Stock Market Volatility and Learning

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Abstract

Consumption based asset pricing models with time-separable preferences can generate realistic amounts of stock price volatility if one allows for small deviations from rational expectations. We consider rational investors who entertain subjective prior beliefs about price behavior that are not equal but close to rational expectations. Optimal behavior then dictates that investors learn about price behavior from past price observations. We show that this imparts momentum and mean reversion into the equilibrium behavior of the price dividend ratio, similar to what can be observed in the data. When estimating the model on U.S. stock price data using the method of simulated moments, it can quantitatively account for the observed stock price volatility, the persistence of the price-dividend ratio, and the predictability of long-horizon returns. For reasonable degrees of risk aversion, the model generates up to one half of the equity premium observed in the data. It also passes a formal statistical test for the overall goodness of fit, provided one excludes the equity premium from the set of moments to be matched.

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Keywords: asset pricing, learning, subjective beliefs, internal rationality
"Investors, their confidence and expectations buoyed by past price increases, bid up speculative prices further, thereby enticing more investors to do the same, so that the cycle repeats again and again."

Irrational Exuberance, Shiller (2005, p. 56)

1 Introduction

The purpose of this paper is to show that a simple asset pricing model is able to quantitatively reproduce a variety of stylized asset pricing facts if one allows for slight deviations from rational expectations. We find it a striking observation that the quantitative asset pricing implications of the standard model are not robust to small departures from rational expectations and that this non-robustness is empirically so encouraging.

We study a simple variant of the Lucas (1978) model with standard time-separable consumption preferences. It is well known that the asset pricing implications of this model under rational expectations (RE) are at odds with basic facts, such as the observed high persistence and volatility of the price dividend ratio, the high volatility of stock returns, the predictability of long horizon excess stock returns, and the risk premium.

We stick to Lucas’ framework but relax the standard assumption that agents have perfect knowledge about the pricing function that maps each history of fundamental shocks into a market outcome for the stock price.¹ We assume instead that investors hold subjective beliefs about all payoff-relevant random variables that are beyond their control; this includes beliefs about model endogenous variables, such as prices, as well as model exogenous variables, such as the dividend and income processes. Given these subjective beliefs, investors maximize utility subject to their budget constraints. We call such agents ‘internally rational’, as they know all internal aspects of their individual decision problem and maximize utility given this knowledge. Furthermore, their system of beliefs is ‘internally consistent’ in the sense

¹Lack of knowledge of the pricing function may arise from a lack of common knowledge of investors’ preferences, price beliefs and dividend beliefs, as is explained in detail in Adam and Marcet (2011).
that it specifies for all periods the joint distribution of all payoff-relevant variables (i.e., dividends, income and stock prices), but these probabilities differ from those implied by the model in equilibrium. We then consider systems of beliefs implying only a small deviation from RE, as we explain further below.

We show that given the subjective beliefs we specify, subjective utility maximization dictates that agents update subjective expectations about stock price behavior using realized market outcomes. Consequently, agents’ stock price expectations influence stock prices and observed stock prices feed back into agents’ expectations. This self-referential aspect of the model turns out to be key for generating stock price volatility of the kind that can be observed in the data. More specifically, the empirical success of the model emerges whenever agents learn about the growth rate of stock prices, i.e., about the capital gains from their investments, using past observations of capital gains.

We first demonstrate the ability of the model to produce data-like behavior by deriving a number of analytical results about the behavior of stock prices that is implied by a general class of belief updating rules encompassing most learning algorithms that have been used in the learning literature. Specifically, we show that learning from market outcomes imparts ‘momentum’ on stock prices around their RE value, which gives rise to sustained deviations of the price dividend ratio from its mean, as can be observed in the data. Such momentum arises because if agents’ expectations about stock price growth increase in a given period, the actual growth rate of prices has a tendency to increase beyond the fundamental growth rate, thereby reinforcing the initial belief of higher stock price growth through the feedback from outcomes to beliefs. At the same time, the model displays ‘mean reversion’ over longer horizons, so that even if subjective stock price growth expectations are very high (or very low) at some point in time, they will eventually return to fundamentals. The model thus displays price cycles of the kind described in the opening quote above.

We then consider a specific system of beliefs which allows for subjective prior uncertainty about the average growth rate of stock market prices, given the values for all exogenous variables. As we show, internal rationality, i.e., standard utility maximization given these beliefs, then dictates that agents’ price growth expectations react to the realized growth rate of market prices. In particular, the subjective prior prescribes that agents should update conditional expectations of one-step-ahead risk-adjusted price growth using a
constant gain model of adaptive learning. This constant gain model belongs to the general class of learning rules that we studied analytically before, therefore displays momentum and mean reversion.

We document in several ways that the resulting beliefs represent only a small deviation from RE beliefs. First, we show that for the special case in which the prior uncertainty about price growth converges to zero, the learning rule delivers RE beliefs and prices under learning converge to RE prices (see section 4.3). In our empirical section, we then find that the asset pricing facts can be explained with a small amount of prior uncertainty. Second, using an econometric test that exhausts the empirical implications of agents’ subjective model of price behavior, we show that agents’ price beliefs would not be rejected by the data (section 6.3). Third, using the same test but applying it to artificial data generated by the estimated model, we show that it is difficult to detect that price beliefs differ from the actual behavior of prices in equilibrium (section 6.3).

To quantitatively evaluate the learning model we first consider how well it matches asset pricing moments individually, just as many papers on stock price volatility have done. We use formal structural estimation based on the method of simulated moments (MSM), adapting results of Duffie and Singleton (1993). We find that the model can individually match all the asset pricing moments we consider, including the volatility of stock market returns, the mean, persistence and volatility of the price dividend ratio and the evidence on excess return predictability over long-horizons. Using t-statistics derived from asymptotic theory we can not reject that any of the individual model moments differs from the moments in the data in one of our estimated models (see Table 2 in section 5.2). The model also delivers an equity premium of up to one half of the value observed in the data. All this is achieved in spite of the fact that we use time-separable CRRA preferences and a degree of relative risk aversion equal to five only.

We also perform a formal econometric test for the overall goodness of fit of our consumption based asset pricing model. This is a considerably more stringent test than implied by individually matching asset pricing moments as in calibration exercises, e.g., Campbell and Cochrane (1999), but a natural one to explore given our MSM strategy. As it turns out, the overall goodness of fit test is much more stringent and rejects the model if one includes both the risk free rate and the mean stock returns, but if we leave out the risk premium by excluding the risk free rate from the estimation, the p-value of the model amounts to a respectable 7.1%.
The general conclusion we obtain is that for moderate risk aversion the model can quantitatively account for all asset pricing facts, except for the equity premium. For a sufficiently high risk aversion as in Campbell and Cochrane (1999) the model can also replicate the equity premium, while under RE it explains only one quarter of the observed value. This is a remarkable improvement relative to the performance of the model under RE and suggests that allowing for small departures from RE is a promising avenue for research more generally.

The paper is organized as follows. In section 2 we discuss the related literature. Section 3 presents the stylized asset pricing facts we seek to match. We outline the asset pricing model in section 4 where we also derive analytic results showing how - for a general class of belief systems - our model can qualitatively deliver the stylized asset pricing facts described before in section 3. Section 5 presents the MSM estimation and testing strategy that we use and documents that the model with subjective beliefs can quantitatively reproduce the stylized facts. Readers interested in obtaining a glimpse of the quantitative performance of our one parameter extension of the RE model may directly jump to Tables 2 and 3 in section 5.2. Section 6 investigates the robustness of our finding to a number of alternative modeling assumptions, as well as the degree to which agents could detect if they are making systematic forecast errors. A conclusion briefly summarizes the main findings.

2 Related Literature

A large body of literature documents that the basic asset pricing model with time-separable preferences and RE has great difficulties in matching the observed volatility of stock returns.\(^2\)

Models of learning have long been considered as a promising avenue to match stock price volatility. Stock price behavior under Bayesian learning has been studied in Timmermann (1993, 1996), Brennan and Xia (2001), Cecchetti, Lam, and Mark (2000), and Cogley and Sargent (2008), among others. Some papers in this vein study agents that have asymmetric information or asymmetric beliefs, examples include Biais, Bossaerts and Spatt (2010) and Dumas, Kurshev and Uppal (2009). Agents in these papers learn about the dividend or income process and then set the asset price equal to the discounted expected sum of dividends. As explained in Adam and Marcet

\(^2\)See Campbell (2003) for an overview.
(2011), this amounts to assuming that agents know exactly how dividends and income map into prices, so that there is a rather asymmetric treatment of the issue of learning: while agents learn about how dividends and income evolve, they are assumed to know perfectly the stock price process, conditional on the realization of dividends and income. As a result, in these models stock prices typically represent redundant information given agents’ assumed knowledge, and there exists no feedback from market outcomes (stock prices) to beliefs. Since agents are then learning about exogenous processes only, their beliefs are ‘anchored’ by the exogenous processes and the volatility effects resulting from learning are generally limited when considering standard time-separable preference specifications. In contrast, we largely abstract from learning about the dividend and income processes and focus on learning about stock price behavior. Price beliefs and actual price outcomes then mutually influence each other. It is precisely this self-referential nature of the learning problem that imparts momentum to expectations and is key in explaining stock price volatility.

