

where \( \varepsilon_{t+1} \) denotes a transitory shock to price growth and \( \beta_{t+1} \) a persistent price growth component that drifts slowly over time according to

\[ \ln \beta_{t+1} = \ln \beta_t + \ln \nu_{t+1}. \]  

(19)

This setup can capture periods with sustained increases in the PD ratio (\( \beta_{t+1} > \beta_d \)) or sustained decreases (\( \beta_{t+1} < \beta_d \)).\(^{35}\) In the limiting case where the variance of the innovation \( \ln \nu_{t+1} \) becomes small, the persistent price growth component behaves almost like a constant, as is the case in the RE solution.

For simplicity, we assume that agents perceive the innovations \( \ln \varepsilon_{t+1} \) and \( \ln \nu_{t+1} \) to

\(^{35}\)We do not incorporate mean-reversion into price growth beliefs in our benchmark setting because we seek to determine the model-endogenous forces that lead to a reversal of price booms and busts, i.e., we do not want to obtain reversals because they are hard-wired into beliefs. While our benchmark represents the most parsimonious way of specifying beliefs that contemplate the possibility of price booms and busts in stock prices, we extend the setup to a setting with mean reverting beliefs in section 11.2.
be jointly normally distributed according to
\[
\begin{pmatrix}
\ln \varepsilon_{t+1} \\
\ln \nu_{t+1}
\end{pmatrix}
\sim iiN \left( \begin{pmatrix}
-\frac{\sigma_\varepsilon^2}{2} \\
-\frac{\sigma_\nu^2}{2}
\end{pmatrix},
\begin{pmatrix}
\sigma_\varepsilon^2 & 0 \\
0 & \sigma_\nu^2
\end{pmatrix}\right).
\] (20)

Since agents observe the change of the asset price, but do not separately observe the persistent and transitory elements driving it, the previous setup defines a filtering problem in which agents need to decompose observed price growth into the persistent and transitory subcomponents, so as to forecast optimally.

To emphasize the importance of learning about price behavior rather than learning about the behavior of dividends or the wage income process, which was the focus of much of an earlier literature on learning in asset markets, e.g., Timmermann (1993, 1996), we continue to assume that agents know the processes (9), i.e., hold rational dividend and wage expectations.

Internal Rationality of Price Beliefs. Among academics there appears to exist a widespread belief that rational behavior and knowledge of the fundamental processes (dividends and wages in our case) jointly dictate a certain process for prices and thus the price beliefs agents can rationally entertain.\textsuperscript{36} This view stipulates that rational behavior implies knowledge of the pricing function (as under RE or Bayesian RE), so that postulating subjective price beliefs as those specified in equation (18) would be inconsistent with the assumption of optimal behavior on the part of agents.

This view is correct in some special cases, for example, when agents are risk neutral and do not face trading constraints but fails to be true more generally. Therefore, agents in our model are ‘internally rational’: their behavior is optimal given an internally consistent system of subjective beliefs about variables that are beyond their control, including prices.

To illustrate this point, consider first risk neutral agents with rational dividend expectations and ignore limits to stock holdings. Forward-iteration on the agents’ own optimality condition (48) then delivers the present value relationship
\[
P_t = E_t \left[ \sum_{i=1}^{T} \delta^i D_{t+i} \right] + \delta^T E_t^{P_t} [P_{t+T}],
\]
which is independent of the agents’ own choices. Provided agents’ price beliefs satisfy a

\textsuperscript{36}We often received this reaction during seminar presentations.
standard transversality condition \( \lim_{T \to \infty} \delta^T E_t^{P_t} [P_{t+T}] = 0 \) for all \( i \), then each rational agent would conclude that there must be a degenerate joint distribution for prices and dividends given by

\[
P_t = E_t \left[ \sum_{i=1}^\infty \delta^i D_{t+i} \right] \text{ a.s.} \tag{21}
\]

Since the r.h.s of the previous equation is fully determined by dividend expectations, the beliefs about the dividend process deliver the price process compatible with optimal behavior. In such a setting, it would be plainly inconsistent with optimal behavior to assume the subjective price beliefs (18)-(19).\(^{37}\)

Next, consider a concave utility function \( u(\cdot) \) satisfying standard Inada conditions. Forward iteration on (48) and assuming an appropriate transversality condition then delivers

\[
P_t u'(C_t^i) = E_t^{P_t} \left[ \sum_{j=1}^\infty \delta^j D_{t+j} u'(C_{t+j}^i) \right] \text{ a.s.} \tag{22}
\]

Unlike in equation (21), the previous equation depends on the agent’s current and future consumption. Equation (22) thus falls short of mapping dividend beliefs into a price outcome. This is so because the agent’s optimal consumption depends on price expectations, so that equation (22) fails to determine what optimizing agents can possibly believe about the price process, given their knowledge about \( W, D \) alone. Indeed, the equilibrium price \( P_t \) will be such that (22) holds, i.e., will be such that the price is equal to the agent’s present value of future dividends, where discounting occurs with the internally rational stochastic discount factor that depends on the agent’s price expectations.

With the considered non-linear utility function, we can thus simultaneously assume that agents maximize utility, hold the subjective price beliefs (18)-(19) and rational beliefs about dividends and wages.

**Learning about the Capital Gains Process.** The beliefs (18) give rise to an optimal filtering problem. To obtain a parsimonious description of the evolution of beliefs, we specify conjugate prior beliefs about the unobserved persistent component \( \ln \beta_t \) at \( t = 0 \). Specifically, agent \( i \)’s prior is

\[
\ln \beta_0 \sim N(\ln m^i_0, \sigma^2), \tag{23}
\]

where prior uncertainty \( \sigma^2 \) is assumed to be equal to its Kalman filter steady state value.

\(^{37}\)See Adam and Marcet (2011) for a discussion of how in the presence of trading constraints, this conclusion breaks down, even with risk-neutral consumption preferences.
i.e.,
\[
\sigma^2 = \frac{-\sigma_v^2 + \sqrt{(\sigma_v^2)^2 + 4\sigma_v^2\sigma_\varepsilon^2}}{2},
\]
and the prior is also assumed independent of all other random variables at all times. Equations (18), (19) and (23), and knowledge of the dividend and wage income processes (9) then jointly specify agents’ probability beliefs \( \mathcal{P}^i \).

The optimal Bayesian filter then implies that the posterior beliefs following some history \( \omega^t \) are given by\(^{38} \)
\[
\ln \beta_t | \omega^t \sim N(\ln m_t^i, \sigma^2),
\]
with
\[
\ln m_t^i = \ln m_{t-1}^i - \frac{\sigma_v^2}{2} + g \left( \ln P_t - \ln P_{t-1} + \frac{\sigma_v^2 + \sigma_\varepsilon^2}{2} - \ln m_{t-1}^i \right)
\]
\[
g = \frac{\sigma_v^2}{\sigma^2 + \sigma_\varepsilon^2}.
\]

Agents’ beliefs can thus be parsimoniously summarized by a single state variable \( m_t^i \) describing agents’ degree of optimism about future capital gains. These beliefs evolve recursively according to equation (26) and imply that
\[
E_{t}^{\mathcal{P}^i} \left[ \frac{P_{t+1}}{P_t} \right] = e^{\ln m_t^i + \sigma^2/2},
\]
which is - up to the presence of a log and exponential transformation and some variance correction terms - identical to the adaptive prediction model (6) considered in section 3.3.

**Nesting PF Equilibrium Expectations.** The subjective price beliefs (18),(19) and (23) generate perfect foresight equilibrium price expectations in the special case in which prior beliefs are centered at the growth rate of dividends, i.e.,
\[
\ln m_0^i = \ln \beta^D,
\]
and when considering the limiting case with vanishing uncertainty, where \( \sigma_v^2, \sigma_\varepsilon^2, \sigma_D^2, \sigma_W^2 \rightarrow 0 \). Agents’ prior beliefs at \( t = 0 \) about price growth in \( t \geq 1 \) then increasingly concentrates at the perfect foresight outcome \( \ln \beta^D \), see equations (18) and (19). With

\(^{38}\)See theorem 3.1 in West and Harrison (1997). Choosing a value for \( \sigma^2 \) different from the steady state value (24) would only add a deterministically evolving variance component \( \sigma_\varepsilon^2 \) to posterior beliefs with the property \( \lim_{t \to \infty} \sigma_\varepsilon^2 = \sigma^2 \), i.e., it would converge to the steady state value.
price and dividend expectations being at their PF value, the perfect foresight price $PD_0 = \delta \beta^{RE}/(1 - \delta \beta^{RE})$ becomes the equilibrium outcome at $t = 0$ in the limit. Importantly, it continues to be possible to study learning dynamics in the limit with vanishing risk: keeping the limiting ratio $\sigma_\epsilon^2/\sigma_g^2$ finite and bounded from zero as uncertainty vanishes, the Kalman gain parameter $g$ defined in (27), remains well-specified in the limit and satisfies $\lim \frac{\sigma_\epsilon^2}{\sigma_g^2} = \lim \frac{g^2}{1-g}$. We will exploit this fact in section 8 when presenting analytical results.

7 Dynamics under Learning

This section explains how equilibrium prices are determined under the subjective beliefs introduced in the previous section and how they evolve over time.

Agents’ stock demand is given by equation (14). Stock demand depends on the belief $m^i_t$, which characterizes agents’ capital gains expectations. These beliefs evolve according to (26). As a benchmark, we shall now assume that all agents hold identical beliefs ($m^i_t = m_t$ for all $i$). While agents may initially hold heterogenous prior beliefs $m^0$, heterogeneity would asymptotically vanish because all agents observe the same price history. The asset dynamics derived under the assumption of identical beliefs thus describe the long-run outcome of the model.

Using this assumption and imposing market clearing in periods $t$ and $t-1$ in equation (14) shows that the equilibrium price in any period $t \geq 0$ solves

$$1 = S \left( 1, \frac{P_t}{D_t}, \frac{W_t}{D_t}, m_t \right),$$

which exploits the fact that the total supply of stocks is equal to one.

The beliefs $m_t$ and the price dividend ratio $P_t/D_t$ are now simultaneously determined via equations (26) and (29). Unfortunately, this simultaneity could give rise to multiple market clearing price and belief pairs, due to a complementarity between realized capital gains and expected future capital gains.\textsuperscript{39} While this multiplicity may be a potentially interesting avenue to explain asset price booms and busts, analyzing price dynamics within such a setting would require introducing non-standard features, such as an equilibrium selection device for periods in which there are multiple solutions to (26) and (29). In-

\textsuperscript{39}Intuitively, a higher PD ratio implies higher realized capital gains and thus higher expectations of future gains via equation (26). Higher expected future gains may in turn induce a higher willingness to pay for the asset, thereby justifying the higher initial PD ratio.
stead, we resort to a standard approach of using only lagged information for updating beliefs.

Appendix A.6 shows that the simultaneity can be overcome by slightly modifying the information structure. The modification is relatively straightforward and consists of assuming that agents observe at any time $t$ information about the lagged temporary price growth component $\varepsilon_{t-1}$ entering equation (18). The appendix then shows that Bayesian updating implies that

$$\ln m_t = \ln m_{t-1} + g \left( \ln P_{t-1} - \ln P_{t-2} - \ln m_{t-1} \right) + g \ln \varepsilon_{t}^1, \quad (30)$$

where updating now occurs using only lagged price growth (even though agents do observe current prices) and where $\ln \varepsilon_{t}^1 \sim i.i.d. \left( -\frac{\sigma_t^2}{2}, \sigma_t^2 \right)$ is a time $t$ innovation to agent’s information set (unpredictable using information available to agents up to period $t - 1$), which reflects the information about the transitory price growth component $\varepsilon_{t-1}$ received in period $t$. This provides microfoundations for the updating with delayed information as in (30), which is used in most of the literature on learning about prices.

With this slight modification, agents’ beliefs $m_t$ are now pre-determined at time $t$, so that the economy evolves according to a uniquely determined recursive process: equation (29) determines the market clearing price for period $t$ given the beliefs $m_t$ and equation (30) determines how time $t$ beliefs are updated following the observation of the new market clearing price.\textsuperscript{40}

### 8 Equilibrium: Analytic Findings

This section derives a closed form solution for the equilibrium asset price for the special case where all agents hold the same subjective beliefs $\mathcal{P}$ and where these beliefs imply no (or vanishing) uncertainty about future prices, dividends and wages. While the absence of uncertainty is unrealistic from an empirical standpoint, it helps us in deriving key insights into how the equilibrium price depends on agents’ beliefs, as well as on how prices and beliefs evolve over time.\textsuperscript{41} The empirically more relevant case with uncertainty will be considered in section 10 using numerical solutions.

\textsuperscript{40}There could still be an indeterminacy arising from the fact that $S(\cdot)$ is non-linear, so that equation (29) may not have a unique solution. We have neither encountered such problems neither in our analytical solution nor when numerically solving the model.

\textsuperscript{41}An analytic solution can be found because in the absence of uncertainty one can evaluate more easily the expectations of nonlinear functions of future variables showing up in agents’ FOCs.
We present a series of results that increasingly adds assumptions on agents’ beliefs system $P$. The next section provides a closed form expression for the equilibrium PD ratio as a function of agents’ subjective expectations about future stock market returns for any belief system $P$ without uncertainty. Section 8.2 then discusses the pricing implications of this result for the subjective capital gains beliefs introduced in section 6. Finally, section 8.3 shows how the interaction between asset price behavior and subjective belief revisions can temporarily de-link asset prices from their fundamental value, i.e., give rise to a self-reinforcing boom and bust cycle in asset prices along which subjective expected returns rise and fall.