A number of papers within the adaptive learning literature study agents who learn about stock prices. Bullard and Duffy (2001) and Brock and Hommes (1998) show that learning dynamics can converge to complicated attractors and that the RE equilibrium may be unstable under learning dynamics. Branch and Evans (2010) study a model where agents’ algorithm to form expectations switches depending on which of the available forecast models is performing best. Branch and Evans (2011) study a model with learning about returns and return risk. Lansing (2010) shows how near-rational bubbles can arise under learning dynamics when agents forecast a composite variable involving future price and dividends. Boswijk, Hommes and Manzan (2007) estimate a model with fundamentalist and chartist traders whose relative shares evolve according to an evolutionary performance criterion. Timmermann (1996) analyzes a case with self-referential learning, assuming that agents use dividends to predict future price. Marcet and Sargent (1992) also study convergence to RE in a model where agents use today’s price to forecast the price tomorrow in a stationary environment with limited information. Cárcceles-Poveda and Giannitsarou (2008) assume that agents know the mean stock price and find that learning does then not significantly alter

\footnote{Stability under learning dynamics is defined in Marcet and Sargent (1989).}

\footnote{Timmerman reports that this form of learning delivers even lower volatility than a settings with learning about the dividend process only. It is thus crucial for our results that agents use information on past price growth behavior to predict future price growth.}
the behavior of asset prices. Chakraborty and Evans (2008) show that a model of adaptive learning can account for the forward premium puzzle in foreign exchange markets.

We contribute relative to the adaptive learning literature by deriving the learning and asset pricing equations from internally rational investor behavior. In addition, we use formal econometric inference and testing to show that the model can quantitatively match the observed stock price volatility. Finally, our paper also shows that the key issue for matching the data is that agents learn about the mean growth rate of stock prices from past stock prices observations.

In contrast to the RE literature, the behavioral finance literature seeks to understand the decision-making process of individual investors by means of surveys, experiments and micro evidence, exploring the intersection between economics and psychology, see Shiller (2005) for a non-technical summary. We borrow from this literature an interest in deviating from RE but we are keen on making only a minimal deviation from the standard approach: we assume that agents behave optimally given an internally consistent system of subjective beliefs that is close (but not equal) to the RE beliefs.

3 Facts

This section describes stylized facts of U.S. stock price data that we seek to replicate in our quantitative analysis. These observations have been extensively documented in the literature, we reproduce them here as a point of reference using a single and updated data base.5

Since the work of Shiller (1981) and LeRoy and Porter (1981) it has been recognized that the volatility of stock prices in the data is much higher than standard RE asset pricing models suggest, given the available evidence on the volatility of dividends. Figure 1 shows the evolution of the price dividend (PD) ratio in the United States, where the PD ratio is defined as the ratio of stock prices over quarterly dividend payments. The PD ratio displays very large fluctuations around its sample mean (the bold horizontal line in the graph): in the year 1932 the quarterly PD ratio takes on values below 30, while in the year 2000 values close to 350. The standard deviation of the PD ratio (σ_PD) is approximately one half of its sample mean (EPD). We report this feature of the data as Fact 1 in Table 1.

5 Details on the data sources are provided in Appendix A.1.
Figure 1: Quarterly U.S. price dividend ratio 1927:2-2012:2

Figure 1 also shows that the deviation of the PD ratio from its sample mean are very persistent, so that the first order quarterly autocorrelation of the PD ratio ($\rho_{PD,-1}$) is very high. We report this as Fact 2 in Table 1.

Related to the excessive volatility of prices is the observation that the volatility of stock returns ($\sigma_{x^*}$) in the data is almost four times the volatility of dividend growth ($\sigma_{\Delta D/D}$). We report the volatility of returns as Fact 3 in Table 1, and the mean and standard deviation of dividend growth at the bottom of the Table.

While stock returns are difficult to predict at short horizons, the PD ratio helps to predict future excess stock returns in the long run. More precisely, estimating the regression

$$X_{t,n} = c_n^1 + c_n^2 PD_t + u_{t,n} ,$$

(1)

where $X_{t,n}$ is the observed real excess return of stocks over bonds from quarter $t$ to quarter $t$ plus $n$ years, and $u_{t,n}$ the regression residual, the estimate $c_n^2$ is found to be negative, significantly different from zero, and the absolute value of $c_n^2$ and the $R$-square of this regression, denoted $R_n^2$, increase with $n$. We choose to include the OLS regression results for the 5-year horizon as Fact
4 in Table 1.\textsuperscript{6}

<table>
<thead>
<tr>
<th>Fact 1</th>
<th>Volatility of PD ratio</th>
<th>$E_{PD}$</th>
<th>123.91</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\sigma_{PD}$</td>
<td></td>
<td>62.43</td>
</tr>
</tbody>
</table>

| Fact 2      | Persistence of PD ratio | $\rho_{PD,-1}$ | 0.97  |

| Fact 3      | Excessive return volatility | $\sigma_{r^\ast}$ | 11.44 |

| Fact 4      | Excess return predictability | $c_5^2$ | -0.0041 |
|             |                           | $R_5^2$  | 0.2102  |

| Fact 5      | Equity premium             | $E_{r^\ast}$ | 2.25  |
|             | Quarterly real stock returns |          |       |
|             | Quarterly real bond returns | $E_{r,b}$ | 0.15  |

| Dividend Behavior | Mean growth | $E_{\Delta D}$ | 0.41  |
|                  | Std. dev. of growth | $\sigma_{\Delta D}$ | 2.88  |

PD ratio is price over quarterly dividend, growth rates & returns are expressed in real terms and quarterly rates of increase

Table 1: U.S. asset pricing facts 1927:2-2012:2

Finally, it is well known that through the lens of standard models real stock returns tend to be too high relative to short-term real bond returns, a fact often referred to as the equity premium puzzle. We report it as **Fact**

\textsuperscript{6}We focus on the 5-year horizon for simplicity, but obtain very similar results for other horizons. Our focus on a single horizon is justified because chapter 20 in Cochrane (2005) shows that Facts 1, 2 and 4 are closely related: up to a linear approximation, the presence of return predictability and the increase in the $R_n^2$ with the prediction horizon $n$ are qualitatively a joint consequence of persistent PD ratios (Fact 2) and i.i.d. dividend growth. It is not surprising, therefore, that our model also reproduces the increasing size of $c_n^2$ and $R_n^2$ with $n$. We match the regression coefficients at the 5-year horizon to check the quantitative model implications.
5 in Table 1, which shows that the average quarterly real return on bonds \( \bar{E}_{r,b} \) is much lower than the corresponding quarterly return on stocks \( \bar{E}_{r,s} \).

Table 1 reports ten statistics. As we show in section 5, we can replicate these statistics using a model that has only four free parameters.

4 The Model

We describe below a Lucas (1978) asset pricing model with agents who hold subjective prior beliefs about stock price behavior. We show that the presence of subjective uncertainty implies that utility maximizing agents update their beliefs about stock price behavior using observed stock price realizations.\(^7\)

Using a generic updating mechanism, section 4.2 shows that such learning gives rise to oscillations of asset prices around their fundamental value and qualitatively contributes to reconciling the Lucas asset pricing model with the empirical evidence. Section 4.3 then introduces a specific system of prior beliefs that gives rise to constant gain learning and that we employ in our empirical work in section 5. Section 4.3 also shows under what conditions this system of beliefs gives rise to small deviations from RE.

4.1 Model Description

The Environment: Consider an economy populated by a unit mass of infinitely-lived investors, endowed with one unit of a stock that can be traded on a competitive stock market and that pays dividend \( D_t \) consisting of a perishable consumption good. Dividends evolve according to

\[
\frac{D_t}{D_{t-1}} = a \varepsilon_t^d
\]

for \( t = 0, 1, 2, \ldots \), where \( \log \varepsilon_t^d \sim \mathcal{iN}(-\frac{s_d^2}{2}, s_d^2) \) and \( a \geq 1 \). This implies \( E(\varepsilon_t^d) = 1, \ E_{\Delta D} \equiv E \left( \frac{D_t - D_{t-1}}{D_{t-1}} \right) = a - 1 \) and \( \sigma_{\Delta D}^2 \equiv \text{var} \left( \frac{D_t - D_{t-1}}{D_{t-1}} \right) = a^2 \left( e^{s_d^2} - 1 \right) \). To capture the fact that the empirically observed consumption process is considerably less volatile than the dividend process and to replicate the correlation between dividend and consumption growth, we assume that each agent receives in addition an endowment \( Y_t \) of perishable consumption

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\(^7\)This draws on results in Adam and Marcet (2011).
goods. Total supply of consumption goods in the economy is then given by
the feasibility constraint \( C_t = Y_t + D_t \). Following the consumption-based
asset pricing literature, we impose assumptions directly on the aggregate
consumption supply process\(^8\)

\[
\frac{C_t}{C_{t-1}} = a \varepsilon_t, \tag{3}
\]

where \( \log \varepsilon_t \sim i\mathcal{N}(\mu, \sigma^2) \) and \( \log \varepsilon^c_t, \log \varepsilon^d_t \) jointly normal. In our empiri-
cal application, we follow Campbell and Cochrane (1999) and choose \( s^c = \frac{1}{7}s^d \)
and the correlation between \( \log \varepsilon^c_t \) and \( \log \varepsilon^d_t \) equal to \( \rho_{c,d} = 0.2 \).

**Objective Function and Probability Space:** Agent \( i \in [0,1] \) has a
standard time-separable expected utility function\(^9\)

\[
E_0^\mathcal{P} \sum_{t=0}^{\infty} \delta^t \left( \frac{C^i_t}{1 - \gamma} \right)^{1-\gamma},
\]

where \( \gamma \in (0, \infty) \) and \( C^i_t \) denotes consumption demand of agent \( i \). The
expectation is taken using a subjective probability measure \( \mathcal{P} \) that assigns
probabilities to all external variables, i.e., all payoff-relevant variables that
are beyond the agent’s control. Importantly, \( C^i_t \) denotes the agent’s con-
sumption demand, while \( C_t \) denotes the aggregate supply of consumption
goods in the economy.

The competitive stock market assumption and the exogeneity of the divi-
dend and income processes imply that investors consider the process for stock
prices \( \{P_t\} \) and the income and dividends processes \( \{Y_t, D_t\} \) as exogenous
to their decision problem. The underlying sample (or state) space \( \Omega \) thus
consists of the space of realizations for prices, dividends and income. Specifi-
cally, a typical element \( \omega \in \Omega \) is an infinite sequence \( \omega = \{P_t, Y_t, D_t\}_{t=0}^{\infty} \). As
usual, we let \( \Omega^t \) denote the set of histories from period zero up to period \( t \)
and \( \omega^t \) its typical element. The underlying probability space is thus given
by \( (\Omega, \mathcal{B}, \mathcal{P}) \) with \( \mathcal{B} \) denoting the corresponding \( \sigma \)-Algebra of Borel subsets
of \( \Omega \), and \( \mathcal{P} \) is the agent’s subjective probability measure over \( (\Omega, \mathcal{B}) \).

The probability measure \( \mathcal{P} \) specifies the joint distribution of \( \{P_t, Y_t, D_t\}_{t=0}^{\infty} \)
at all dates and is fixed at the outset. While the measure is fixed, investors’

\(^8\)The process for \( Y_t \) is then implied by feasibility.

\(^9\)We assume standard preferences so as to highlight the effect of learning on asset price
volatility.
beliefs about unknown parameters describing the stochastic processes of these variables, as well as investors’ conditional expectations of future values of these variables will change over time in a way that is derived from \( \mathcal{P} \) and that will depend on realized data. This specification thus encompasses settings in which agents are learning about the stochastic processes describing \( P_t, Y_t, \text{ and } D_t \). Moreover, unlike in the anticipated utility framework proposed in Kreps (1998), agents are fully aware of the fact that beliefs in the future will get revised. While the probability measure \( \mathcal{P} \) is not equal to the distribution of \( \{ P_t, Y_t, D_t \}_{t=0}^{\infty} \) implied by the model in equilibrium, it will be chosen in a way such that it is ‘close’ to it in a sense that we make precise in sections 4.3 and 6.3.

Expected utility is then defined as

\[
E_0^\mathcal{P} \sum_{t=0}^{\infty} \delta^t \left( C_t^i \right)^{1-\gamma} \frac{1}{1-\gamma} \int_{\Omega} \sum_{t=0}^{\infty} \delta^t \frac{C_t^i(\omega^t)^{1-\gamma}}{1-\gamma} d\mathcal{P}(\omega).
\]  

(4)

Our specification of the probability space is more general than the one used in other modeling approaches because we include also price histories in the realization \( \omega^t \). Standard practice is to assume instead that agents know the exact mapping \( P_t(\cdot) \) from a history of income and dividends to equilibrium asset prices \( P_t(Y^t, D^t) \), so that market prices carry only redundant information. This allows - without loss of generality - to exclude prices from the underlying state space. This practice is standard in models of rational expectations, models with rational bubbles, in Bayesian RE models as those described in the second paragraph of section 2, and models incorporating robustness concerns. This standard practice amounts to imposing a singularity in the joint density over prices, income and dividends, which is equivalent to assuming that agents know exactly the equilibrium pricing function \( P_t(\cdot) \).

While being a convenient modeling device, assuming exact knowledge of this function is at the same time very restrictive, it assumes that agents have a very detailed knowledge of how prices are formed. This makes it of interest to study the implication of (slightly) relaxing the assumption that agents know the function \( P_t(\cdot) \). Adam and Marcket (2011) show that rational behavior is indeed perfectly compatible with agents not knowing the exact form of the equilibrium pricing function \( P_t(\cdot) \).\(^{10}\)

\(^{10}\)Specifically, they show that with incomplete markets, i.e., in the absence of state contingent forward markets for stocks, agents cannot simply learn the equilibrium mapping
**Choice Set and Constraints:** Agents make contingent plans for consumption $C^i_t$, bondholdings $B^i_t$ and stockholdings $S^i_t$, i.e., choose the functions

$$(C^i_t, S^i_t, B^i_t) : \Omega^i \rightarrow \mathbb{R}^3$$

for all $t \geq 0$. Agent’s choices are subject to the budget constraint

$$C^i_t + P_t S^i_t + B^i_t \leq (P_t + D_t) S^i_{t-1} + (1 + r_{t-1}) B^i_{t-1} + Y_t$$

for all $t \geq 0$, where $r_{t-1}$ denotes the real interest rate on riskless bonds issued in period $t - 1$ and maturing in period $t$. The initial endowments are given by $S^i_{-1} = 1$ and $B^i_{-1} = 0$, so that bonds are in zero net supply. To avoid Ponzi schemes and to insure existence of a maximum the following bounds are assumed to hold

$$\underline{S} \leq S^i_t \leq \overline{S}$$

$$\underline{B} \leq B^i_t \leq \overline{B}$$

We only assume the bounds $\underline{S}, \overline{S}, \underline{B}, \overline{B}$ are finite and they satisfy $\underline{S} < 1 < \overline{S}$, $\underline{B} < 0 < \overline{B}$.

**Maximizing Behavior (Internal Rationality):** The investor’s problem then consists of choosing the sequence of functions $\{C^i_t, S^i_t, B^i_t\}_{t=0}^\infty$ to maximize (4) subject to the budget constraint (6) and the asset limits (7), where all constraints have to hold for all $t$ almost surely in $\mathcal{P}$. Later on, the probability measure $\mathcal{P}$ will be specified through some perceived law of motion describing the agent’s view about the evolution of $(P, Y, D)$ over time, together with a prior distribution about the parameters governing this law of motion. Optimal behavior will then entail learning about these parameters, in the sense that agents update their posterior beliefs about the unknown parameters in the light of new price, income and dividend observations. For the moment this learning problem remains ‘hidden’ in the belief structure $\mathcal{P}$.

**Optimality Conditions:** Since the objective function is concave and the feasible set is convex, the agent’s optimal plan is characterized by the first order conditions

$$(C^i_t)^{-\gamma} P_t = \delta E^P_t \left[(C^i_{t+1})^{-\gamma} P_{t+1}\right] + \delta E^P_t \left[(C^i_{t+1})^{-\gamma} D_{t+1}\right]$$

$$(C^i_t)^{-\gamma} = \delta (1 + r_t) E^P_t \left[(C^i_{t+1})^{-\gamma}\right].$$

$P_t(\cdot)$ by observing market prices. Furthermore, if the preferences and beliefs of agents in the economy fail to be common knowledge, then agents cannot deduce the equilibrium mapping from their own optimization conditions.
These conditions are standard except for the fact that the conditional expectations are taken with respect to the subjective probability measure \( \mathcal{P} \).\(^{11}\)

## 4.2 Asset Pricing Implications: Analytical Results

This section presents analytical results that explain why the asset pricing model with subjective beliefs can explain the asset pricing facts presented in Table 1.

Before doing so we briefly review the well-known result that under RE the model is at odds with these asset pricing facts. A routine calculation shows that the unique RE solution of the model is given by

\[
P_{t}^{RE} = \frac{\delta a^{1-\gamma} \rho_{\varepsilon}}{1 - \delta a^{1-\gamma} \rho_{\varepsilon}} D_t, \tag{10}
\]

where

\[
\rho_{\varepsilon} = E \left[ \left( \varepsilon_{t+1}^{c} - \bar{\varepsilon}_{t+1} \right)^{-\gamma} \varepsilon_{t+1}^{d} \right] = e^{\gamma(1+\gamma)\frac{\alpha}{2} e^{-\gamma \rho_{c,d} \varepsilon^{c} \varepsilon^{d}}.}
\]

The PD ratio is then constant, return volatility equals approximately the volatility of dividend growth and there is no (excess) return predictability, so that the model misses Facts 1 to 4 listed in Table 1. This holds independently of the parameterization of the model. Furthermore, even for very high degrees of relative risk aversion, say \( \gamma = 80 \), the model implies a fairly small risk premium. This emerges because of the low correlation between the innovations to consumption growth and dividend growth in the data (\( \rho_{c,d} = 0.2 \)).\(^{12}\) The model thus also misses Fact 5 in Table 1.

We now characterize the equilibrium outcome under learning. One may be tempted to argue that \( C_{t+j}^{u} \) can be substituted by \( C_{t+j} \) for \( j = 0, 1 \) in the first

\(^{11}\)We write these Euler equations with equality because in equilibrium the interior solution holds. Adam and Marcet (2011) argue that since the Euler equations are expected to hold with inequality under the agents’ subjective plans, a system of beliefs about prices is perfectly consistent with utility maximization and a system of beliefs about dividends. We come back to this issue in section 6.4.