8.1 Main Result

The following proposition summarizes our main finding:

**Proposition 2** Suppose $u(C) = C^{1-\gamma}/(1 - \gamma)$, agents’ beliefs $P$ imply no uncertainty about future prices, dividends and wages, and

$$\lim_{T \to \infty} E^P_t R_T > 1 \quad \text{and} \quad \lim_{T \to \infty} E^P_t \left( \sum_{j=1}^{T} \left( \prod_{i=1}^{j} \frac{1}{R_{t+i}} \right) W_{t+j} \right) < \infty, \quad (31)$$

then the equilibrium PD ratio in period $t$ is given by

$$\frac{P_t}{D_t} = \left( 1 + \frac{W_t}{D_t} \right) \sum_{j=1}^{\infty} \left( \left( \frac{1}{\delta} \right)^j \left( E^P_t \prod_{i=1}^{j} \frac{1}{R_{t+i}} \right) \frac{\gamma-1}{\gamma} \right)$$

$$- \frac{1}{D_t} E^P_t \left( \prod_{j=1}^{\infty} \left( \prod_{i=1}^{j} \frac{1}{R_{t+i}} \right) W_{t+j} \right) \quad (32)$$

Conditions (31) insure that the infinite sums in the pricing equation (32) converge.\(^{43}\)

Under the additional assumption that agents hold rational wage and dividend expecta-

\(^{42}\)The proof can be found in appendix A.7.

\(^{43}\)These are satisfied, for example, for the expectations associated with the perfect foresight RE solution. Equation (32) then implies that the PD ratio equals the perfect foresight PD ratio (17), as is easily verified. Conditions (31) are equally satisfied for the subjective beliefs defined in section 6, when considering the case with vanishing uncertainty $(\sigma_{\epsilon}, \sigma_{\nu}^2, \sigma_D^2, \sigma_W^2) \to 0$. 

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tions, equation (32) simplifies further to

\[
\frac{P_t}{D_t} = (1 + \rho) \sum_{j=1}^{\infty} \left( (\beta^D)^j \left( E_t^P \prod_{i=1}^{j} \frac{1}{R_{t+i}} \right)^{1/\gamma} \right)
\]

\[
-\rho \left( \sum_{j=1}^{\infty} (\beta^D)^j \left( E_t^P \prod_{i=1}^{j} \frac{1}{R_{t+i}} \right) \right). \tag{33}
\]

We now discuss the implications of equation (33), focusing on the empirically relevant case where \( \rho > 0 \) and \( \gamma > 1 \).

Consider first the upper term on the r.h.s. of equation (33), which is decreasing in the expected asset returns. This emerges because for \( \gamma > 1 \) the wealth effect of a change in return expectations then dominates the substitution effect, so that expected asset demand and therefore the asset price has a tendency to decrease as return expectations increase. The negative wealth effect thereby increases in strength if the ratio of wage to dividend income (\( \rho \)) increases. This is the case because higher return expectations also reduce the present value of wage income.

Next, consider the lower term on the r.h.s. of equation (33), including the negative sign pre-multiplying it. This term depends positively on the expected returns and captures a substitution effect that is associated with increased return expectations. This substitution effect only exists if \( \rho > 0 \), i.e., only in the presence of non-dividend income, and it is increasing in \( \rho \). It implies that increased return expectations are associated with increased stock demand and thus with a higher PD ratio in equilibrium. It is this term that allows the model to match the positive correlation between expected returns and the PD ratio.

This substitution effect is present even in the limiting case with log consumption utility (\( \gamma \to 1 \)). The upper term on the r.h.s. of equation (33) then vanishes because the substitution and wealth effects associated with changes in expected returns cancel each other, but the lower term still induces a positive relationship between prices and return expectations. The substitution effect is also present for \( \gamma > 1 \) and can then dominate the negative wealth effect arising from the upper term on the r.h.s. of (33). Consider, for example, the opposite limit with \( \gamma \to \infty \). Equation (33) then delivers

\[
\frac{P_t}{D_t} = \sum_{j=1}^{\infty} \left( 1 + \rho \sum_{j=1}^{\infty} \left( 1 - (\beta^D)^j \right) \right) \left( E_t^P \prod_{i=1}^{j} \frac{1}{R_{t+i}} \right) .
\]

Since \( \beta^D > 1 \), there is a positive relationship between prices and expected asset returns, whenever \( \rho \) is sufficiently large. The two limiting results (\( \gamma \to 1 \) and \( \gamma \to \infty \)) thus suggest
that for sufficiently large $\rho$ the model can generate a positive relationship between return expectations and the PD ratio, in line with the evidence obtained from survey data.

### 8.2 PD Ratio and Expected Capital Gains

We now specialize the results in section 8.1 to the belief system introduced in section 6.\(^{44}\)

Equation (33) implies a non-linear relationship between the PD ratio and the subjective capital gain expectations $m_t$, but one cannot obtain a closed-form solution for the PD ratio as a function of the capital gains expectations.\(^{45}\) Figure 5 depicts the relationship between the PD ratio and $m_t$ using the parameterization employed in our quantitative application later on, but abstracting from future uncertainty.\(^{46}\)

Figure 5 shows that there is a range of price growth beliefs in the neighborhood of the perfect foresight value ($m_t = \beta D$) over which the PD ratio depends positively on expected price growth, similar to the positive relationship between expected returns and the PD ratio derived analytically in the previous section. Over this range, the substitution effect dominates the wealth effect because our calibration implies that dividend income finances only a small share of total consumption (approximately 4.3%). As a result, stock market wealth is only a small share of the total present value of household wealth (the same 4.3%) when beliefs assume their perfect foresight value ($m_t = \beta D$).

Figure 5 also reveals that there exists a capital gains belief beyond which the PD ratio starts to decrease. Mathematically, this occurs because if $m_t \to \infty$, expected returns also increase without bound\(^{47}\), so that $E_t^P \prod_{i=1}^{T} \frac{1}{R_{t+i}} \to 0$. From equation (33) one then obtains $P_t = D_t$.

The economic intuition for the existence of a maximum PD ratio is as follows: for

\[^{44}\text{Appendix A.8 proves that condition (31) is satisfied for all beliefs } m_t > 0.\]

\[^{45}\text{More precisely, with vanishing uncertainty the beliefs from section 6 imply } E_t^P [P_{t+i}] = (m_t)^i P_t,\]

which together with perfect foresight about dividends allows expressing agents’ expectations of future inverse returns as a function of $m_t$ and the current PD ratio:

\[E_t^P \frac{1}{R_{t+i}} = \frac{E_t^P P_{t+i-1}}{E_t^P P_{t+i} + E_t^P D_{t+i}} = \frac{(m_t)^i P_t}{(m_t)^i P_t + (\beta D)^i}.\]

Substituting this into (33) one can solve numerically for $P_t/D_t$ as a function of $m_t$.

\[^{46}\text{The parameterization assumes a moderate degree of risk aversion } \gamma = 2, \text{ a quarterly discount factor of } \delta = 0.995, \text{ quarterly real dividend growth equal to the average postwar growth rate of real dividends } \beta D = 1.0048, \text{ and } \rho = 22 \text{ to match the average dividend-consumption ratio in the U.S. over 1946-2011, see section 10 for further details.}\]

\[^{47}\text{This follows from } E_t^P R_{t+i+1} = E_t^P \frac{P_{t+i+1} + D_{t+i+1}}{P_{t+i}} > E_t^P \frac{P_{t+i+1}}{P_{t+i}} = m_t.\]
higher $m_t$ the present value of wage income is declining, as increased price growth optimism implies higher expected returns\footnote{This is shown in appendix A.10, which depicts the relationship between expected capital gains and expected returns at various forecast horizons.} and therefore a lower discount factor. This can be seen by noting that the FOC (48) can alternatively be written as

$$1 = \delta E_t^P \left[ \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma} R_{t+1} \right],$$

which implies that increased return expectations $E_t^P R_{t+1}$ imply a lower discount factor $\delta E_t^P [(C_{t+1}/C_t)^{-\gamma}]$.\footnote{This holds true under the maintained assumption of no or vanishing uncertainty.} With increased optimism, the present value of wage income thus falls. At the same time, stock market wealth initially increases strongly. Indeed, at the maximum PD ratio, stock market wealth amounts to approximately 4.5 times the value it assumes in the perfect foresight solution, see figure 5. This relative wealth shift has the same effect as a decrease in the wage to non-wage income ratio. As argued in section 8.1, for sufficiently small values of $\rho$ the income effect starts to dominate the substitution effect, so that prices start to react negatively to increased return optimism.

### 8.3 Endogenous Boom and Bust Dynamics

We now explain how the interplay between price realizations and belief updating can temporarily de-link asset prices from their fundamental values. This process emerges endogenously and takes the form of a sustained asset price boom along which expected returns rise and that ultimately results in a price bust along which expected returns fall. This feature allows the model to generate volatile asset prices and to capture the positive correlation between expected returns and the PD ratio.

Consider figure 5 and a situation in which agents become optimistic, in the sense that their capital gains expectations $m_t$ increase slightly above the perfect foresight value $m_{t-1} = \beta^D$ entertained in the previous period.\footnote{In the model with uncertainty, such upward revisions can be triggered by fundamentals, e.g., by an exceptionally high dividend growth realization in the previous period, which is associated with an exceptionally high price growth realization.} Figure 5 shows that this increase in expectations leads to an increase in the PD ratio, i.e., $P_t/D_t > P_{t-1}/D_{t-1}$. Moreover, due to the relatively steep slope of the PD function, realized capital gains will strongly exceed the initial increase in expected capital gains. The belief updating equation (30) then implies further upward revisions in price growth expectations and thus further capital gains, leading to a sustained asset price boom in which the PD ratio and return expectations
jointly move upward.

The price boom comes to an end when expected price growth reaches a level close to where the PD function in figure 5 reaches its maximum. At this point, stock prices grow at most at the rate of dividends \((\beta^D)\), but agents hold considerably more optimistic expectations about future capital gains \((m_t > \beta^D)\). Investors’ high expectations will thus be disappointed, which subsequently leads to a reversal. During a price boom, expected price growth and actual price growth thus mutually reinforce each other. The presence of an upper bound in prices implies, however, that the boom must come to an end. Price growth must thus eventually become very low, sending actual and expected stock price growth eventually down.

The previous dynamics are also present in the stochastic model considered in the next sections. They introduce low frequency movements in the PD ratio, allowing the model to replicate boom and bust dynamics and thereby empirically plausible amounts of asset price volatility, despite assuming standard consumption preferences. These dynamics also generate a positive correlation between the PD ratio and expected returns.

In the deterministic model, however, the dynamics for the PD ratio tend to be temporary phenomena because beliefs tend to converge to the perfect foresight equilibrium.

**Lemma 1** Consider the limiting case without uncertainty and suppose investors hold rational dividend and wage beliefs.

1. For any \(m_t > 0\), we have \(\lim_{t \to \infty} m_t = \beta^D\), whenever \(\lim_{t \to \infty} m_t\) exists.

2. For \(m_t\) sufficiently close to \(\beta^D\) and \(g < \frac{1}{2}\), we have \(\lim_{t \to \infty} m_t = \beta^D\) if

\[
-1 < \frac{\beta^D}{PD(\beta^D)} \frac{\partial PD(m)}{\partial m} \bigg|_{m=\beta^D} < 1,
\]

where \(PD(m)\) is the equilibrium PD ratio associated with beliefs \(m\), as implied by equation (33).

The first result in the lemma provides a global convergence result. It shows that whenever beliefs settle down, they must settle down on the perfect foresight equilibrium.

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Footnote 45 explains how this mapping can be computed.

\(^{51}\)In the model with noise, fundamental shocks, e.g., a low dividend growth realization, can cause the process to end well before reaching this point.

\(^{52}\)While the arguments above only show that expected capital gains correlate positively with the PD ratio, Appendix A.10 shows that expected capital gains and expected returns comove positively, so that expected returns also comove positively with the PD ratio.

\(^{53}\)The proof of lemma 1 can be found in appendix A.9.
value. When this is the case, equilibrium prices also converge to the perfect foresight value. While technically one cannot rule out convergence to deterministic or chaotic cycles, the second result in the lemma shows that locally beliefs do converge to the perfect foresight equilibrium, whenever the elasticity of the PD ratio with respect to price growth believes is below one in absolute value and the gain parameter $g$ not too large. Condition (34) is satisfied, for example, for the parameterization of the estimated models reported in table 7.

To illustrate the global belief dynamics further, figure 6 depicts how beliefs evolve over time using the parameterization of the estimated models from table 7. The arrows in the figure indicate, starting from any point $(m_t, m_{t-1})$ in the plane, the direction in which the belief pair $(m_t, m_{t-1})$ evolves. The black dot indicates the position of the perfect foresight equilibrium $(m_t = m_{t-1} = \beta^D)$, which is a rest point of the dynamics. In line with what we find when simulating the model, figure 6 strongly suggests that beliefs globally converge to the perfect foresight equilibrium in the absence of stochastic disturbances.