\(^{12}\)Under RE, the risk free rate is given by \( 1 + r = \left( \delta a^{-\gamma} e^{\gamma(1+\gamma)\frac{\alpha}{2}} \right)^{-1} \) and the expected equity return equals \( E_t[(P_{t+1} + D_{t+1})/P_t] = (\delta a^{-\gamma} \rho_{\varepsilon})^{-1} \). For \( \rho_{c,d} = 0 \) there is thus no equity premium, independently of the value for \( \gamma \).
order conditions (8) and (9), simply because \( C_t^i = C_t \) holds in equilibrium for all \( t \).\(^{13}\) However, outside of strict rational expectations we may have \( E_t^P_C [C_{t+1}] \neq E_t^P [C_{t+1}] \) even if in equilibrium \( C_t^i = C_t \) holds ex-post.\(^{14}\) To understand how this arises, consider the following simple example: suppose agents know the aggregate process for \( D_t \) and \( Y_t \). In this case, \( E_t^P_C [C_{t+1}] \) is a function only of the exogenous variables \( (Y^t, D^t) \). At the same time, \( E_t^P [C_{t+1}] \) is generally a function of price realizations also, since in the eyes of the agent, optimal future consumption demand depends on future prices and, therefore, also on today’s prices whenever agents are learning about price behavior. As a result, in general \( E_t^P_C [C_{t+1}] \neq E_t^P [C_{t+1}] \), so that one cannot routinely substitute individual by aggregate consumption on the right-hand side of agent’s first order conditions (8) and (9).

Nevertheless, if in any given period \( t \) the optimal plan for period \( t+1 \) from the viewpoint of the agent is such that \( (P_{t+1}(1 - S_{t+1}^i) - B_{t+1}^i) / (Y_t + D_t) \) is expected to be small according to the agent’s expectations \( E_t^P \), then agents with beliefs \( P \) realize in period \( t \) that \( C_{t+1}^i / C_t^i \approx C_{t+1} / C_t \).\(^{15}\) One can then rely on the approximations

\[
E_t^P \left[ \left( \frac{C_{t+1}^i}{C_t^i} \right)^{-\gamma} (P_{t+1} + D_{t+1}) \right] \approx E_t^P \left[ \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma} (P_{t+1} + D_{t+1}) \right] \quad (11)
\]

\[
E_t^P \left[ \left( \frac{C_{t+1}^i}{C_t^i} \right)^{1-\gamma} \right] \approx E_t^P \left[ \left( \frac{C_{t+1}}{C_t} \right)^{1-\gamma} \right] \quad (12)
\]

The following assumption provides sufficient conditions for this to be the case:

**Assumption 1** We assume that \( Y_t \) is sufficiently large and that \( E_t^P P_{t+1}/D_t < \overline{M} \) for some \( \overline{M} < \infty \) so that, given finite asset bounds \( \underline{S}, \overline{S}, \underline{B}, \overline{B} \), the approximations (11) and (12) hold with sufficient accuracy.

Intuitively, for high enough income \( Y_t \), the agent’s asset trading decisions matter little for the agents’ stochastic discount factor \( \left( \frac{C_{t+1}^i}{C_t^i} \right)^{-\gamma} \), allowing us

\(^{13}\)\( C_t^i = C_t \) follows from market clearing and the fact that all agents are identical.

\(^{14}\)This is the case because the preferences and beliefs of agents are not assumed to be common knowledge, so that agents do not know that \( C_t^i = C_t \) must hold in equilibrium.

\(^{15}\)This follows from the flow budget constraint for period \( t+1 \) and the fact that \( S_t^i = 1 \), \( B_t^i = 0 \) and \( C_t^i = C_t \) in equilibrium in period \( t \).
to approximate individual consumption in $t + 1$ by aggregate consumption in $t + 1$. The bound on subjective price expectations imposed in Assumption 1 is justified by the fact that the price dividend ratio will be bounded in equilibrium, so that the objective expectation $E_t P_{t+1}/D_t$ will also be bounded.

With Assumption 1, the risk-free interest rate solves

$$1 = \delta(1 + r_t)E_t^p \left[ \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma} \right],$$

(13)

Furthermore, defining the subjective expectations of risk-adjusted stock price growth

$$\beta_t \equiv E_t^p \left( \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma} \frac{P_{t+1}}{P_t} \right),$$

(14)

and subjective expectations of risk-adjusted dividend growth

$$\beta_t^D \equiv E_t^p \left( \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma} \frac{D_{t+1}}{D_t} \right),$$

the first order condition for stocks (8) implies that the equilibrium stock price under subjective beliefs is given by

$$P_t = \frac{\delta \beta_t^D}{1 - \delta \beta_t} D_t,$$

(15)

provided $\beta_t < \delta^{-1}$. The equilibrium stock price is thus increasing in (subjective) expected risk-adjusted dividend growth and also increasing in expected risk-adjusted price growth.

For the special case when agents know the RE growth rates $\beta_t = \beta_t^D = a^{1-\gamma} \rho_c$ for all $t$, equation (15) delivers the RE price outcome (10). Furthermore, when agents hold subjective beliefs about risk-adjusted dividend growth but objectively rational beliefs about risk-adjusted price growth then $\beta_t^P = \beta_t$ and (15) delivers the pricing implications derived in the Bayesian RE asset pricing literature, as surveyed in section 2.

\(^{16}\)Note that independent from their tightness, the asset holding constraints never prevent agents from marginally trading or selling securities in any period $t$ along the equilibrium path, where $S_t^1 = 1$ and $B_t^1 = 0$ holds for all $t$.

\(^{17}\)To see this note that $P_{t+1}/D_{t+1} < \bar{P}D$ implies $E_t[P_{t+1}]/D_t < a\bar{P}D < \infty$ where $a$ denotes the mean dividend growth rate.
To highlight the fact that the improved empirical performance of the present asset pricing model derives exclusively from the presence of subjective beliefs about risk-adjusted price growth, we shall entertain assumptions that are orthogonal to those made in the Bayesian RE literature. Specifically, we assume that agents know the true process for risk-adjusted dividend growth:

**Assumption 2** Agents know the process for risk adjusted dividend growth, i.e., $\beta_t^D \equiv a^{1-\gamma} \rho_e$ for all $t$.

Under this assumption the asset pricing equation (15) simplifies to:

$$P_t = \frac{\delta a^{1-\gamma} \rho_e}{1 - \delta \beta^t} D_t.$$ \hspace{1cm} (16)

### 4.2.1 Stock Price Behavior under Learning

We now derive a number of analytical results regarding the behavior of asset prices over time. We start out with a general observation about the volatility of prices and thereafter derive results about the behavior of prices over time for a general belief updating scheme.

The asset pricing equation (16) implies that fluctuations in subjective price expectations can contribute to the fluctuations in actual prices. As long as the correlation between $\beta_t$ and the last dividend innovation $\varepsilon_t^d$ is small (as occurs for the updating schemes for $\beta_t$ that we consider in this paper), equation (16) implies

$$\text{var} \left( \ln \frac{P_t}{P_{t-1}} \right) \simeq \text{var} \left( \ln \frac{1 - \delta \beta_{t-1}}{1 - \delta \beta_t} \right) + \text{var} \left( \ln \frac{D_t}{D_{t-1}} \right).$$ \hspace{1cm} (17)

The previous equation shows that even small fluctuations in subjective price growth expectations can significantly increase the variance of price growth, and thus the variance of stock price returns, if $\beta_t$ fluctuates around values close to but below $\delta^{-1}$.

To determine the behavior of asset prices over time, one needs to take a stand on how the subjective price expectations $\beta_t$ are updated over time.

---

\(^{18}\)Some readers may be tempted to believe that entertaining subjective price beliefs while entertaining subjective beliefs about the dividend process is inconsistent with individual rationality. Adam and Marcet (2011) show, however, that there exists no such contradiction, as long as the preferences and beliefs of agents in the economy are not common knowledge.
To improve our understanding for the empirical performance of the model and to illustrate that the results we obtain in our empirical application do not depend on the specific belief system considered, we now derive analytical results for a general nonlinear belief updating scheme.

Given that $\beta_t$ denotes the subjective one-step-ahead expectation of risk adjusted stock price growth, it appears natural to assume that the measure $\mathcal{P}$ implies that rational agents revise $\beta_t$ upwards (downwards) if they underpredicted (overpredicted) the risk adjusted stock price growth ex-post. This prompts us to consider measures $\mathcal{P}$ that imply updating rules of the form\footnote{Note that $\beta_t$ is determined from observations up to period $t - 1$ only. This simplifies the analysis and it avoids simultaneity of price and forecast determination. This lag in the information is common in the learning literature. Difficulties emerging with simultaneous information sets in models of learning are discussed in Adam (2003).}

$$\Delta \beta_t = f_t \left( \left( \frac{C_{t-1}}{C_{t-2}} \right)^{-\gamma} \frac{P_{t-1}}{P_{t-2}} - \beta_{t-1}; \beta_{t-1} \right)$$

for given non-linear updating functions $f_t : R^2 \rightarrow R$ with the properties

$$f_t(0; \beta) = 0$$

$$f_t (\cdot; \beta) \text{ increasing}$$

$$0 < \beta + f_t (x; \beta) < \beta^U$$

for all $(t, x), \beta \in (0, \beta^U)$ and for some constant $\beta^U \in (a^{1-\gamma} \rho_\xi, \delta^{-1})$. Properties (19) and (20) imply that $\beta_t$ is adjusted in the same direction as the last prediction error, where the strength of the adjustment may depend on the current level of beliefs, as well as on calendar time (e.g., on the number of observations available to date). Property (21) is needed to guarantee that positive equilibrium prices solving (16) always exist.

In section 4.3 below we provide an explicit system of beliefs $\mathcal{P}$ in which agents optimally update beliefs according to a special case of equation (18). Updating rule (18) is more general and nests also other systems of beliefs, as well as a range of learning schemes considered in the literature on adaptive learning, such as the widely used least squares learning or switching gains learning used by Marcet and Nicolini (2003).

To derive the equilibrium behavior of price expectations and price realizations over time, we first use (16) to determine realized price growth

$$\frac{P_t}{P_{t-1}} = \left( a + \frac{a \delta}{1 - \delta} \Delta \beta_t \right) \xi_t.$$  

(22)
Combining the previous equation with the belief updating rule (18) one obtains
\[
\Delta \beta_{t+1} = f_{t+1}(T(\beta_t, \Delta \beta_t)(\varepsilon_{t}^{\delta} - \beta_t; \beta_t),
\]
where
\[
T(\beta, \Delta \beta) = a^{1-\gamma} + \frac{a^{1-\gamma} \Delta \beta}{1 - \delta \beta}.
\]
Given initial conditions \((Y_0, D_0, P_{-1})\), and initial expectations \(\beta_0\), equation (23) completely characterizes the equilibrium evolution of the subjective price expectations \(\beta_t\) over time. Given that there is a one-to-one relationship between \(\beta_t\) and the PD ratio, see equation (16), the previous equation also characterizes the evolution of the equilibrium PD ratio under learning. High (low) price growth expectations are thereby associated with high (low) values for the equilibrium PD ratio.

The properties of the second order difference equation (23) can be illustrated in a 2-dimensional phase diagram for the dynamics of \((\beta_t, \beta_{t-1})\), which is shown in Figure 2 for the case where the shocks \((\varepsilon_{t}^{\delta})^{\gamma} \varepsilon_{t}^{d}\) assume their unconditional mean value \(\rho_{e}\). The effects of different shock realizations for the dynamics will be discussed separately below.

The arrows in Figure 2 indicate the direction in which the vector \((\beta_t, \beta_{t-1})\) evolves over time according to equation (23), and the solid lines indicate the boundaries of these areas. Since we have a difference equation rather than a differential equation, we cannot plot the evolution of expectations exactly, as the difference equation gives rise to discrete jumps in the vector \((\beta_t, \beta_{t-1})\) over time. Yet, if agents update beliefs only relatively weakly in response to forecast errors, as will be the case for our estimated model later on, then for some areas in the figure these jumps will be correspondingly small, as we now explain.

Consider, for example, region A in the diagram. In this area \(\beta_t < \beta_{t-1}\) and \(\beta_t\) keeps decreasing, showing that there is momentum in price changes. This holds true even if \(\beta_t\) is already at or below its fundamental value \(a^{1-\gamma} \rho_{e}\). Provided the updating gain is small, beliefs in region A will slowly move above the 45° degree line in the direction of the lower left corner of the graph. Yet, once they enter area B, \(\beta_t\) starts increasing, so that in the next period beliefs will discretely jump into area C. In region C we have \(\beta_t > \beta_{t-1}\) and \(\beta_t\) keeps increasing, so that beliefs then display upward momentum. This manifests

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20 Appendix A.2 explains in detail the construction of the phase diagram.

21 The vertical solid line close to \(\delta^{-1}\) is meant to illustrate the restriction \(\beta < \delta^{-1}\).
itself in an upward and rightward move of beliefs over time, until these reach area $D$. There, beliefs $\beta_t$ start decreasing, so that they move in one jump back into area $A$, thereby displaying mean reversion. The elliptic movements of beliefs around $a^{1-\gamma}\rho_\varepsilon$ imply that expectations (and thus the PD ratio) are likely to oscillate in sustained and persistent swings around the RE value.

The effects of the stochastic disturbances $(\varepsilon_t^\gamma\varepsilon_t^d)$ is to shift the curve labeled "$\beta_{t+1} = \beta_t$" in Figure 2. Specifically, for realizations $(\varepsilon_t^\gamma\varepsilon_t^d) > \rho_\varepsilon$ this curve is shifted upwards. As a result, beliefs are more likely to increase, which is the case for all points below this curve. Conversely, for $(\varepsilon_t^\gamma\varepsilon_t^d) < \rho_\varepsilon$ this curve shifts downward, making it more likely that beliefs decrease from the current period to the next.

The previous results show that learning causes beliefs and the PD ratio to stochastically oscillate around its RE value. Such behavior will be key in explaining the observed volatility and the serial correlation of the PD ratio, i.e., Facts 1 and 2 in Table 1. Also, from the discussion around equation (17) it should be clear that such behavior makes stock returns more volatile than dividend growth, which contributes to replicating Fact 3. As discussed in Cochrane (2005), a serially correlated and mean reverting PD ratio gives
rise to excess return predictability, i.e., it contributes to matching Fact 4.

The momentum of changes in beliefs around the RE value of beliefs, as well as the overall mean reverting behavior can be more formally captured in the following results:

**Momentum:** If $\Delta \beta_t > 0$ and

$$\beta_t \leq a^{1-\gamma} (\varepsilon^e_t)^{-\gamma} \varepsilon^d_t,$$

then $\Delta \beta_{t+1} > 0$. This also holds if all inequalities are reversed.\(^{22}\)

Therefore, up to a linear approximation of the updating function $f$,

$$E_{t-1}[\Delta \beta_{t+1}] > 0,$$

whenever $\Delta \beta_t > 0$ and $\beta_t \leq a^{1-\gamma} \rho_\varepsilon$. Therefore beliefs have a tendency to increase (decrease) further following an initial increase (decrease), whenever beliefs are at or below (above) the RE value.

The following result shows formally that stock prices would eventually return to their (deterministic) RE value in the absence of further disturbances, and that such reverting behavior occurs monotonically: \(^{23}\)

**Mean reversion:** Consider an arbitrary initial belief $\beta_t \in (0, \beta^U)$. In the absence of further disturbances ($\varepsilon^d_{t+j} = \varepsilon^e_{t+j} = 0$ for all $j \geq 0$)

$$\limsup_{t \to \infty} \beta_t \geq a^{1-\gamma} \geq \liminf_{t \to \infty} \beta_t.$$  

Furthermore, if $\beta_t > a^{1-\gamma}$, there is a period $t' \geq t$ such that $\beta_t$ is non-decreasing between $t$ and $t'$ and non-increasing between $t'$ and $t''$, where $t''$ is the first period where $\beta_{t''}$ is arbitrarily close to $a^{1-\gamma}$. Symmetrically, if $\beta_t < a^{1-\gamma}$.

The previous result implies that - absent any shocks - $\beta_t$ cannot stay away from the RE value forever. Beliefs either converge to the deterministic RE value (when $\limsup = \liminf$) or stay fluctuating around it forever (when $\limsup > \liminf$). Any initial deviation, however, is eventually eliminated

\(^{22}\)The result follows from the fact that condition (24) implies that the first argument in the $f$ function on the right-hand side of equation (23) is positive.

\(^{23}\)See Appendix A.3 for the proof under an additional technical assumption.
with the reversion process being monotonic. This result also implies that an upper bound on price beliefs can not be an absorbing point: if beliefs $\beta_t$ go up and they get close to the upper bound $\beta^U$ they will eventually bounce off this upper bound and go back down towards the RE value.

Summing up, the previous results show that for a general set of belief updating rules, stock prices and beliefs fluctuate around their RE values in a way that helps to qualitatively account for Facts 1 to 4 listed in Table 1.

### 4.3 Optimal Belief Updating: Constant Gain Learning

We now introduce a fully specified probability measure $\mathcal{P}$ and derive the optimal belief updating equation it implies. We employ this belief updating equation in our empirical work in section 5. We show below a precise sense in which this system of beliefs represents a small deviation from RE.

In line with Assumption 2, we consider agents who hold rational expectations about the dividend and aggregate consumption processes. At the same time, we allow for subjective beliefs about risk-adjusted stock price growth by allowing agents to entertain the possibility that risk-adjusted price growth may contain a small and persistent time-varying component. This is motivated by the observation that in the data there are periods in which the PD ratio increases persistently, as well as periods in which the PD ratio falls persistently, see figure 1. In an environment with unpredictable innovations to dividend growth, this implies the existence of persistent and time-varying components in stock price growth. For this reason, we consider agents who think that the process for risk-adjusted stock price growth is the sum of a persistent component $b_t$ and of a transitory component $\varepsilon_t$

\[
\left( \frac{C_t}{C_{t-1}} \right)^{-\gamma} \frac{P_t}{P_{t-1}} = b_t + \varepsilon_t
\]

\[
b_t = b_{t-1} + \xi_t
\]

for $\varepsilon_t \sim i.i.d. N(0, \sigma^2_{\varepsilon})$, $\xi_t \sim i.i.d. N(0, \sigma^2_{\xi})$, independent of each other.\(^{24}\)

The previous setup encompasses the RE equilibrium beliefs as a special case. Namely, when agents believe $\sigma^2_{\varepsilon} = 0$ and assign probability one to $b_0 = a^{1-\gamma}\rho$, we have that $\beta_t = a^{1-\gamma}\rho$ for all $t \geq 0$ and prices are as given by RE equilibrium prices in all periods.