### 9 Matching Asset Pricing Moments

This section evaluates the ability of the model to replicate key asset pricing moments when using dividend and wage shocks as fundamental driving forces. The set of data moments that we seek to replicate is listed in the second column of table 7. The first eight asset pricing moments listed in the table are standardly used in the asset pricing literature to summarize the main features of stock price volatility. It is well known that it is difficult for RE models with time separable utility functions to jointly match these moments. We augment this standard set of moments by the correlation between the PD ratio and expected stock returns, as implied by the UBS survey data, denoted $\text{Corr}[\text{PD}_t, E^{\text{P}}_t R_{t+1}]$ in the table. The first eight moments in table 7 include the mean, standard deviation and autocorrelation of the quarterly PD ratio ($E[\text{PD}], \text{Std}[\text{PD}]$ and $\text{Corr}[\text{PD}_t, \text{PD}_{t-1}]$, respectively), the mean and standard deviation of quarterly real stock returns ($E[r^s]$ and $\text{Std}[r^s]$), the risk-free interest rate ($E[r^b]$) and the regression coefficient.

---

55. The two estimated models in table 7 have a slightly different gain parameter $g$, but this leads to negligible differences in the vector graph.

56. To increase readability of the graph, the length of the arrows $l$ is nonlinearly rescaled by dividing by $l^{3/4}$ and then linearly adjusted, so as to fit into the picture.

57. While we refer to the data statistics in column two of table 7 as data ‘moments’, most of these statistics are functions of data moments.

58. These moments are also considered in Adam, Marcet, and Nicolini (2014).
(c) and R-square value ($R^2$) obtained from regressing five year ahead excess returns on the quarterly PD ratio.\(^{59}\)

Numerically solving the non-linear asset pricing model with subjective beliefs turns out to be computationally time-consuming, despite the fact that we extensively rely on parallelization in the solution algorithms.\(^{60}\) For this reason, we match the model to the data by calibrating most parameters and estimate only a key subset using the simulated method of moments (SMM).

Table 6 reports the calibrated parameters and the calibration targets.\(^{61}\) The mean and standard deviation of dividend growth ($\beta^D$ and $\sigma_D$) are chosen to match the corresponding empirical moments of the U.S. dividend process. The ratio of non-dividend to dividend income ($\rho$) is chosen to match the average dividend-consumption ratio in the U.S. for 1946-2011.\(^{62}\) The standard deviation of wage innovations ($\sigma_W$) and the covariance between wage and dividend innovations ($\sigma_{DW}$) are chosen, in line with Campbell and Cochrane (1999), such that the correlation between consumption and dividend growth is 0.2 and the standard deviation of consumption growth is one seventh of the the standard deviation of dividend growth.\(^{63}\) The perceived uncertainty in stock price growth ($\sigma_e$) is set equal to the empirical standard deviation of stock price growth.\(^{64}\)

This leaves us with three remaining parameters: the updating parameter $g$, the time discount factor $\delta$ and the risk aversion parameter $\gamma$. Letting $\theta = (g, \delta, \gamma)$ denote this set of parameters and $\Theta$ the set of admissible values, the SMM estimate $\hat{\theta}$ is given by

$$\hat{\theta} = \arg \min_{\theta \in \Theta} \left[ \hat{S} - S(\theta) \right]^{\prime} \Sigma \left[ \hat{S} - S(\theta) \right],$$  \(35\)

where $\hat{S}$ is the set of moments in the data to be matched (the ones listed in table 7), $S(\theta)$ the corresponding moment from the model and $\Sigma$ a weighting matrix.

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\(^{59}\) The regression also contains a constant whose value is statistically insignificant and not reported in the table.

\(^{60}\) The numerical solution is obtained by numerically determining the stock demand function (14) solving the FOC (48) under the subjectively perceived dividend, wage and price dynamics, where agents understand that their beliefs evolve according to (30). We verify that in the limiting case without uncertainty, our numerical solution algorithm recovers the analytical solution derived in proposition 2. Furthermore, in the case with uncertainty, we insure the accuracy of the numerical solution by verifying that the Euler equation errors are in the order of $10^{-5}$ over the relevant area of the state space. Insuring this requires a considerable amount of adjustment by hand of the grid points and grid size used for spanning the model’s state space. Further details of the solution approach are described in appendix A.11. The MatLab code used for solving the model is available upon request.

\(^{61}\) The targets are chosen to match features of the fundamental processes emphasized in the asset pricing literature.

\(^{62}\) See appendix A.3 for further details.

\(^{63}\) For details on how this can be achieved, see appendix A.3.

\(^{64}\) Since the gain parameter $g$ will be small, the contribution of $\sigma_e^2$ in (18) is negligible.
We pursue two estimation approaches, one that implements efficient SMM and an alternative one that emphasizes more directly the replication of the data moments. Both approaches have advantages and disadvantages, as we discuss below, but ultimately deliver similar estimates and model moments. All estimations exclude the risk free rate from the set of moments to be matched, as the model has a hard time in fully replicating the equity premium. We thus only report the risk-free interest rate implied by the estimated models.

Our first estimation approach chooses the weighting matrix $\tilde{\Sigma}$ in equation (35) to be equal to the inverse of the estimated covariance matrix of the data moments $\hat{\Sigma}$, as required for efficient SMM estimation. While an efficient weighting matrix is desirable for estimation, it forces us to exclude two additional model moments from the estimation. First, the weighting matrix turns out to be approximately singular; following Adam, Marcet and Nicolini (2014) we thus exclude the excess return regression coefficient ($c$) from the set of estimated moments. Second, we also have to exclude the correlation between the PD ratio and expected returns, so as to insure that we have a sufficiently large sample to estimate $\tilde{\Sigma}$ reliably.

The second estimation approach uses a diagonal weighting matrix $\tilde{\Sigma}$ in equation (35), with the diagonal entries consisting of the inverse of the individually estimated variances of the corresponding data moments in $\hat{\Sigma}$. This allows using the full set of moments in the estimation (except for the risk free rate) and emphasizes more directly matching the moments, as typically pursued in the asset pricing literature.

An unconstrained minimization of the objective function (35) over $\theta = (g, \delta, \gamma)$ turns out to be numerically unstable and computationally too costly. For this reason, we impose one additional restriction on the parameter space $\Theta$. Specifically, we impose $\delta (\beta^D)^{-\gamma} = 0.995 (\beta^D)^{-2}$, where $\beta^D$ assumes the value from table 6. This additional restriction is inspired by the fact that we know that the model can perform reasonably well for $(\delta, \gamma) = (0.995, 2)$ and resolves numerical instabilities in our numerical solution routines. In any case, this additional restriction can only constrain the empirical performance of the model, so that the goodness of fit results presented below constitute a lower bound on what the model can potentially achieve.

Table 7 reports the estimation outcome in terms of implied model moments, estimated

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65See section 5 and the online appendix in Adam, Marcet and Nicolini (2014) for details on how to use a systematic criterion for excluding moments, details on how to estimate $\tilde{\Sigma}$, how to compute $\hat{S}(\theta)$ using small samples and asymptotics.
parameters and t-ratios. In terms of estimated parameters, the discount factor is estimated to be close to one and relative risk aversion is slightly above 2. The estimated gains ($\hat{g}$) are very close to the values obtained in the empirical section 3.3.

For both estimation approaches, the model matches the data moments rather well. In particular, the model easily replicates the positive correlation between the PD ratio and expected stock returns (Corr[$PD_t, E_t^P R_{t+1}$]), in line with the value found in the survey data. This is achieved, even though this moment was not used in the estimation using the efficient weighting matrix. The model also performs well in terms of producing sufficient volatility for the PD ratio (Std[$PD$]) and stock returns (Std[$r^s$]). If anything, the model tends to produce too much volatility. The model also succeeds in replicating the mean and autocorrelation of the PD ratio (E[$PD$] and Corr[$PD_t, PD_{t-1}$]) and the evidence on excess return predictability ($c$ and $R^2$), even though the regression coefficient $c$ was not included in the set of moments to be matched.

The most significant shortcoming of the model concerns the equity premium. While it matches the average stock return (E[$r^s$]), it predicts a too high value for the risk free rate (E[$r^b$]) and thus only half of the equity premium observed in the data. Given the low estimated value for the degree of relative risk aversion ($\hat{\gamma}$), this is still a considerable success. Nevertheless, exploring whether model performance could be improved along this dimension by adding further model features appears to be an interesting avenue for further research.

Abstracting from the risk free rate, the estimation using the diagonal weighting matrix generates t-ratios at or below one for all remaining moments. The estimation using the full matrix performs similarly well, but implies too much volatility for stock returns and the PD ratio. The empirical performance of the subjective belief model is overall remarkable, especially when compared to the performance of the rational expectations version of the model. Table 8 reports the model moments implied by the RE model, using the same parameters as for the estimated models in table 7. The t-statistics then all increase in absolute terms, with some increases being quite dramatic. With objective price beliefs, the model produces insufficient asset price volatility (too low values for Std[$PD$] and Std[$r^s$]) and the wrong sign for the correlation between the PD ratio and expected stock returns (Corr[$PD_t, E_t^P R_{t+1}$]). It also gives rise to a tiny equity premium only. These

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66 The t-ratio is the ratio of the gap between the model and the data moment over the standard deviation of the moment in the data, as implied by the weighting matrix. For excluded moments we use the individually estimated standard deviations in the numerator. For the case of an efficient weighting matrix, the t-ratio for moments included is computed according to the proper covariance matrix that delivers a standard normal distribution, see the online appendix in Adam Marcet and Nicolini (2014).
features are rather robust across alternative parameterizations of the RE model and highlight the strong quantitative improvement obtained by incorporating subjective belief dynamics. Taken together, these results show that - according to our model - stock price volatility is not difficult to explain in the presence of subjective price growth dynamics.

Figure 7 illustrates how the subjective belief model improves empirical performance. The figure depicts the equilibrium PD ratio (y-axis) as a function of agents’ capital gain beliefs (x-axis). It graphs this relationship once for the model with uncertainty (black line) and once for the vanishing noise limit analyzed in the previous section (red line). While the presence of price, dividend and wage risk lowers the equilibrium PD ratio compared to a setting without risk, the functional form of the relationship remains qualitatively unchanged. The intuition from the vanishing noise limit thus carries over to the model with noise: the model continues to give rise to occasional boom and bust dynamics in asset prices.

10 Historical PD Ratio and Survey Evidence

This section shows that the estimated model from the previous section can successfully replicate the low-frequency movements in the time series of the postwar U.S. PD ratio, as well as the available time series of survey expectations.

To illustrate this point, we use the estimated model from column 3 in table 7 and feed the historically observed price growth observations into the model’s belief updating equation (30), so as to obtain a model-implied belief process. The resulting belief sequence is depicted in figure 8. We then use these model implied beliefs together with the equilibrium pricing function of the model to derive a model implied series for the PD ratio. Figure 9 graphs the model implied PD ratio together with the PD ratio observed in the data. It reveals that the model captures a lot of the low-frequency variation in the historically observed PD ratio. It captures particularly well the variations before the year 2000, including the strong run-up in the PD ratio from the mid 1990’s to year 2000. The model also predicts a strong decline of the PD ratio after the year 2000, but overpredicts

\footnote{For the stochastic solution, the equilibrium PD in figure 7 is determined from the market clearing condition (29) assuming $W_t/D_t = \rho$, to be comparable with the value this variable assumes in the vanishing risk limit. The figure assumes $\gamma = 2$ and $\delta = 0.995$, as in figure 5.}

\footnote{We set the initial price growth belief at the start of the sample equal to $\ln m_{Q:1946} = -1.11\%$, in line with the values employed in constructing figure 3, which allowed matching the UBS survey expectations. Since we cannot observe the shocks $\ln \epsilon_t^Q$ in (30), we set them equal to zero. Gaps between the model predicted and actual PD ratio may thus partly be due to these shocks.}

\footnote{We thereby set the wage-dividend ratio $W_t/D_t$ equal to its steady state value.}
the decline relative to the data.

Figure 10 depicts the model-implied price growth expectations and those implied by the UBS survey.\textsuperscript{70} While the model fits overall the survey data well, it predicts after the year 2003 considerably lower capital gains expectations, which partly explains why the model underpredicts the PD ratio in figure 9 towards the end of the sample period. Yet, the expectations gap in figure 10 narrows considerably after the year 2004, while this fails to be the case in figure 9. Underprediction of expected price growth thus explains only partly the deterioration of the fit of the PD ratio towards the end of the sample period.

The gap after the year 2000 emerging in figure 9 is hardly surprising, given the empirical evidence presented in figure 4, which shows that the relationship between the PD ratio and the expectations implied by equation (6) has shifted upward in the data following the year 2000. While we can only speculate about potential reasons causing this shift, the exceptionally low real interest rates implemented by the Federal Reserve following the reversal of the tech stock boom and following the collapse of the subsequent housing boom may partly contribute to the observed discrepancy. Formally incorporating the asset pricing effects of monetary policy decisions is, however, beyond the scope of the present paper.

\section{Robustness Analysis}

This section explores the robustness of our main findings to changing key model parameters and to using more general price belief systems. The next section studies the pricing effects of alternative model parameters. Section 11.2 discusses the implications of more general price belief systems that incorporate mean-reversion of the PD ratio.

\subsection{Model Parameterization}

This section studies how the model’s ability to generate boom and bust dynamics in stock prices depends on key model parameters. The equilibrium pricing function for our baseline parameterization, depicted in figure 5, allows for self-reinforcing stock price boom and bust dynamics because the price dividend ratio increases initially strongly with capital gain optimism. This section explores the robustness of this feature by studying how the equilibrium pricing is affected by the coefficient of relative risk aversion ($\gamma$), the discount factor ($\delta$), and the average wage to dividend income ratio ($\rho$).