\(^{24}\)Notice that we use the notation $C_t = Y_t + D_t$ so that equation (25) contains only payoff-relevant variables that are beyond the agent’s control.
In what follows we allow for a non-zero variance \( \sigma^2_\xi \), i.e., for the presence of a persistent time-varying component in price growth. The setup then gives rise to a learning problem because agents observe only the realizations of risk-adjusted price growth, but not the persistent and transitory component separately. The learning problem consist of optimally filtering out the persistent component of price growth \( b_t \). Assuming that agents prior beliefs \( b_0 \) are centered at the RE value and given by

\[
b_0 \sim N(a^{1-\gamma} \rho_\xi, \sigma^2_0) \]

and setting \( \sigma^2_0 \) equal to the steady state Kalman filter uncertainty about \( b_t \), which is given by

\[
\sigma^2_0 = \frac{-\sigma^2_\xi + \sqrt{(\sigma^2_\xi)^2 + 4\sigma^2_\xi^2}}{2},
\]

agents’ posterior beliefs at any time \( t \) are given by

\[
b_t \sim N(\beta_t, \sigma_0),
\]

with optimal updating implying that \( \beta_t \), defined in equation (14), recursively evolves according to

\[
\beta_t = \beta_{t-1} + \frac{1}{\alpha} \left( \left( \frac{C_{t-1}}{C_{t-2}} \right)^{-\gamma} P_{t-1} - \beta_{t-1} \right).
\]

The optimal (Kalman) gain is given by \( 1/\alpha = (\sigma^2_0 + \sigma^2_\xi) / (\sigma^2_0 + \sigma^2_\xi + \sigma^2_\xi) \) and captures the strength with which agents optimally update their posteriors in response to surprises.\(^{25}\)

These beliefs constitute a small deviation from RE beliefs in the limiting case with vanishing innovations to the random walk process \( \sigma^2_\xi \to 0 \). Agents’ prior uncertainty then vanishes \( (\sigma^2_0 \to 0) \) and the optimal gain converges to zero \( (1/\alpha \to 0) \). As a result, \( \beta_t \to a^{1-\gamma} \rho_\xi \) in distribution for all \( t \), so that one recovers the RE equilibrium value for risk-adjusted price growth expectations. This shows that for a given distribution of asset price, agents’ beliefs are close to RE beliefs for a small value of the gain parameter \( (1/\alpha) \).\(^{26}\)

\(^{25}\)In line with equation (18) we incorporate information with a lag, so as to eliminate the simultaneity between prices and price growth expectations. The lag in the updating equation could be justified by a specific information structure where agents observe some of the lagged transitory shocks to risk-adjusted stock price growth.

\(^{26}\)We show below that continues to be true when using the equilibrium distribution of asset prices generated by these beliefs."
For our empirical application, we need to slightly modify the updating equation (26) to guarantee that the bound \( \beta_t < \beta^U \) holds for all periods and equilibrium prices always exist. The exact way in which this bound is imposed matters little for our empirical result, because the moments we compute do not change much as long as \( \beta_t \) is close to \( \beta^U \) only rarely over the sample length considered. To impose this bound, we consider in our empirical application a concave, increasing and differentiable function \( w : R_+ \to (0, \beta^U) \) and modify the belief updating equation (26) to\(^{27}\)

\[
\beta_t = w \left( \beta_{t-1} + \frac{1}{\alpha} \left[ \left( \frac{C_{t-1}}{C_{t-2}} \right)^{-\gamma} \frac{P_{t-1}}{P_{t-2}} - \beta_{t-1} \right] \right),
\]

where

\[
w(x) = x \text{ if } x \in (0, \beta^L)
\]

for some \( \beta^L \in (a^{1-\gamma} \rho, \beta^U) \). Beliefs thus continue to evolve according to (26), as long as they are below the threshold \( \beta^L \), while for higher beliefs we have that \( w(x) \leq x \). The modified algorithm (27) satisfies the constraint (21) and can be interpreted as an approximate implementation of a Bayesian updating scheme where agents have a truncated prior that puts probability zero on \( b_t > \beta^U \).\(^{28,29}\)

We now show that for a small value of the gain \( (1/\alpha) \) agents’ beliefs are close to RE beliefs when using the equilibrium distribution of prices generated by these beliefs. More precisely, the setup gives rise to a stationary

\(^{27}\)The exact functional form for \( w \) that we use in the estimation is shown in appendix A.5.

\(^{28}\)The issue of bounding beliefs so as to insure that expected utility remains finite is present in many applications of both Bayesian and adaptive learning to asset prices. The literature has typically dealt with this issue by using a "projection facility", assuming that agents simply ignore observations that would imply updating beliefs beyond the required bound, see Timmermann (1993, 1996), Marcet and Sargent (1989), or Evans and Honkapohja (2001). This approach, has two problems, first, it does not arise from Bayesian updating, second, it introduces a discontinuity in the simulated moments and creates difficulties for our MSM estimation in section 5, prompting us to pursue the differentiable approach to bounding beliefs described above.

\(^{29}\)Adam, Beutel, Marcet (2013) solve the model without Assumption 1, which is computationally much more costly, so that one cannot use MSM to estimate the model. They use the optimal consumption plan that solves the individual investor’s problem to price the asset and show how a projection facility then arises endogenously from the wealth effect in the standard case that relative risk aversion is larger than one.
and ergodic equilibrium outcome in which risk-adjusted stock price growth expectations have a distribution that is increasingly centered at the RE value $a^{1-\gamma} \rho_s$ as the gain parameter becomes vanishingly small. From equation (16) it then follows that actual equilibrium prices also become increasingly concentrated at their RE value, so that the difference between beliefs and outcomes becomes vanishingly small as $1/\alpha \to 0$.

**Stationarity, Ergodicity, and Small Deviations from RE:** Suppose agents’ posterior beliefs evolve according to equation (27) and equilibrium prices are determined according to equation (16). Then $\beta_t$ is geometrically ergodic for $\alpha$ sufficiently large. Furthermore, as $1/\alpha \to 0$, we have $E[\beta_t] \to a^{1-\gamma} \rho_s$ and $VAR(\beta_t) \to 0$.

The proof is based on results from Duffie and Singleton (1993) and contained in appendix A.4. Geometric ergodicity implies the existence of a unique stationary distribution for $\beta_t$ that is ergodic and that is reached from any initial condition. Geometric ergodicity is required for estimation by MSM.

We explore further in section 6 the connection between agents’ beliefs and model outcomes.

## 5 Quantitative Model Performance

This section evaluates the quantitative performance of the asset pricing model with subjective price beliefs and shows that it can robustly replicate Facts 1 to 4 listed in Table 1. We formally estimate and test the model using the method of simulated moments (MSM). This approach to structural estimation and testing helps us focusing on the ability of the model to explain the specific moments of the data described in Table 1.\(^3\) Regarding Fact 5 in Table 1, the model gives rise to an equity premium that is much higher than under

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\(^3\)A popular alternative approach in the asset pricing literature has been to test if agents’ first order conditions hold in the data. Hansen and Singleton (1982) pioneered this approach for RE models and Bossaerts (2004) provides an approach that can be applied to models of learning. We pursue the MSM estimation approach here because it naturally provides additional information on how the formal test for goodness of fit of the model relates to the model’s ability to match the moments of interest. The results are then easily interpretable, they point out which parts of the model fit well and which parts do not, thus providing intuition about possible avenues for improving the model fit.
RE, but for reasonable degrees of relative risk aversion it tends to fall short of the risk premium observed in the data.

We first evaluate the model’s ability to explain the individual moments, which is the focus of much of the literature on matching stock price volatility. We find that the model can explain the individual moments well. Using t-statistics based on formal asymptotic distribution we find that in some versions of the model all t-statistics are at or below two in absolute value, even with a moderate relative risk aversion of $\gamma = 5$. Moreover, with this degree of risk aversion, the model can explain up to 50% of the equity premium.

We then turn to the more demanding task of testing if all the moments are accepted jointly by computing Chi-square test statistics. Due to their stringency such test statistics are rarely reported in the consumption based asset pricing literature. A notable exception is Bansal, Kiku and Yaron (2013) who test the overidentifying restrictions of a long-run risk model. Different from our approach, they test equilibrium conditions instead of matching statistics. Also, they use a diagonal weighting matrix instead of the optimal weighting matrix in the objective function (29) introduced below.

We find that with a relative risk aversion of $\gamma = 5$ the model fails to pass an overall goodness of fit as long as one includes the equity premium. Yet, the test reaches a moderate p-value of 2.5% once we exclude the risk free rate from the set of moments to be matched, confirming that it is the equity premium which poses a quantitative challenge to the model.\textsuperscript{31} With a relative risk aversion of $\gamma = 3$, the p-value increases even further to 7.1%, when again excluding the risk free rate.

Finally, we allow for a very high risk aversion coefficient. Specifically, we set $\gamma = 80$, which is the steady state value of relative risk aversion used in Campbell and Cochrane (1999).\textsuperscript{32} The model then replicates all moments in Table 1, including the risk premium. In particular, the model generates a quarterly equity premium of 2.0%, slightly below the 2.1% per quarter observed in US data, while still replicating all other asset pricing moments.

The next section explains the method of simulated moments (MSM) approach for estimating the model and the formal statistical test for evaluating the goodness of fit. The subsequent section reports on the estimation and

\textsuperscript{31}The literature suggests a number of other model ingredients, that - once added - would allow generating a higher equity premium. For example, ambiguity aversion as in Collard et al. (2011), initially pessimistic expectations as in Cogley and Sargent (2008), or habits in consumption preferences.

\textsuperscript{32}This value is reported on p. 244 in their paper.
test outcomes.

5.1 MSM Estimation and Statistical Test

This section outlines the MSM approach and the formal test for evaluating the fit of the model. This is a simple adaptation of standard MSM to include matching of statistics that are functions of simple moments by using the delta method, see appendix A.6 for details.

For a given value of the coefficient of relative risk aversion, there are four free parameters left in the model, comprising the discount factor $\delta$, the gain parameter $1/\alpha$, and the mean and standard deviation of dividend growth, denoted by $a$ and $\sigma_{DP}^2$, respectively. We summarize these in the parameter vector

$$\theta \equiv \left( \delta, 1/\alpha, a, \sigma_{DP}^2 \right).$$

The four parameters will be chosen so as to match some or all of the ten sample moments in Table 1.\(^{33}\)

$$\left( \hat{E}_r, \hat{E}_{PD}, \hat{\sigma}_r, \hat{\sigma}_{PD}, \hat{\rho}_{PD,-1}, \hat{c}_2^2, \hat{R}_s^2, \hat{E}_{v^2}, \hat{E}_{\Delta P}, \hat{\sigma}_{\Delta P}^2 \right). \quad (28)$$

Let $\hat{S}_N \in \mathbb{R}^s$ denote the subset of sample moments in (28) that will be matched in the estimation, with $N$ denoting the sample size and $s \leq 10.$\(^{34}\) Furthermore, let $\hat{S}(\theta)$ denote the moments implied by the model for some parameter value $\theta$. The MSM parameter estimate $\hat{\theta}_N$ is defined as

$$\hat{\theta}_N \equiv \arg \min_{\theta} \left[ \hat{S}_N - \hat{S}(\theta) \right]' \hat{\Sigma}^{-1}_{\hat{S}S,N} \left[ \hat{S}_N - \hat{S}(\theta) \right], \quad (29)$$

where $\hat{\Sigma}_{\hat{S}S,N}$ is an estimate of the variance-covariance matrix of the sample moments $\hat{S}_N$. The MSM estimate $\hat{\theta}_N$ chooses the model parameter such that the model moments $\hat{S}(\theta)$ fit the observed moments $\hat{S}_N$ as close as possible in terms of a quadratic form with weighting matrix $\hat{\Sigma}^{-1}_{\hat{S}S,N}$. We estimate $\hat{\Sigma}_{\hat{S}S,N}$

\(^{33}\)Many elements listed in (28) are not sample moments but they are non-linear functions of sample moments. For example, the $R^2$ coefficient is a function of sample moments. This means we have to use the delta method to adapt standard MSM, see appendix A.6. It would be more precise to refer to the elements in (28) as ‘sample statistics’, as we do in the appendix. For simplicity we avoid this terminology in the main text.

\(^{34}\)As discussed before, we exclude the risk premium from some estimations, in those cases $s < 10$. 

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from the data in a standard way. Adapting standard results from MSM, one can prove that for a given list of moments included in $\tilde{S}_N$, the estimate $\tilde{\theta}_N$ is consistent and it is the best estimate amongst those obtained with different weighting matrices.

The MSM estimation approach also provides an overall test of the model. Under the null hypothesis that the model is correct, we have

$$\tilde{W}_N \equiv N \left[ \tilde{S}_N - \tilde{S}(\tilde{\theta}_N) \right]^T \tilde{\Sigma}_{S,N}^{-1} \left[ \tilde{S}_N - \tilde{S}(\tilde{\theta}_N) \right] \rightarrow \chi^2_{s-4} \text{ as } N \rightarrow \infty \quad (30)$$

where convergence is in distribution. Furthermore, we obtain a proper asymptotic distribution for each element of the deviations $\tilde{S}_N - \tilde{S}(\tilde{\theta}_N)$, so that we can build t-statistics that indicate which moments are better matched in the estimation.

In our application we find a nearly singular $\tilde{\Sigma}_{S,N}$. As shown in appendix A.6, asymptotic results require this matrix to be invertible. The near-singularity indicates that one statistic is nearly redundant, i.e., carries practically no additional information. The appendix describes a procedure for selecting the redundant statistic; it suggests that we drop the coefficient from the five year ahead excess return regression $\hat{c}^5$ from the estimation. In the empirical section below, the value of the regression coefficient implied by the estimated model is always such that the t-statistic for this moment remains below two. This happens despite the fact that information about $\hat{c}^5_2$ has not been used in the estimation.

### 5.2 Estimation Results

Table 2 reports estimation outcomes when assuming $\gamma = 5$. The second and third column in the table report the asset pricing moments from the data and the estimated standard deviation for each of these moments, respectively. Columns four and five then show the model moments and the t-statistics, respectively, when estimating the model using all asset pricing moments (except for $c^5_2$ which has been excluded for reasons explained in the previous section).

The estimated model reported in columns four and five of Table 2 quantitatively replicates the volatility of stock returns ($\sigma_{r^*}$), the large volatility and high persistence of the PD ratio ($\sigma_{PD}, \rho_{PD-1}$), as well as the excess return predictability evidence ($c^2_5, R^2_5$). This is a remarkable outcome given the assumed time-separable preference structure. The model has some difficulty
in replicating the mean stock return and dividend growth, but t-statistics for all other moments have an absolute value well below two, and more than half of the t-statistics are below one.

The last two columns in Table 2 report the estimation outcome when dropping the mean stock return \( E_{rs} \) from the estimation and restricting \( \delta \) to one, which tends to improve the ability of the model to match individual moments. All t-statistics are then close to or below two, including the t-statistics for the mean stock return and for \( c_2^2 \) that have not been used in the estimation, and the majority of the t-statistics is below one. This estimation outcome shows that the subjective beliefs model successfully matches individual moments with a relatively low degree of risk aversion. The model also delivers an equity premium of 1% per quarter, i.e., nearly half of the value observed in US data (2.1% per quarter).

<table>
<thead>
<tr>
<th></th>
<th>US data</th>
<th>Estimated model ((c_2^2 \text{ not included}))</th>
<th>Estimated model ((c_2^2, E_{rs} \text{ not included}))</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Data Moment</td>
<td>Std. dev.</td>
<td>Model moment ( \tilde{S}_i(\theta) )</td>
</tr>
<tr>
<td>Quarterly mean stock return ( E_{rs} )</td>
<td>2.25</td>
<td>0.34</td>
<td>1.27</td>
</tr>
<tr>
<td>Quarterly mean bond return ( E_{rb} )</td>
<td>0.15</td>
<td>0.19</td>
<td>0.39</td>
</tr>
<tr>
<td>Mean PD ratio ( E_{PD} )</td>
<td>123.91</td>
<td>21.36</td>
<td>122.50</td>
</tr>
<tr>
<td>Std.dev. stock return ( \sigma_{rs} )</td>
<td>11.44</td>
<td>2.71</td>
<td>10.85</td>
</tr>
<tr>
<td>Std.dev. PD ratio ( \sigma_{PD} )</td>
<td>62.43</td>
<td>17.60</td>
<td>67.55</td>
</tr>
<tr>
<td>Autocorr. PD ratio ( \rho_{PD,-1} )</td>
<td>0.97</td>
<td>0.01</td>
<td>0.95</td>
</tr>
<tr>
<td>Excess return reg. coefficient ( c_2^2 )</td>
<td>-0.0041</td>
<td>0.0014</td>
<td>-0.0066</td>
</tr>
<tr>
<td>( R^2 ) of excess return regression ( R_5^2 )</td>
<td>0.2102</td>
<td>0.0825</td>
<td>0.2132</td>
</tr>
<tr>
<td>Mean dividend growth ( E_{\Delta D/D} )</td>
<td>0.41</td>
<td>0.17</td>
<td>0.00</td>
</tr>
<tr>
<td>Std. dev. dividend growth ( \sigma_{\Delta D/D} )</td>
<td>2.88</td>
<td>0.82</td>
<td>2.37</td>
</tr>
<tr>
<td>Discount factor ( \delta_N )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Gain coefficient ( 1/\hat{\alpha}_N )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Test statistic ( \hat{W}_N )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>p-value of ( \hat{W}_N )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Table 2: Estimation outcome for \( \gamma = 5 \)**
The measure for the overall goodness of fit $\hat{W}_N$ and its p-value are reported in the last two rows of Table 2. The statistic is computed using all moments that are included in the estimation. The reported values of $\hat{W}_N$ are off the chart of the $\chi^2$ distribution, implying that the overall fit of the model is rejected, even if all moments are matched individually. This indicates that some of the joint deviations observed in the data are unlikely to happen given the observed second moments. It also shows that the overall goodness of fit test is considerably more stringent.

To show that the equity premium is indeed the source of the difficulty for passing the overall test, columns 4 and 5 in Table 3 reports results obtained when we repeat the estimation excluding the risk-free rate $E_{r^b}$ instead of the stock returns $E_{r^b}$ from the estimation. The estimation imposes the constraint $\hat{\delta}_N \leq 1$ since most economists believe that values above 1 are unacceptable. This constraint turns out to be binding. The t-statistics for the individual moments included in the estimation are then quite low, but the model fails to replicate the low value for the bond return $E_{r^b}$, which has not been used in the estimation. Despite larger t-statistics, the model now comfortably passes the overall goodness of fit test at the one percent level, as the p-value for the reported $\hat{W}_N = 12.87$ statistic is 2.5%. The last two columns in Table 3 repeat the estimation when imposing $\gamma = 3$ and $\hat{\delta}_N = 1$. The performance in terms of matching the moments is then very similar with $\gamma = 5$, but the p-value of the $\hat{W}_N$ statistic increases to 7.1%.