\textsuperscript{70}See footnote 30 for how to compute price growth expectations from the UBS survey.
Figure 11 depicts the equilibrium pricing function for alternative parameter choices. Each panel plots the pricing function from our baseline parameterization, as well as those generated by increasing or decreasing the values for $\gamma$, $\delta$ and $\rho$. The top panel, for example, shows that lowering (increasing) the coefficient of relative risk aversion increases (reduces) the hump in the PD function and moves it to the left (right), thereby causing asset price booms to become more (less) likely and larger in size. Similar effects are associated with increasing (decreasing) the wage to dividend income ratio ($\rho$) and with increasing (decreasing) the discount factor ($\delta$). Overall, figure 11 shows that the model can produce hump-shaped equilibrium PD functions over a fairly wide set of parameter specifications.

11.2 Generalized Belief System

This section considers a generalized price belief system, which implies that investors expect the PD ratio to eventually mean-revert over time. This is motivated by the fact that the price belief system (18)-(19) employed in the main part of the paper can imply that agents expect the log of the PD ratio to have a permanent drift, which may be implausible on a priori grounds. This section shows that the pricing implications of the model do not depend on the expected long run behavior for the PD ratio and that very similar equilibrium pricing functions can be generated by belief systems that imply persistent changes in the PD ratio but where the PD ratio is ultimately mean reverting.

It is important to note that the survey data provide little support for mean reverting price growth expectations over the available forecast horizons and thus for mean reversion in the expected PD ratio. In particular, the last four rows in table 5a show that the regression coefficient $\hat{c}$ obtained from regressing the expected 10 year ahead capital gain forecasts from the Shiller survey on the PD ratio is more than 10 times larger than the corresponding regression coefficient obtained from regressing the one year ahead forecast on the PD ratio. The expected annualized ten year price growth thus reacts stronger to movements in the PD ratio than the expected one year price growth. This shows that one can incorporate only a mild degree of mean reversion into subjective price growth beliefs, as the belief system would otherwise become inconsistent with the survey evidence.

To analyze the effects of mean reversion in price beliefs, we now assume that investors
perceive prices to evolve according to

\[
\ln P_{t+1} = \ln \beta_{t+1} + \ln P_t + (1 - \eta_{PD}) \ln PD + \ln \varepsilon_{t+1} \tag{36}
\]
\[
\ln \beta_{t+1} = (1 - \eta_\beta) \ln \beta^D + \eta_\beta \ln \beta_t + \ln \nu_{t+1}, \tag{37}
\]

where \( \ln PD \) denotes the perceived long run mean of the log PD ratio and \( \eta_{PD}, \eta_\beta \in [0, 1] \) are given parameters. For \( \eta_{PD} = \eta_\beta = 1 \) these equations deliver the price belief system (18)-(19) studied in the main part of the paper.\textsuperscript{71} For \( \eta_\beta < 1 \), equation (37) implies that agents expect mean reversion in the persistent price growth component \( \ln \beta_t \) towards the mean growth rate of dividends \( (\ln \beta^D) \); if in addition \( \eta_{PD} < 1 \), equation (36) implies that agents expect \( \ln P_t/D_t \) to eventually return to its long run mean \( \ln PD \).\textsuperscript{72}

Suppose that \( \eta_\beta < 1 \) and \( \eta_{PD} < 1 \), that agents are optimistic about future capital gains, i.e., they believe \( \beta_t \) to be above \( \beta^D \), and that agents observe a PD ratio above its long-run mean \( (P_t/D_t > PD) \). Provided \( \eta_\beta \) and \( \eta_{PD} \) are sufficiently close to one, equations (36)-(37) imply that agents expect a fairly persistent boom in the PD ratio, as is the case with the standard belief system (18)-(19). Yet, unlike with the standard belief system, they also expect a price bust further down the road, because the PD ratio is expected to eventually return to its long run value.

Equations (36) and (37) jointly imply that optimal belief updating about the unobserved persistent stock price growth component \( \ln \beta_t \) is described by a generalized version of equation (30), which states that the posterior mean \( \ln m_t \equiv E^D (\ln \beta_t \mid P^t) \) in steady state evolves according to\textsuperscript{73}

\[
\ln m_t = (1 - \eta_\beta) \ln \beta^D + \eta_\beta \ln m_{t-1} + g \left( \begin{array}{c}
\ln P_{t-1} - \ln P_{t-2} - \eta_\beta \ln m_{t-1} \\
- \eta_{PD} \ln (PD - \ln P_{t-1}/D_{t-1})
\end{array} \right) + g \ln \varepsilon_t. \tag{38}
\]

Using the generalized belief and updating equations, figure 12 depicts the impact of different perceived values for \( \eta_{PD} \) and \( \eta_\beta \) on the equilibrium pricing function when setting the perceived long-run mean \( \ln PD \) in (36) equal to the perfect foresight value of the \( PD \)

\textsuperscript{71} As before, we assume that agents have rational expectations about the dividend and wage income processes.

\textsuperscript{72} This can be seen by subtracting (8) from (36).

\textsuperscript{73} Since equation (36) only introduces an additional observable variable into (18) and equation (37) only adds a known constant and known mean reversion coefficient relative to (19), the arguments delivering equation (30) as optimal Bayesian updating directly generalize to equation (38).
The figure’s top panel plots the effects of decreasing \( \eta_{PD} \) below one, while keeping \( \eta_\beta = 0.9999 \). For the considered values of \( \eta_{PD} \), agents expect the log PD ratio to mean revert by 1%, 2%, or 3% per year towards its long run mean (\( \ln PD \)). The figure also plots the outcome when there is virtually no mean reversion (\( \eta_{PD} = 0.9999 \)). The top panel of figure 12 shows that by introducing mean reversion, one pushes the peak of the equilibrium PD ratio to the right and also lowers its height. The shift to the right occurs because agents only expect a persistent boom in the PD ratio if the increase in the PD ratio implied by the persistent growth component (\( \ln \beta_i \)) outweighs the mean reversion generated by the negative feedback from the deviation of the PD from its long run value in equation (36). The downward shift in the PD ratio occurs because mean reversion causes investors to expect lower and eventually negative returns sooner.

The second panel in figure 12 depicts the effects of decreasing \( \eta_\beta \) below one, while keeping \( \eta_{PD} = 0.9999 \). As before, we consider values for \( \eta_\beta \) that imply virtually no mean reversion and mean reversion by 1%, 2% and 3% per year towards the long run value (\( \ln \beta^D \)). The panel shows that the pricing implications are very similar to those of decreasing \( \eta_{PD} \).

Finally, the bottom panel depicts the effects of jointly decreasing \( \eta_\beta \) and \( \eta_{PD} \). The pricing implication of such simultaneous changes turn out to be considerably stronger, compared to the case where only one of the persistence parameters decreases. Figure 13 illustrates why this sharp difference occurs. It graphs the expected path of the PD ratio for different parameter combinations (\( \eta_{PD}, \eta_\beta \)).\(^{75}\) If either \( \eta_{PD} \) or \( \eta_\beta \) are close to 1, agents expect a rather prolonged stock price boom that is expected to revert only in the distant future. Expected stock returns are thus high for many periods before they turn negative. This differs notably from the case where both persistence parameters fall significantly below one (\( \eta_{PD} = \eta_\beta = 0.974 \)). The expected stock price boom is then much smaller and considerably more short-lived, so that returns are lower and expected to become negative earlier. From the discussion following equation (33), it should be clear that the implied path for expected returns then cannot sustain a high PD ratio as an equilibrium outcome.

Summing up, the model continues to given rise to hump-shaped equilibrium pricing

\(^{74}\)We assume that \( \sigma_\varepsilon^2 \) and \( \sigma_\beta^2 \) assume the same values as in the baseline specification, so that the gain parameter also remains unchanged at \( g = 0.02515 \). This parameterization makes sure that - for the values of \( \eta_\beta \) and \( \eta_{PD} \) considered below - the perceived standard deviation for stock price growth implied by equations (37)-(36) approximately matches the standard deviation of stock price growth in the data.

\(^{75}\)The figure assumes that the equilibrium PD ratio initially equals \( PD_0 = 150 \), i.e., it is above its long run value, and that agents are mildly optimistic about future capital gains with \( \ln m_0 = 1\% > \ln \beta^P \).
functions, even if agents ultimately expect mean reversion in the PD ratio, provided the generalized belief system implies that agents expect elevated capital gains to be sufficiently persistent.

12 Conclusions

We present a model with rationally investing agents that gives rise to market failures in the sense that the equilibrium stock price deviates from its fundamental value. These deviations take the form of asset price boom and bust cycles that are fueled by the belief-updating dynamics of investors who behave optimally given their imperfect knowledge about the behavior of stock prices. Investors update beliefs about market behavior using observed market outcomes and Bayes' law, causing their subjective expectations about future capital gains to commove positively with the price-dividend ratio, consistent with the evidence available from investor surveys. As we argue, this feature cannot be replicated within asset pricing models that impose rational price expectations.

We relax slightly the RE assumption but maintain full rationality of investors. The fact that a fairly small deviation from a standard asset pricing model significantly improves the empirical fit of the model strongly suggests that issues of learning are important when accounting for stock price fluctuations.

If asset price dynamics are to a large extent influenced by investors’ subjective belief dynamics, i.e., by subjective optimism and pessimism, then the asset price fluctuations observed in the data are to considerable extent inefficient. Due to a number of simplifying assumptions, this did not yield adverse welfare implications within the present setup.\textsuperscript{76} For models incorporating investor heterogeneity, endogenous output or endogenous stock supply, such fluctuations can give rise to significant distortions that affect welfare. Exploring these within a setting that gives rise to quantitatively credible amounts of asset price fluctuations appears to be an interesting avenue for further research. Such research will in turn lead to further important questions, such as whether policy can and should intervene with the objective to stabilize asset prices.

\textsuperscript{76}This is true if one evaluates welfare using ex-post realized consumption.
<table>
<thead>
<tr>
<th>UBS Gallup</th>
<th>Nominal Return Exp.</th>
<th>Real Ret. Exp. (SPF)</th>
<th>Real Ret. Exp. (Michigan)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Average</td>
<td>Median</td>
<td>Average</td>
</tr>
<tr>
<td>Own portfolio, &gt;100k US$</td>
<td>0.80</td>
<td>(0.01)</td>
<td>0.79</td>
</tr>
<tr>
<td>Own portfolio, all investors</td>
<td>0.80</td>
<td>(0.01)</td>
<td>0.79</td>
</tr>
<tr>
<td>Stock market, &gt;100k US$</td>
<td>0.90</td>
<td>(0.004)</td>
<td>0.90</td>
</tr>
<tr>
<td>Stock market, all investors</td>
<td>0.90</td>
<td>(0.004)</td>
<td>0.90</td>
</tr>
</tbody>
</table>

Table 1: Correlation between PD ratio and 1-year ahead expected return measures (UBS Gallup Survey, robust p-values in parentheses)

<table>
<thead>
<tr>
<th>Shiller Survey</th>
<th>Nominal Capital Gain Exp.</th>
<th>Real Capital Gain Exp. (SPF)</th>
<th>Real Capital Gain Exp. (Michigan)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Horizon</td>
<td>Average</td>
<td>Median</td>
<td>Average</td>
</tr>
<tr>
<td>1 month</td>
<td>0.46</td>
<td>(0.01)</td>
<td>0.45</td>
</tr>
<tr>
<td>3 months</td>
<td>0.57</td>
<td>(0.01)</td>
<td>0.54</td>
</tr>
<tr>
<td>6 months</td>
<td>0.58</td>
<td>(0.01)</td>
<td>0.54</td>
</tr>
<tr>
<td>1 year</td>
<td>0.43</td>
<td>(0.03)</td>
<td>0.38</td>
</tr>
<tr>
<td>10 years</td>
<td>0.74</td>
<td>(0.01)</td>
<td>0.66</td>
</tr>
</tbody>
</table>

Table 2: Correlation between PD ratio and expected stock price growth (Shiller’s Individual Investors’ Survey, robust p-values in parentheses)

<table>
<thead>
<tr>
<th>CFO Survey</th>
<th>Nominal Return Exp.</th>
<th>Real Return Exp. (SPF)</th>
<th>Real Return Exp. (Michigan)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Average</td>
<td>Median</td>
<td>Average</td>
</tr>
<tr>
<td>1 year</td>
<td>0.71</td>
<td>(0.00)</td>
<td>0.62</td>
</tr>
</tbody>
</table>

Table 3: Correlation between PD ratio and 1-year ahead expected stock return measures (CFO Survey, robust p-values in parentheses)
<table>
<thead>
<tr>
<th>Variables</th>
<th>Time Period</th>
<th>Stock Index</th>
<th>Correlation</th>
</tr>
</thead>
<tbody>
<tr>
<td>PD, 1 year-ahead real return</td>
<td>UBS Gallup sample</td>
<td>S&amp;P 500</td>
<td>-0.66 (0.08)</td>
</tr>
<tr>
<td>PD, 1 year-ahead real price growth</td>
<td>Shiller 1 year sample</td>
<td>Dow Jones</td>
<td>-0.42 (0.06)</td>
</tr>
<tr>
<td>PD, 10 year-ahead real price growth</td>
<td>Shiller 10 year sample</td>
<td>Dow Jones</td>
<td>-0.88 (0.16)</td>
</tr>
<tr>
<td>PD, 1 year-ahead real return</td>
<td>CFO sample</td>
<td>S&amp;P 500</td>
<td>-0.46 (0.06)</td>
</tr>
</tbody>
</table>

Table 4: Correlation between PD and actual real returns/capital gains (robust p-value in parentheses)

<table>
<thead>
<tr>
<th>Survey measure</th>
<th>$\hat{c} \cdot 10^3$</th>
<th>$\hat{c} \cdot 10^3$</th>
<th>p-value</th>
<th>$\hat{c} \cdot 10^3$</th>
<th>$\hat{c} \cdot 10^3$</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$H_0: \hat{c} = \hat{\lambda}$</td>
<td>$H_0: \hat{c} = \hat{\lambda}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>S&amp;P 500, real returns</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>UBS*, &gt;100k, 1 yr, SPF</td>
<td>0.56</td>
<td>-2.93</td>
<td>0.0000</td>
<td>0.44</td>
<td>-2.93</td>
<td>0.0000</td>
</tr>
<tr>
<td>UBS*, &gt;100k, 1 yr, Michigan</td>
<td>0.55</td>
<td>-2.93</td>
<td>0.0000</td>
<td>0.43</td>
<td>-2.93</td>
<td>0.0000</td>
</tr>
<tr>
<td>UBS*, all, 1 yr, SPF</td>
<td>0.54</td>
<td>-2.93</td>
<td>0.0000</td>
<td>0.45</td>
<td>-2.93</td>
<td>0.0000</td>
</tr>
<tr>
<td>UBS*, all, 1 yr, Michigan</td>
<td>0.53</td>
<td>-2.93</td>
<td>0.0000</td>
<td>0.44</td>
<td>-2.93</td>
<td>0.0000</td>
</tr>
<tr>
<td>CFO, 1 yr, SPF</td>
<td>0.29</td>
<td>-1.88</td>
<td>0.0003</td>
<td>0.24</td>
<td>-1.74</td>
<td>0.0176</td>
</tr>
<tr>
<td>CFO, 1 yr, Michigan</td>
<td>0.26</td>
<td>-1.88</td>
<td>0.0001</td>
<td>0.32</td>
<td>-1.74</td>
<td>0.0135</td>
</tr>
<tr>
<td>Dow Jones, real price growth</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Survey Average</td>
<td>Survey Median</td>
<td></td>
<td>Survey Average</td>
<td>Survey Median</td>
<td></td>
</tr>
<tr>
<td>Shiller, 1 yr, SPF</td>
<td>0.23</td>
<td>-1.48</td>
<td>0.0001</td>
<td>0.23</td>
<td>-1.48</td>
<td>0.0002</td>
</tr>
<tr>
<td>Shiller, 1 yr, Michigan</td>
<td>0.28</td>
<td>-1.48</td>
<td>0.0001</td>
<td>0.29</td>
<td>-1.48</td>
<td>0.0001</td>
</tr>
<tr>
<td>Shiller, 10 yrs, SPF</td>
<td>4.11</td>
<td>-6.48</td>
<td>0.0000</td>
<td>5.49</td>
<td>-6.48</td>
<td>0.0000</td>
</tr>
<tr>
<td>Shiller, 10 yrs, Michigan</td>
<td>3.51</td>
<td>-6.48</td>
<td>0.0000</td>
<td>4.89</td>
<td>-6.48</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

* stock market return expectations

Table 5a: Forecast rationality test (returns)
<table>
<thead>
<tr>
<th>Survey measure</th>
<th>$\hat{c} \cdot 10^3$</th>
<th>$\hat{c} \cdot 10^3$</th>
<th>p-value</th>
<th>$\hat{c} \cdot 10^3$</th>
<th>$\hat{c} \cdot 10^3$</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>S&amp;P 500, real excess returns</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Survey Average</td>
<td>Survey Median</td>
<td></td>
<td>Survey Average</td>
<td>Survey Median</td>
<td></td>
</tr>
<tr>
<td>UBS*, &gt;100k, 1 yr, SPF</td>
<td>0.25</td>
<td>-3.02</td>
<td>0.0000</td>
<td>0.14</td>
<td>-3.02</td>
<td>0.0000</td>
</tr>
<tr>
<td>UBS*, &gt;100k, 1 yr, Michigan</td>
<td>0.25</td>
<td>-3.02</td>
<td>0.0000</td>
<td>0.14</td>
<td>-3.02</td>
<td>0.0000</td>
</tr>
<tr>
<td>UBS*, all, 1 yr, SPF</td>
<td>0.23</td>
<td>-3.02</td>
<td>0.0000</td>
<td>0.15</td>
<td>-3.02</td>
<td>0.0000</td>
</tr>
<tr>
<td>UBS*, all, 1 yr, Michigan</td>
<td>0.23</td>
<td>-3.02</td>
<td>0.0000</td>
<td>0.15</td>
<td>-3.02</td>
<td>0.0000</td>
</tr>
<tr>
<td>CFO, 1 yr, SPF</td>
<td>0.04</td>
<td>-1.97</td>
<td>0.0016</td>
<td>0.12</td>
<td>-1.66</td>
<td>0.0571</td>
</tr>
<tr>
<td>CFO, 1 yr, Michigan</td>
<td>0.04</td>
<td>-1.97</td>
<td>0.0016</td>
<td>0.12</td>
<td>-1.66</td>
<td>0.0570</td>
</tr>
<tr>
<td>Dow Jones, real excess price growth</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Survey Average</td>
<td>Survey Median</td>
<td></td>
<td>Survey Average</td>
<td>Survey Median</td>
<td></td>
</tr>
<tr>
<td>Shiller, 1 yr, SPF</td>
<td>-0.05</td>
<td>-1.68</td>
<td>0.0008</td>
<td>-0.04</td>
<td>-1.68</td>
<td>0.0009</td>
</tr>
<tr>
<td>Shiller, 1 yr, Michigan</td>
<td>-0.05</td>
<td>-1.68</td>
<td>0.0008</td>
<td>-0.04</td>
<td>-1.68</td>
<td>0.0009</td>
</tr>
<tr>
<td>Shiller, 10 yrs, SPF</td>
<td>2.25</td>
<td>-7.98</td>
<td>0.0000</td>
<td>3.62</td>
<td>-7.98</td>
<td>0.0000</td>
</tr>
<tr>
<td>Shiller, 10 yrs, Michigan</td>
<td>2.08</td>
<td>-7.98</td>
<td>0.0000</td>
<td>3.46</td>
<td>-7.98</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

* stock market return expectations

Table 5b: Forecast rationality test (excess returns)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Calibration Target</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta^D$</td>
<td>1.0048</td>
<td>average quarterly real dividend growth</td>
</tr>
<tr>
<td>$\sigma_D$</td>
<td>0.0192</td>
<td>std. deviation quarterly real dividend growth</td>
</tr>
<tr>
<td>$\rho$</td>
<td>22</td>
<td>average consumption-dividend ratio</td>
</tr>
<tr>
<td>$\sigma_{DW}$</td>
<td>$-3.74 \cdot 10^{-4}$</td>
<td>jointly chosen s.t. $\text{corr}<em>t(C</em>{t+1}/C_t, D_{t+1}/D_t) = 0.2$</td>
</tr>
<tr>
<td>$\sigma_W$</td>
<td>0.0197</td>
<td>and $\text{std}<em>t(C</em>{t+1}/C_t) = \frac{1}{2} \text{std}<em>t(D</em>{t+1}/D_t)$</td>
</tr>
<tr>
<td>$\sigma_e$</td>
<td>0.0816</td>
<td>std. deviation of quarterly real stock price growth</td>
</tr>
</tbody>
</table>

Table 6: Model calibration
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Moment</td>
<td>Moment t-ratio</td>
<td>Moment t-ratio</td>
</tr>
<tr>
<td>$E[PD]$</td>
<td>139.7</td>
<td>117.4</td>
<td>111.0</td>
</tr>
<tr>
<td>$Std[PD]$</td>
<td>65.3</td>
<td>87.0</td>
<td>75.7</td>
</tr>
<tr>
<td>$Corr[PD_t, PD_{t-1}]$</td>
<td>0.98</td>
<td>0.98</td>
<td>0.98</td>
</tr>
<tr>
<td>$Std[r^s]$</td>
<td>8.01</td>
<td>9.03</td>
<td>7.86</td>
</tr>
<tr>
<td>$c$</td>
<td>-0.0041</td>
<td>-0.0052</td>
<td>-0.0048</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.24</td>
<td>0.19</td>
<td>0.16</td>
</tr>
<tr>
<td>$E[r^s]$</td>
<td>1.89</td>
<td>1.91</td>
<td>1.83</td>
</tr>
<tr>
<td>$E[r^p]$</td>
<td>0.13</td>
<td>0.98</td>
<td>0.97</td>
</tr>
<tr>
<td>UBS Survey Data:</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$Corr[PD_t, E_t R_{t+1}]$</td>
<td>0.79</td>
<td>0.85</td>
<td>0.84</td>
</tr>
<tr>
<td>Estimates:</td>
<td></td>
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</tr>
<tr>
<td>$g$</td>
<td></td>
<td>0.02515</td>
<td>0.02315</td>
</tr>
<tr>
<td>$\delta$</td>
<td></td>
<td>0.9955</td>
<td>0.9955</td>
</tr>
<tr>
<td>$\gamma$</td>
<td></td>
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<td>2.1</td>
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</table>

Table 7: Asset pricing moments (estimated model)

<table>
<thead>
<tr>
<th></th>
<th>U.S. Data Moment</th>
<th>RE Model Moment</th>
<th>t-ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E[PD]$</td>
<td>139.7</td>
<td>106.4</td>
<td>1.34</td>
</tr>
<tr>
<td>$Std[PD]$</td>
<td>65.3</td>
<td>4.17</td>
<td>1.42</td>
</tr>
<tr>
<td>$Corr[PD_t, PD_{t-1}]$</td>
<td>0.98</td>
<td>-0.0058</td>
<td>&gt;100</td>
</tr>
<tr>
<td>$Std[r^s]$</td>
<td>8.01</td>
<td>4.49</td>
<td>8.75</td>
</tr>
<tr>
<td>$c$</td>
<td>-0.0041</td>
<td>-0.0125</td>
<td>6.68</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.24</td>
<td>0.11</td>
<td>1.00</td>
</tr>
<tr>
<td>$E[r^s]$</td>
<td>1.89</td>
<td>1.50</td>
<td>0.83</td>
</tr>
<tr>
<td>$E[r^p]$</td>
<td>0.13</td>
<td>1.50</td>
<td>-8.23</td>
</tr>
<tr>
<td>UBS Survey Data:</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$Corr[PD_t, E_t R_{t+1}]$</td>
<td>0.79</td>
<td>-1.00</td>
<td>26.38</td>
</tr>
</tbody>
</table>

Table 8: Asset pricing moments, RE model
Figure 1: Quarterly PD Ratio of the S&P 500

Figure 2: PD ratio and investors’ expected returns (UBS Gallup Survey)
Figure 3: UBS survey expectations versus adaptive prediction model

Figure 4: PD ratio S&P 500 vs. adaptive price growth predictions
Figure 5: PD ratio and expected capital gains (vanishing noise)

Figure 6: Illustration of global belief dynamics

Figure 7: The effects of uncertainty on the equilibrium PD ratio
Figure 8: Price growth expectations implied by Bayesian updating and historical price growth information

Figure 9: PD ratio - model vs. data

Figure 10: Price growth expectations: UBS survey vs. Bayesian updating model
Figure 11: Effects of ($\gamma, \rho, \delta$) on the equilibrium PD function

Figure 12: Equilibrium pricing function (generalized belief system)
Figure 13: Persistence parameters and expected PD path (generalized belief system)
A Appendix (not for publication)

A.1 Data Sources

Stock price data: our stock price data is for the United States and has been downloaded from ‘The Global Financial Database’ (http://www.globalfinancialdata.com). The period covered is Q1:1946-Q1:2012. The nominal stock price series is the ‘SP 500 Composite Price Index (w/GFD extension)’ (Global Fin code ‘_SPXD’). The daily series has been transformed into quarterly data by taking the index value of the last day of the considered quarter. To obtain real values, nominal variables have been deflated using the ‘USA BLS Consumer Price Index’ (Global Fin code ‘CPUSAM’). The monthly price series has been transformed into a quarterly series by taking the index value of the last month of the considered quarter. Nominal dividends have been computed as follows

\[ D_t = \left( \frac{I^D(t)/I^D(t-1)}{I^{ND}(t)/I^{ND}(t-1)} - 1 \right) I^{ND}(t) \]

where \( I^{ND} \) denotes the ‘SP 500 Composite Price Index (w/GFD extension)’ described above and \( I^D \) is the ‘SP 500 Total Return Index (w/GFD extension)’ (Global Fin code ‘_SPXTRD’). We first computed monthly dividends and then quarterly dividends by adding up the monthly series. Following Campbell (2003), dividends have been deseasonalized by taking averages of the actual dividend payments over the current and preceding three quarters.

Stock market survey data: The UBS survey is the UBS Index of Investor Optimism, which is available (against a fee) at


The quantitative question on stock market expectations has been surveyed over the period Q2:1998-Q4:2007 with 702 responses per month on average and has thereafter been suspended. For each quarter we have data from three monthly surveys, except for the first four quarters and the last quarter of the survey period where we have only one monthly survey per quarter. The Shiller survey data covers individual investors over the period Q1:1999Q1-Q4:2012 and has been kindly made available to us by Robert Shiller at Yale University. On average 73 responses per quarter have been recorded for the question on stock price growth. Since the Shiller data refers to the Dow Jones, we used the PD ratio for the Dow Jones, which is available at http://www.djaverages.com/, to compute correlations. The CFO survey is collected by Duke University and CFO magazine and collects responses from U.S. based CFOs over the period Q3:2000-Q4:2012 with on average 390 responses per quarter, available at http://www.cfosurvey.org/.


A.2 Details of the t-Test in Section 3.2

Under the RE hypothesis, equations (2) and (4) both hold for the same parameters \( a^N, c^N \), given any horizon \( N \). These two equations define a standard SUR model. Dependent variables are \( E_t^N \) and \( R_t^{N+N} \), where the latter is the \( N \)-period rate of return and \( E_t^N \) is
the observed survey expectation at time \( t \), explanatory variables in both equations are \( x_t = (1, \frac{P_t}{T}) \), satisfying the orthogonality conditions (3) and (5). For expositional clarity we relabel the true parameters in equation (4) as \((\alpha^N, \epsilon^N)\). The aim is to design efficient estimators of the true parameters \( \beta^N_0 \equiv (\alpha^N, \epsilon^N, \tilde{\alpha}^N, \tilde{\epsilon}^N) \) and to test the hypothesis \( H_0 : \tilde{\epsilon}^N = \epsilon^N \).

The OLS estimator equation by equation \( \beta_T \) is defined by

\[
\beta_T = \left[ \begin{array}{c}
\hat{\alpha}^N_t \\
\hat{\epsilon}^N_t \\
\hat{a}^N_t \\
\hat{c}^N_t \\
\end{array}
\right] = \left( \begin{array}{c}
\sum_{t=1}^{T} x_t x'_t \otimes I_2 \\
\sum_{t=1}^{T} x_t \\
\end{array}
\right) \left[ \begin{array}{c}
\epsilon^N_t \\
\beta^N_t \\
\end{array}
\right],
\]

where \( I_2 \) is a \( 2 \times 2 \) identity matrix. A standard result is that under stationarity and ergodicity of all the observed variables and if \( E(x_t x'_t) \) is invertible, OLS equation by equation is consistent and efficient among the set of estimators that use only orthogonality conditions (3) and (5).

As is well known, if we add the assumption of strong ergodicity and bounded second moments, the asymptotic distribution as \( T \to \infty \) is given by

\[
\sqrt{T} (\beta_T - \beta_0) \to N \left( 0, \left[ E(x_t x'_t) \otimes I_2 \right]^{-1} S_w \left[ E(x_t x'_t) \otimes I_2 \right]^{-1} \right),
\]

where

\[
S_w = \Gamma_0 + \sum_{k=1}^{\infty} \Gamma_k + \Gamma_k'
\]

\[
\Gamma_k = E \left( \left[ \begin{array}{c}
u \mu_t \\
u \epsilon_t \\
\end{array} \right] \left[ \begin{array}{c}u \mu_{t-k} \epsilon_{t-k} \\
\end{array} \right] \right),
\]

where \( u \mu_t \equiv u^N_t + \mu_t^N \) and \( u \epsilon_t \equiv u^N_t + \epsilon_{t+N}^N \). To build the test-statistic, we now only need to find an estimator for \( \text{var-cov} \) matrix in (39).

We can estimate \( E(x_t x'_t) \) by \( \frac{1}{T} \sum_{t=1}^{T} x_t x'_t \). Since \( u \mu_t \) and \( u \epsilon_t \) are not forecasting errors, there is no reason why \( \Gamma_k \) should be zero for any \( k \), so we estimate \( S_w \) using Newey-West.

It is possible to exploit the special form of the error \( u \epsilon_t \) to estimate the \( \Gamma_k \) terms in the case of no measurement error, when \( \mu_t^N = 0 \). It is interesting to look at this case to explore robustness of the \( p \)-values. Partition each \( \Gamma_k \) into four \( 2 \times 2 \) matrices, with \( \Gamma_{ij,k} \) denoting the \((i, j)\)th element of this partition. Then, letting \( \hat{\mu}_t \) and \( \hat{\epsilon}_t \) denote the calculated errors of each equation, we use standard estimators

\[
\Gamma_{11,k,T} = \frac{1}{T-k} \sum_{t=1}^{T-k} \hat{\mu}_t \hat{\mu}_{t-k} x_t x_{t-k},
\]

\[
\Gamma_{12,k,T} = \frac{1}{T-k} \sum_{t=1}^{T-k} \hat{\mu}_t \hat{\epsilon}_{t-k} x_t x_{t-k}
\]

Since \( \epsilon_{t+N} \) is a forecast error using information up to \( t \), under the assumption of no measurement error we have

\[
\Gamma_{21,k} = E([u_t + \epsilon_{t+N}] u_{t-k} x_t x'_{t-k}) = \Gamma_{11,k} \text{ for all } k \geq 0,
\]

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so $\Gamma_{11,k,T}$ is an estimate of $\Gamma_{21,k}$. Furthermore, we have

$$\Gamma_{22,k} = E(u_{t+k} x_{t+k}) = \Gamma_{12,k} + E(\varepsilon_{t+N-k} x_{t+k}) \quad \text{for all } k$$

where the second equality follows from $E(\varepsilon_{t+N} u_{t-k} x_{t+k}) = 0$. Moreover, since $\varepsilon_{t+N}$ is orthogonal to $\varepsilon_{t+N-k} x_{t+k}$ for $k \geq N$ we have $\Gamma_{22,k} = \Gamma_{12,k}$ for $k \geq N$. Therefore, we can use the relationship

$$\Gamma_{22,k,T} = \Gamma_{21,k,T} + \frac{1}{T-k} \sum_{t=1}^{T-k} (\bar{u}_{t-k} - \bar{u}_t) (\bar{\varepsilon}_{t-k} - \bar{\mu}_t) x_{t+k} \quad \text{for } k < N$$

$$\Gamma_{22,k,T} = \Gamma_{21,k,T} \quad \text{for } k \geq N$$

which allows using the estimated $\Gamma_{21,k,T}$ as our estimate for $\Gamma_{22,k}$.

It turns out that when using the var-cov matrix using this procedure the $p$-values are virtually unchanged.

### A.3 Parameterization of the Wage Process

To calibrate $\rho$ we compute the average dividend-consumption share in the U.S. from 1946-2011, using the ‘Net Corporate Dividends’ and the ‘Personal Consumption Expenditures’ series from the Bureau of Economic Analysis. This delivers an average ratio of $\rho = 22$. Following Campbell and Cochrane (1999) we then choose the standard deviation of one-step-ahead consumption growth innovations to be $1/7$ of that of one-step-ahead dividend growth innovations, i.e.,

$$\sqrt{\frac{\text{var}_{t}(\ln C_{t+1} - \ln C_t)}{\text{var}_{t}(\ln D_{t+1} - \ln D_t)}} = \frac{1}{7},$$

and the correlation between one-step-ahead consumption and dividend growth to be equal to 0.2, i.e.

$$\frac{\text{cov}_{t}(\ln C_{t+1} - \ln C_t, \ln D_{t+1} - \ln D_t)}{\sqrt{\text{var}_{t}(\ln C_{t+1} - \ln C_t) \text{var}_{t}(\ln D_{t+1} - \ln D_t)}} = 0.2$$

To achieve this we need to compute the required variance and covariances. We have

$$\text{var}_{t}(\ln D_{t+1} - \ln D_t) = \sigma_D^2$$

$$\text{var}_{t}(\ln C_{t+1} - \ln C_t) = \text{var}_{t}(\ln(D_{t+1} + W_{t+1}) - \ln(D_t + W_t))$$

$$= \text{var}_{t}(\ln(D_{t+1} + \rho D_{t+1} \varepsilon_{t+1}^W))$$

$$= \text{var}_{t}(\ln D_{t+1} + \ln(1 + \rho \varepsilon_{t+1}^W))$$

$$= \text{var}_{t}(\ln D_{t+1}) + 2\text{cov}_{t}(\ln D_{t+1}, \ln(1 + \rho \varepsilon_{t+1}^W)) + \text{var}_{t}(\ln(1 + \rho \varepsilon_{t+1}^W))$$

$$= \sigma_D^2 + 2\text{cov}_{t}(\ln(1 + \rho \varepsilon_{t+1}^W)) + \text{var}_{t}(\ln(1 + \rho \varepsilon_{t+1}^W)) \quad (40)$$
and

\[
\text{cov}(\ln C_{t+1} - \ln C_t, \ln D_{t+1} - \ln D_t) = \text{cov}(\ln C_{t+1}, \ln \varepsilon_{t+1}^D) \\
= \text{cov}(\ln (D_{t+1} + W_{t+1}), \ln \varepsilon_{t+1}^D) \\
= \text{cov}(\ln D_{t+1} + \ln(1 + \rho \varepsilon_{t+1}^W), \ln \varepsilon_{t+1}^D) \\
= \text{cov}(\ln \varepsilon_{t+1}^D + \ln(1 + \rho \varepsilon_{t+1}^W), \ln \varepsilon_{t+1}^D) \\
= \sigma_D^2 + \text{cov}(\ln(1 + \rho \varepsilon_{t+1}^W), \ln \varepsilon_{t+1}^D) \tag{41}
\]

Linearly approximating \(\ln(1 + \rho \varepsilon_{t+1}^W)\) around the unconditional mean \(\varepsilon_{t+1}^W = 1\) delivers

\[
\ln(1 + \rho \varepsilon_{t+1}^W) \approx c + \frac{\rho}{1 + \rho} \ln \varepsilon_{t+1}^W + O(2)
\]

where \(c\) is a constant and \(O(2)\) a second order approximation error. Using this approximation we have

\[
\text{var}(\ln C_{t+1} - \ln C_t) \approx \sigma_D^2 + 2 \frac{\rho}{1 + \rho} \sigma_{DW} + \left(\frac{\rho}{1 + \rho}\right)^2 \sigma_W^2 \tag{42}
\]

So that

\[
\frac{\text{cov}(\ln C_{t+1} - \ln C_t)}{\sqrt{\text{var}(\ln C_{t+1} - \ln C_t) \text{var}(\ln D_{t+1} - \ln D_t)}} \approx \sqrt{1 + 2 \frac{\rho}{1 + \rho} \frac{\sigma_{DW}}{\sigma_D^2} + \left(\frac{\rho}{1 + \rho}\right)^2 \frac{\sigma_W^2}{\sigma_D^2}} = \frac{1}{7} \tag{43}
\]

Using the approximation we also have

\[
\frac{\text{cov}(\ln C_{t+1}, \ln \varepsilon_{t+1}^D)}{\sqrt{\text{var}(\ln C_{t+1} - \ln C_t) \text{var}(\ln D_{t+1} - \ln D_t)}} \approx \frac{\sigma_D^2 + \frac{\rho}{1 + \rho} \sigma_{WD}}{\sqrt{\left(\sigma_D^2 + 2 \frac{\rho}{1 + \rho} \sigma_{WD} + \left(\frac{\rho}{1 + \rho}\right)^2 \sigma_W^2\right) \sigma_D^2}} = 0.2 \tag{44}
\]

Using (43) to substitute the root in the denominator in (44) we get

\[
\frac{\sigma_D^2 + \frac{\rho}{1 + \rho} \sigma_{WD}}{\frac{\sigma_W^2}{7}} = 0.2 \iff \sigma_{WD} = -\frac{68 \rho}{70} \sigma_D^2 \tag{45}
\]

Using (43) we then get

\[
\sigma_W^2 = -\frac{48}{49} \left(\frac{1 + \rho}{\rho}\right)^2 \sigma_D^2 - 2 \frac{1 + \rho}{\rho} \sigma_{WD} \\
= -\frac{236}{245} \left(\frac{1 + \rho}{\rho}\right)^2 \sigma_D^2 \tag{46}
\]
A.4 Existence of Optimum, Sufficiency of FOCs, Recursive Solution

Existence of Optimum & Sufficiency of FOCs. The choice set in (7) is compact and non-empty. The following condition then insures existence of optimal plans:

**Condition 1** The utility function $u(\cdot)$ is bounded above and for all $i \in [0, 1]$

$$E_0^P \sum_{t=0}^{\infty} \delta^t u(W_t + D_t) > -\infty. \quad (47)$$

The expression on the left-hand side of condition (47) is the utility associated with never trading stocks ($S_i^t = 1$ for all $t$). Since this policy is always feasible, condition (47) guarantees that the objective function in (7) is also bounded from below, even if the flow utility function $u(\cdot)$ is itself unbounded below. The optimization problem (7) thus maximizes a bounded continuous utility function over a compact set, which guarantees existence of a maximum.

Under the assumptions made in the main text (utility function given by (12), knowledge of (9) and $\delta \beta^{RE} < 1$), condition 1 holds, as can be seen from the following derivation:

$$E_0^P \sum_{t=0}^{\infty} \delta^t u(W_t + D_t) = E_0 \sum_{t=0}^{\infty} \delta^t u(W_t + D_t)$$

$$= E_0 \sum_{t=0}^{\infty} \delta^t ((1 + \rho \varepsilon_t^W)D_t)^{1-\gamma}$$

$$= ((1 + \rho \varepsilon_0^W)D_0)^{1-\gamma} + E_0 \sum_{t=1}^{\infty} \delta^t ((1 + \rho \varepsilon_t^W)D_t)^{1-\gamma}$$

$$= ((1 + \rho \varepsilon_0^W)D_0)^{1-\gamma} + E_0 \sum_{t=1}^{\infty} \left( \delta (\beta^D)^{1-\gamma} \right)^t \left( (1 + \rho \varepsilon_t^W)\varepsilon_t^D \prod_{k=1}^{t-1} \varepsilon_k^D \right)^{1-\gamma}$$

$$= ((1 + \rho \varepsilon_0^W)D_0)^{1-\gamma} + \left( (1 + \rho \varepsilon_0^W)D_0 \right)^{1-\gamma} + E \left[ ((1 + \rho \varepsilon_0^W)\varepsilon_t^D)^{1-\gamma} \right] \cdot \sum_{t=1}^{\infty} \left( \delta (\beta^D)^{1-\gamma} \right)^t \left( e^{\sigma_D^2 \gamma (\gamma-1)} \right)^{t-1}$$

$$= ((1 + \rho \varepsilon_0^W)D_0)^{1-\gamma} + \frac{E \left[ ((1 + \rho \varepsilon_0^W)\varepsilon_t^D)^{1-\gamma} \right]}{e^{\sigma_D^2 \gamma (\gamma-1)}} \cdot \sum_{t=1}^{\infty} \left( \delta \beta^{RE} \right)^t$$

Since (7) is a strictly concave maximization problem the maximum is unique. With the utility function being differentiable, the first order conditions

$$u'(C_i^t) = \delta E_t^P \left[ u'(C_i^{t+1}) \frac{P_{t+1} + D_{t+1}}{P_t} \right] \quad (48)$$

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plus a standard transversality condition are necessary and sufficient for the optimum.

**Recursive Solution.** We have a recursive solution whenever the optimal stockholding policy can be written as a time-invariant function $S_i^t = S_i^t(x_t)$ of some state variables $x_t$. We seek a recursive solution where $x_t$ contains appropriately rescaled variables that do not grow to infinity. With this in mind, we impose the following condition:

**Condition 2** The flow utility function $u(\cdot)$ is homogeneous of degree $\eta \geq 0$. Furthermore, the beliefs $\mathcal{P}^i$ imply that $\theta_t \equiv \left( \frac{D_t}{\beta_t}, \frac{P_t}{\beta_t}, \frac{W_t}{\beta_t} \right)$ has a state space representation, i.e., the conditional distribution $\mathcal{P}^i(\theta_{t+1}|\omega^t)$ can be written as

$$ \mathcal{P}^i(\theta_{t+1}|\omega^t) = \mathcal{F}^i(m_t^i) \quad (49) $$

$$ m_t^i = \mathcal{R}^i(m_{t-1}^i, \theta_t) \quad (50) $$

for some finite-dimensional state vector $m_t^i$ and some time-invariant functions $\mathcal{F}^i$ and $\mathcal{R}^i$.

Under Condition 2 problem (7) can then be re-expressed as

$$ \max_{\{S_i \in \mathcal{S}\}} E_0^D \sum_{t=0}^{\infty} \delta^t D_t u \left( S_{t-1}^i \left( \frac{P_t}{D_t} + 1 \right) - S_t^i - \frac{P_t}{D_t} + \frac{W_t}{D_t} \right) \quad (51) $$

given $S_{t-1}^i = 1$, where $D_t$ is a time-varying discount factor satisfying $D_{-1} = 1$ and

$$ D_t = D_{t-1} \left( \beta^D \delta^D \right)^\eta. $$

The return function in (51) depends only on the exogenous variables contained in the vector $\theta_t$. Since the beliefs $\mathcal{P}^i$ are assumed to be recursive in $\theta_t$, standard arguments in dynamic programming guarantee that the optimal solution to (51) takes the form (14). This formulation of the recursive solution is useful, because scaling $P_t$ and $W_t$ by the level of dividends eliminates the trend in these variables, as desired. This will be useful when computing numerical approximations to $S_i^t(\cdot)$. The belief systems $\mathcal{P}^i$ introduced in section 6 will satisfy the requirements stated in condition 2.

**A.5 Proof of Proposition 1**

In equilibrium $S_i^t = 1$ for all $t \geq 0$, so that the budget constraint implies

$$ C_t^i = D_t + W_t = (1 + \rho_e^W)D_t. $$

Substituting into the agent’s first order condition delivers

$$ P_t = \delta E_t \left[ \left( \frac{(1 + \rho_e^W D_{t+1})}{(1 + \rho_e^W D_t)} \right)^{-\gamma} (P_{t+1} + D_{t+1}) \right]. \quad (52) $$

Assuming that the following transversality condition holds

$$ \lim_{j \to \infty} E_t \left[ \delta^j \left( \frac{1 + \rho_e^W D_{t+j}}{1 + \rho_e^W D_t} \right)^{-\gamma} P_{t+j} \right] = 0, \quad (53) $$

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one can iterate forward on (52) to obtain

\[
P_t = E_t \left[ \sum_{j=1}^{\infty} \delta^j \left( \frac{1 + \rho \varepsilon_{t+j}^W}{1 + \rho \varepsilon_t^W} \right)^{-\gamma} \left( \frac{D_{t+j}}{D_t} \right)^{1-\gamma} \right],
\]

Using \( D_{t+j} / D_t = (\beta^D)^j \prod_{k=1}^j \epsilon_{t+k}^D \) one has

\[
\frac{P_t}{D_t} = (1 + \rho \varepsilon_t^W)^\gamma \sum_{j=1}^{\infty} (\delta(\beta^D)^{1-\gamma})^j E_t \left[ (1 + \rho \varepsilon_{t+j}^W)^{-\gamma} (\prod_{k=1}^j \epsilon_{t+k}^D)^{1-\gamma} \right]
\]

\[
= (1 + \rho \varepsilon_t^W)^\gamma \sum_{j=1}^{\infty} (\delta(\beta^D)^{1-\gamma})^j E_t \left[ (1 + \rho \varepsilon_{t+j}^W)^{-\gamma} (\epsilon_{t+j}^D)^{1-\gamma} \right] E_t \left[ \left( \prod_{k=1}^{j-1} \epsilon_{t+k}^D \right)^{1-\gamma} \right]
\]

\[
= (1 + \rho \varepsilon_t^W)^\gamma E_t \left[ (1 + \rho \varepsilon_{t+j}^W)^{-\gamma} (\epsilon_{t+j}^D)^{1-\gamma} \right] e^{\gamma(1-\gamma)\sigma_c^2 / 2} \frac{\delta \beta_{RE}}{1 - \delta \beta_{RE}},
\]

as claimed in proposition 1.

### A.6 Bayesian Foundations for Lagged Belief Updating

We now present a slightly modified information structure for which Bayesian updating gives rise to the lagged belief updating equation (30). Specifically, we generalize the perceived price process (18) by splitting the temporary return innovation \( \ln \varepsilon_{t+1} \) into two independent subcomponents:

\[
\ln P_{t+1} - \ln P_t = \ln \beta_{t+1} + \ln \varepsilon_{t+2} + \ln \varepsilon_{t+1}
\]

with \( \ln \varepsilon_{t+2} \sim iiN(-\frac{\sigma_{\varepsilon^2}}{2}, \sigma_{\varepsilon^2}), \ln \varepsilon_{t+1} \sim iiN(-\frac{\sigma_{\varepsilon^2}}{2}, \sigma_{\varepsilon^2}) \) and

\[
\sigma_{\varepsilon}^2 = \sigma_{\varepsilon^1}^2 + \sigma_{\varepsilon^2}^2.
\]

We then assume that in any period \( t \) agents observe the prices, dividends and wages up to period \( t \), as well as the innovations \( \varepsilon_t^1 \) up to period \( t \). Agents’ time \( t \) information set thus consists of \( I_t = \{ P_t, D_t, W_t, \varepsilon_t^1, P_{t-1}, D_{t-1}, W_{t-1}, \varepsilon_{t-1}^1, \ldots \} \). By observing the innovations \( \varepsilon_t^1 \), agents learn - with a one period lag - something about the temporary components of price growth. The process for the persistent price growth component \( \ln \beta_t \) remains as stated in equation (19), but we now denote the innovation variance by \( \sigma_{\varepsilon}^2 \) instead of \( \sigma_{\varepsilon^1}^2 \).

As before, \( \ln m_t \) denotes the posterior mean of \( \ln \beta_t \) given the information available at time \( t \). We prove below the following result:

**Proposition 3** Fix \( \sigma_{\varepsilon}^2 > 0 \) and consider the limit \( \sigma_{\varepsilon^2}^2 \to 0 \), with \( \sigma_{\varepsilon^2}^2 = \sigma_{\varepsilon^2} g^2 / (1 - g) \). Bayesian updating then implies

\[
\ln m_t = \ln m_{t-1} + g (\ln P_{t-1} - \ln P_{t-2} - \ln m_{t-1}) - g \ln \varepsilon_t^1 \tag{54}
\]

The modified information structure thus implies that only lagged price growth rates enter the current state estimate, so that beliefs are predetermined, precisely as assumed in equation (30). Intuitively, this is so because lagged returns become infinitely more informative relative to current returns as \( \sigma_{\varepsilon^2}^2 \to 0 \), which eliminates the simultaneity
problem. For non-vanishing uncertainty \( \sigma^2 \) the weight of the last observation actually remains positive but would still be lower than that given to the lagged return observation, see equation (57) in the proof below and the subsequent discussion for details.

We now sketch the proof of the previous proposition. Let us define the following augmented information set \( \tilde{I}_{t-1} = I_{t-1} \cup \{ \varepsilon^1_t \} \). The posterior mean for \( \beta_t \) given \( \tilde{I}_{t-1} \), denoted \( \ln m_{i|\tilde{I}_{t-1}} \) is readily recursively determined via

\[
\ln m_{i|\tilde{I}_{t-1}} = \ln m_{i|I_{t-1}} - \frac{\sigma^2}{2} + g \left( \ln P_{t-1} - \ln P_{t-2} - \ln \varepsilon^1_t + \frac{\sigma^2 \sigma^2}{2} - \ln m_{t-1|\tilde{I}_{t-1}} \right) \quad (55)
\]

and the steady state posterior uncertainty and the Kalman gain by

\[
\sigma^2 = \frac{-\sigma^2 + \sqrt{(\sigma^2)^2 + 4\sigma^2 \sigma^2}}{2} \\
\tilde{g} = \frac{\sigma^2}{\sigma^2} \\
\quad (56)
\]

Standard updating formulas for normal distributions then imply that the posterior mean of \( \ln \beta_t \) using information set \( I_t \) can be derived by updating the posterior mean based on \( \tilde{I}_{t-1} \) according to

\[
\ln m_{i|I_t} = \ln m_{i|\tilde{I}_{t-1}} + \frac{\sigma^2}{\sigma^2 + \sigma^2_{I_t} + \sigma^2_{I_t} + \sigma^2_{I_t}} \left( \ln P_t - \ln P_{t-1} + \frac{\sigma^2 + \sigma^2 + \sigma^2}{2} - \ln m_{i|\tilde{I}_{t-1}} \right) \\
\quad (57)
\]

Since \( \frac{\sigma^2}{\sigma^2 + \sigma^2_{I_t} + \sigma^2_{I_t} + \sigma^2_{I_t}} < \frac{\sigma^2}{\sigma^2} = \tilde{g} \), the weight of the price observation dated \( t \) is reduced relative to the earlier observation dated \( t-1 \) because it is ‘noisier’. Now consider the limit \( \sigma^2 \to 0 \) and along the limit choose \( \sigma^2_{I_t} = \sigma^2 - \sigma^2_{I_t} \) and \( \sigma^2_{I_t} = \frac{g^2}{1-g} \sigma^2_{I_t} \), as assumed in the proposition. Equation (57) then implies that \( \ln m_{i|I_{t+1}} = \ln m_{i|\tilde{I}_{t-1}} \), i.e., the weight of the last observation price converges to zero. Moreover, from \( \sigma^2_{I_t} = \frac{g^2}{1-g} \sigma^2_{I_t} \) and (56) we get \( \tilde{g} = g \). Using these results, equation (55) can exactly be written as stated by equation (54) in the main text.

### A.7 Proof of Proposition 2

The proof relies on the fact that in a situation without uncertainty the expectation of a non-linear function of ‘random’ variables is identical to the non-linear function of the expectation of these random variables, i.e., for some continuous non-linear function \( f(\cdot, \cdot) \) and some random variables \( X_{t+j}, Y_{t+j} \) we have under the stated assumptions \( E^p_f(X_{t+j}, Y_{t+j}) = f(E^p_t X_{t+j}, E^p_t Y_{t+j}) \). Simplifying notation (and slightly abusing it) we let \( X_{t+j} = E^p_t X_{t+j} \) for all \( j \geq 1 \), so that \( X_{t+j} \) below denotes the subjective expectation conditional on information at time \( t \) of the variable \( X \) at time \( t + j \). The first order conditions (48) can then be written as

\[
1 = \left( \frac{C_{t+1+j}}{C_{t+j}} \right)^{-\gamma} \delta R_{t+1+j} \iff \frac{C_{t+1+j}}{P_{t+1+j} + D_{t+1+j}} = \delta^\gamma \left( R_{t+1+j} \right)^{\frac{1-\gamma}{\gamma}} \frac{C_{t+j}}{P_{t+j}} \\
\quad (58)
\]

58
for all \( j \geq 0 \). The budget constraint implies

\[
S_{t-1}(P_t + D_t) = C_t - W_t + S_t P_t \implies S_{t-1} = \frac{C_t - W_t}{P_t + D_t} + \frac{P_t}{P_t + D_t} S_t
\]

Iterating forward on the latter equation gives

\[
S_{t-1} = \frac{C_t - W_t}{P_t + D_t} + \frac{P_t}{P_t + D_t} \left( \frac{C_{t+1} - W_{t+1}}{P_{t+1} + D_{t+1}} + \frac{P_{t+1}}{P_{t+1} + D_{t+1}} \frac{C_{t+2} - W_{t+2}}{P_{t+2} + D_{t+2}} + \ldots \right)
\]

Repeatedly using equation (58) gives

\[
S_{t-1} = \frac{C_t - W_t}{P_t + D_t} + \frac{P_t}{P_t + D_t} \left( \delta^j \left( R_{t+1} \right) \frac{i_{t+1}}{P_{t+1}} - \frac{W_{t+1}}{P_{t+1} + D_{t+1}} \right)
\]

\[
+ \frac{P_{t+1}}{P_{t+1} + D_{t+1}} \left( \delta^j \left( R_{t+2} \right) \frac{i_{t+2}}{P_{t+2}} - \frac{W_{t+2}}{P_{t+2} + D_{t+2}} \right) + \ldots
\]

\[
= \frac{C_t}{P_t + D_t} + \delta^j \left( R_{t+1} \right) \frac{i_{t+1}}{P_t + D_t} + \frac{C_{t+1}}{P_{t+1} + D_{t+1}} + \frac{1}{W_{t+1}} + \frac{1}{W_{t+2}} - \ldots
\]

\[
= \frac{C_t}{P_t + D_t} + \left( \delta^j \right)^2 \left( R_{t+2} R_{t+1} \right) \frac{i_{t+2}}{P_t + D_t} + \ldots
\]

\[
- \frac{1}{P_t + D_t} \left( \sum_{j=0}^{\infty} W_{t+j} \prod_{i=1}^{j} \frac{1}{R_{t+i}} \right)
\]

\[
= \frac{D_t}{P_t + D_t} + \frac{C_t}{P_t + D_t} \left( \sum_{j=1}^{\infty} \left( \delta^j \right)^j \left( \prod_{i=1}^{j} \frac{1}{R_{t+i}} \right) \right)
\]

\[
- \frac{1}{P_t + D_t} \left( \sum_{j=1}^{\infty} \left( \prod_{i=1}^{j} \frac{1}{R_{t+i}} \right) W_{t+j} \right)
\]

(59)

Imposing on the previous equation \( S_{t-1} = 1 \) (the market clearing condition for period \( t - 1 \) if \( t > 1 \), or the initial condition for period \( t = 0 \)) and \( C_t = D_t + W_t \) (the market clearing condition for period \( t \geq 0 \)) one obtains the result stated in the proposition under the convention that \( R_{t+i} = E_t^P R_{t+i} \).

### A.8 Verification of Conditions (31)

For the vanishing noise limit of the beliefs specified in section 6 we have

\[
E_t^P[P_{t+j}] = (m_t)^j P_t
\]

\[
E_t^P[D_{t+j}] = (\beta^D)^j D_t
\]

\[
E_t^P[W_{t+j}] = (\beta^D)^j W_t
\]
We first verify the inequality on the l.h.s. of equation (31). We have

\[
\lim_{T \to \infty} E_{t}^{P}[R_{T}] = m_{t} + \lim_{T \to \infty} \left( \frac{\beta^{D}}{m_{t}} \right)^{T-1} \frac{\beta^{D} D_{t}}{P_{t}},
\]

so that for \( m_{t} > 1 \) the limit clearly satisfies \( \lim_{T \to \infty} E_{t}^{P}[R_{T}] > 1 \) due to the first term on the r.h.s.; for \( m_{t} < 1 \) the second term on the r.h.s. increases without bound, due to \( \beta^{D} > 1 \), so that \( \lim_{T \to \infty} E_{t}^{P}[R_{T}] > 1 \) also holds.

In a second step we verify that the inequality condition on the r.h.s. of equation (31) holds for all subjective beliefs \( m_{t} > 0 \). We have

\[
\lim_{T \to \infty} E_{t}^{P} \left( \sum_{j=1}^{T} \left( \prod_{i=1}^{j} \frac{1}{R_{t+i}} \right) W_{t+j} \right) = \lim_{T \to \infty} W_{t} E_{t}^{P} \left( \sum_{j=1}^{T} (\beta^{D})^{j} \left( \prod_{i=1}^{j} \frac{1}{R_{t+i}} \right) \right) = \lim_{T \to \infty} W_{t} \sum_{j=1}^{T} X_{j} \tag{60}
\]

where

\[
X_{j} = \frac{(\beta^{D})^{j}}{\prod_{i=1}^{j}(m_{t}) + \left( \frac{\beta^{D}}{m_{t}} \right)^{j-1} \beta^{D} D_{t} \frac{P_{t}}{P_{i}}} \geq 0 \tag{61}
\]

A sufficient condition for the infinite sum in (60) to converge is that the terms \( X_{j} \) are bounded by some exponentially decaying function. The denominator in (61) satisfies

\[
\prod_{i=1}^{j}(m_{t}) + \left( \frac{\beta^{D}}{m_{t}} \right)^{j-1} \beta^{D} D_{t} \frac{P_{t}}{P_{i}} \geq (m_{t})^{j} + \left( \frac{\beta^{D}}{m_{t}} \right)^{j \left( \frac{1}{2} \right)} \beta^{D} D_{t} \frac{P_{t}}{P_{i}} \tag{62}
\]

where the first term captures the the pure products in \( m_{t} \), the second term the pure products in \( \left( \frac{\beta^{D}}{m_{t}} \right)^{j-1} \beta^{D} D_{t} \frac{P_{t}}{P_{i}} \), and all cross terms have been dropped. We then have

\[
X_{j} = \frac{(\beta^{D})^{j}}{\prod_{i=1}^{j}(m_{t}) + \left( \frac{\beta^{D}}{m_{t}} \right)^{j-1} \beta^{D} D_{t} \frac{P_{t}}{P_{i}}} \leq \frac{(\beta^{D})^{j}}{(m_{t})^{j} + \left( \frac{\beta^{D}}{m_{t}} \right)^{j \left( \frac{1}{2} \right)} \beta^{D} D_{t} \frac{P_{t}}{P_{i}}} = \frac{1}{(m_{t})^{j} + \left( \frac{\beta^{D}}{m_{t}} \right)^{j \left( \frac{1}{2} \right)} \frac{1}{\beta^{D} \frac{D_{t}}{P_{i}}}},
\]

where all terms in the denominator are positive. For \( m_{t} \geq \beta^{D} > 1 \) we can use the first
term in the denominator to exponentially bound $X_j$, as $X_j \leq \left(\frac{\beta^D}{m_t}\right)^j$; for $m_t < \beta^D$ we can use the second term:

$$X_j \leq \frac{1}{\left(\frac{\beta^D}{m_t}\right)^j \left(\frac{\beta^D}{m_t}\right)^{j-1} \frac{D_t}{P_t}} = \frac{1}{\left(\frac{\beta^D}{m_t}\right)^{\frac{j}{2}} \left(\frac{\beta^D}{m_t}\right)^{\frac{j-1}{2}} \frac{D_t}{P_t}}$$

Since $m_t < \beta^D$ there must be a $J < \infty$ such that

$$\left(\frac{\beta^D}{m_t}\right)^{\frac{j}{2}} \geq \frac{\beta^D}{m_t} > 1$$

for all $j \geq J$, so that the $X_j$ are exponentially bounded for all $j \geq J$.

### A.9 Proof of Lemma 1

Proof of lemma 1: We start by proving the first point in the lemma. The price, dividend and belief dynamics in the deterministic model are described by the following equations

$$\begin{align*}
\ln m_t &= \ln m_{t-1} + g (\ln P_{t-1} - \ln P_{t-2} - \ln m_{t-1}) \\
\ln P_t - \ln D_t &= f(\ln m_t) \\
\ln D_t - \ln D_{t-1} &= \beta^D,
\end{align*}$$

(63)

where $f(\cdot)$ is a continuous function, implicitly defined by the log of the $P_t/D_t$ solution to equation 33. Substituting the latter two equations into the first delivers

$$\ln m_t - \ln m_{t-1} = g \left( f(\ln m_{t-1}) - f(\ln m_{t-2}) + \ln \beta^D - \ln m_{t-1} \right).$$

If $\ln m_t$ converges, then the l.h.s. of the previous equation must converge to zero. Since $f(\cdot)$ is continuous, this means that $m_{t-1}$ must converge to $\beta^D$, as claimed.

We now prove the second point in the lemma. The belief dynamics implied by the second order difference equation (63) can expressed as a two-dimensional first order difference equation using the mapping $F : R^2 \to R^2$, defined as

$$F(x) = \begin{pmatrix} x_1 + g(\ln P D(e^{x_1}) - \ln P D(e^{x_2}) + \ln \beta^D - x_1) \\ x_2 \end{pmatrix},$$

so that

$$\begin{pmatrix} \ln m_t \\ \ln m_{t-1} \end{pmatrix} = F \begin{pmatrix} \ln m_{t-1} \\ \ln m_{t-2} \end{pmatrix}.$$ 

Clearly, $F$ has a fixed point at the RE solution, i.e., $(\ln \beta^D, \ln \beta^D)' = F(\ln \beta^D, \ln \beta^D)'$. Moreover, $m_t$ locally converges to the RE beliefs if and only if

$$\frac{\partial F(\ln \beta^D, \ln \beta^D)}{\partial x'} = \begin{pmatrix} 1 + g (\xi - 1) & -g \xi \\ 1 & 0 \end{pmatrix}$$

(64)

has all eigenvalues less than one in absolute value, where $\xi \equiv \frac{\partial \ln P D(e^{\ln m})}{\partial \ln m} \Big|_{m=\beta^D} =$
\[
\lambda = \frac{1 + g(\xi - 1) \pm \sqrt{(1 + g(\xi - 1))^2 - 4g \xi \xi}}{2}.
\]

The eigenvalues of the matrix in equation (64) are

\[
\lambda = \frac{1 + g(\xi - 1) \pm \sqrt{(1 + g(\xi - 1))^2 - 4g \xi}}{2}.
\]

From \(|\xi| < 1\) and \(g < \frac{1}{2}\) follows that \((1 + g(\xi - 1))^2 - 4g \xi > 1 - 2g\xi - 2g \geq 0\), so that all eigenvalues are real. As is easily verified, we have \(\lambda^+ < 1\) because

\[
1 + g(\xi - 1) < 2 - \sqrt{(1 + g(\xi - 1))^2 - 4g \xi} \iff
\sqrt{(1 + g(\xi - 1))^2 - 4g \xi} < 1 - g(\xi - 1) \iff
(1 + g(\xi - 1))^2 - 4g \xi < (1 - g(\xi - 1))^2 \iff
2g(\xi - 1) - 4g \xi < -2g(\xi - 1) \iff
-g < 0
\]

and \(\lambda^- < 1\) because

\[
-1 + g(\xi - 1) < \sqrt{(1 + g(\xi - 1))^2 - 4g \xi}
\]

where the l.h.s. is negative and the r.h.s. positive. We have \(\lambda^+ > -1\) if and only if

\[
1 + g(\xi - 1) > -2 - \sqrt{(1 + g(\xi - 1))^2 - 4g \xi}
\]

From \(|\xi| < 1\) and \(g < \frac{1}{2}\) the l.h.s. is weakly positive, while the r.h.s. is strictly negative. We have \(\lambda^- > -1\) if and only if

\[
3 + g(\xi - 1) > \sqrt{(1 + g(\xi - 1))^2 - 4g \xi} \iff
(2 + (1 + g(\xi - 1))^2 > (1 + g(\xi - 1))^2 - 4g \xi \iff
1 + (1 + g(\xi - 1)) > -g \xi \iff
2 + g(2\xi - 1) > 0
\]

The last equation holds since \(|\xi| < 1\) and \(g < \frac{1}{2}\). This shows that the eigenvalues of (64) are all inside the unit circle.

A.10 Capital Gains Expectations and Expected Returns: Further Details

Figure 14 depicts how expected returns at various horizons depend on agent’s expected price growth expectations using the same parameterization as used in figure 5. It shows that expected returns covary positively with capital gains expectations for \(m_t \geq \beta^D\), as has been claimed in the main text. The flatish part at around \(m_t - 1 \approx 0.01\) arises because in that area the PD ratio increases strongly, so that the dividend yield falls. Only for pessimistic price growth expectations \((m_t < \beta^D)\) and long horizons of expected returns we find a negative relationship. The latter emerges because with prices expected to fall, the dividend yield will rise and eventually result in high return expectations.
A.11 Numerical Solution Algorithm

Algorithm: We solve for agents’ state-contingent, time-invariant stockholdings (and consumption) policy (14) using time iteration in combination with the method of endogenous grid points. Time iteration is a computationally efficient, e.g., Aruoba et al. (2006), and convergent solution algorithm, see Rendahl (2013). The method of endogenous grid points, see Carroll (2006), economizes on a costly root finding step which speeds up computations further.

Evaluations of Expectations: Importantly, agents evaluate the expectations in the first order condition (48) according to their subjective beliefs about future price growth and their (objective) beliefs about the exogenous dividend and wage processes. Expectations are approximated via Hermite Gaussian quadrature using three interpolation nodes for the exogenous innovations.

Approximation of Optimal Policy Functions: The consumption/stockholding policy is approximated by piecewise linear splines, which preserves the nonlinearities arising in particular in the PD dimension of the state space. Once the state-contingent consumption policy has been found, we use the market clearing condition for consumption goods to determine the market clearing PD ratio for each price-growth belief $m_t$.

Accuracy: Carefully choosing appropriate grids for each belief is crucial for the accuracy of the numerical solution. We achieve maximum (relative) Euler errors on the order of $10^{-3}$ and median Euler errors on the order of $10^{-5}$ (average: $10^{-4}$).

Using our analytical solution for the case with vanishing noise, we can assess the accuracy of our solution algorithm more directly. Setting the standard deviations of exogenous disturbances to $10^{-16}$ the algorithm almost perfectly recovers the equilibrium PD ratio of the analytical solution: the error for the numerically computed equilibrium PD ratio for any price growth belief $m_t$ on our grid is within 0.5 % of the analytical solution.

Figure 14: Expected return as a function of expected capital gain
References