Figure 3 shows realizations of the time series outcomes for the PD ratio generated from simulating the estimated model from Table 3 with $\gamma = 5$, for the same number of quarters as numbers of observations in our data sample. The simulated time series display price booms and busts, similar to the ones displayed in figure 1 for the actual data, so that the model also passes an informal ‘eyeball test’.

The estimated gain coefficients in Tables 2 and 3 are fairly small. The estimate in Table 3 implies that agents’ risk-adjusted return expectations respond only 0.7% in the direction of the last observed forecast error, suggesting that the system of price beliefs in our model represents indeed only a small deviation from RE beliefs. Under strict RE the reaction to forecast

\[^{35}\text{The } \chi^2 \text{ distribution has 5 degrees of freedom for the estimations in Table 2, where the last two columns drop a moment, but also fix } \delta = 1. \text{ For the estimation in Table 3, we exclude } c_2^3 \text{ and } E_{r^b} \text{ from the estimation, but the constraint } \hat{\delta} \leq 1 \text{ is either binding or imposed, so that we also have 5 degrees of freedom. Similarly, we have five degrees of freedom for the estimation in Table 4.}\]
errors is zero, but the model then provides a very bad match with the data: it counterfactually implies \( \sigma_{r^*} \approx \sigma_{\Delta D/D}, \sigma_{PD} = 0, \) and \( R_5^2 = 0. \)

<table>
<thead>
<tr>
<th></th>
<th>US data</th>
<th>Estimated model ( \gamma = 5 ) (( c_2^5, E_{r^h} ) not included)</th>
<th>Estimated model ( \gamma = 3 ) (( c_2^5, E_{r^h} ) not included)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Data moment</td>
<td>Model moment, ( \hat{S}_i(\hat{\theta}) )</td>
<td>Model moment, ( \hat{S}_i(\hat{\theta}) )</td>
</tr>
<tr>
<td>Quarterly mean stock return ( E_{r^*} )</td>
<td>2.25</td>
<td>1.32</td>
<td>2.50</td>
</tr>
<tr>
<td>Quarterly mean bond return ( E_{r^h} )</td>
<td>0.15</td>
<td>1.09</td>
<td>-4.90</td>
</tr>
<tr>
<td>Mean PD ratio ( E_{PD} )</td>
<td>123.91</td>
<td>109.66</td>
<td>0.69</td>
</tr>
<tr>
<td>Std.dev. stock return ( \sigma_{r^*} )</td>
<td>11.44</td>
<td>5.34</td>
<td>2.25</td>
</tr>
<tr>
<td>Std.dev. PD ratio ( \sigma_{PD} )</td>
<td>62.43</td>
<td>40.09</td>
<td>1.33</td>
</tr>
<tr>
<td>Autocorrel. PD ratio ( \rho_{PD,-1} )</td>
<td>0.97</td>
<td>0.96</td>
<td>0.30</td>
</tr>
<tr>
<td>Excess return reg. coefficient ( c_5^2 )</td>
<td>-0.0041</td>
<td>-0.0050</td>
<td>0.64</td>
</tr>
<tr>
<td>R² of excess return regression ( R_5^2 )</td>
<td>0.2102</td>
<td>0.2282</td>
<td>-0.22</td>
</tr>
<tr>
<td>Mean dividend growth ( E_{\Delta D/D} )</td>
<td>0.41</td>
<td>0.22</td>
<td>1.14</td>
</tr>
<tr>
<td>Std. dev. dividend growth ( \sigma_{\Delta D/D} )</td>
<td>2.88</td>
<td>1.28</td>
<td>1.95</td>
</tr>
<tr>
<td>Discount factor ( \delta_N )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Gain coefficient ( 1/\hat{\alpha}_N )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Test statistic ( \hat{W}_N )</td>
<td>12.87</td>
<td></td>
<td></td>
</tr>
<tr>
<td>p-value of ( \hat{W}_N )</td>
<td></td>
<td>2.5%</td>
<td></td>
</tr>
</tbody>
</table>

**Table 3: Estimation outcome for \( \gamma = 5 \) and \( \gamma = 3 \)**

To further examine what it takes to match the risk premium and to compare more carefully our results with the performance of other models in the literature, we now assume a high degree of risk aversion of \( \gamma = 80, \) in line with the steady state degree of risk aversion assumed in Campbell and Cochrane (1999). Furthermore, we use all asset pricing moments listed in equation (28) for estimation, except for \( c_5^2. \) The estimation results are reported in Table 4. The learning model then successfully replicates all moments in the data, including the risk-premium: all the t-statistics for the individual moments are below two in absolute value, with most of them even assuming values below one. For sufficiently high risk-aversion we thus match all individual moments, so that the model performance is comparable to that of Campbell
and Cochrane (1999), but achieved with a time-separable preference specification. Yet, the p-value for the test statistic $\hat{W}_N$ in Table 4 is again off the charts, implying that the model fails the overall goodness of fit test. This highlights that the $\hat{W}_N$ test statistic is a much stricter test than imposed by matching moments individually.

<table>
<thead>
<tr>
<th></th>
<th>US data</th>
<th>Estimated model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Data moment</td>
<td>Model moment (c² not included)</td>
</tr>
<tr>
<td>Quarterly mean stock return $E_{r_s}$</td>
<td>2.25</td>
<td>2.11</td>
</tr>
<tr>
<td>Quarterly mean bond return $E_{r_b}$</td>
<td>0.15</td>
<td>0.11</td>
</tr>
<tr>
<td>Mean PD ratio $E_{PD}$</td>
<td>123.91</td>
<td>115.75</td>
</tr>
<tr>
<td>Std.dev. stock return $\sigma_{r_s}$</td>
<td>11.44</td>
<td>16.31</td>
</tr>
<tr>
<td>Std.dev. PD ratio $\sigma_{PD}$</td>
<td>62.43</td>
<td>71.15</td>
</tr>
<tr>
<td>Autocorrel. PD ratio $\rho_{PD,-1}$</td>
<td>0.97</td>
<td>0.95</td>
</tr>
<tr>
<td>Excess return reg. coefficient $c_2^2$</td>
<td>-0.0041</td>
<td>-0.0061</td>
</tr>
<tr>
<td>R² of excess return regression $R_2^2$</td>
<td>0.2102</td>
<td>0.2523</td>
</tr>
<tr>
<td>Mean dividend growth $E_{\Delta D/D}$</td>
<td>0.41</td>
<td>0.19</td>
</tr>
<tr>
<td>Std. dev. dividend growth $\sigma_{\Delta D/D}$</td>
<td>2.88</td>
<td>4.41</td>
</tr>
<tr>
<td>Discount factor $\hat{\delta}_N$</td>
<td></td>
<td>0.998</td>
</tr>
<tr>
<td>Gain coefficient $1/\hat{\alpha}_N$</td>
<td></td>
<td>0.0021</td>
</tr>
<tr>
<td>Test statistic $\hat{W}_N$</td>
<td></td>
<td>28.8</td>
</tr>
<tr>
<td>p-value of $\hat{W}_N$</td>
<td></td>
<td>0.0%</td>
</tr>
</tbody>
</table>

Table 4: Estimation outcome for $\gamma = 80$

Interestingly, the learning model gives rise to a significantly larger risk premium than its RE counterpart.\textsuperscript{36} For the estimated parameter values in Table 4, the quarterly real risk premium under RE is less than 0.5%, which falls short of the 2.0% emerging in the model with learning.\textsuperscript{37} Surprisingly, the model generates a small positive ex-post risk premium for stocks even

\textsuperscript{36}The RE counterpart is the model with the same parameterization, except for $1/\alpha = 0$.

\textsuperscript{37}The learning model and the RE model imply the same risk free rate, as we assumed that agents have objective beliefs about the aggregate consumption and dividend process.
Figure 3: Simulated PD ratio, estimated model from Table 3 ($\gamma = 5$)
when investors are risk neutral \((\gamma = 0)\). This may be surprising, since we did not introduce any feature in the model to generate a risk premium. To understand why this occurs, note that the realized gross stock return between period 0 and period \(N\) can be written as the product of three terms

\[
\prod_{t=1}^{N} \frac{P_t + D_t}{P_{t-1}} = \prod_{t=1}^{N} \frac{D_t}{D_{t-1}} \cdot \left( \frac{PD_N + 1}{PD_0} \right) \cdot \prod_{t=1}^{N-1} \frac{PD_t + 1}{PD_t}.
\]

The first term \((R_1)\) is independent of the way prices are formed, thus cannot contribute to explaining the emergence of an equity premium in the model with learning. The second term \((R_2)\), which is the ratio of the terminal over the starting value of the PD ratio could potentially generate an equity premium but is on average below one in our simulations of the learning model, while it is slightly larger than one under RE.\(^{38}\) The equity premium in the learning model must thus be due to the last component \((R_3)\). This term is convex in the PD ratio, so that a model that generates higher volatility of the PD ratio (but the same mean value) will also give rise to a higher equity premium. Therefore, because our learning model generates a considerably more volatile \(PD\) ratio, it also gives rise to a larger ex-post risk premium.

### 6 Robustness of Results

This section discusses the robustness of our findings with regard to different learning specifications and parameter choices (section 6.1), briefly explains how the present theory could be applied to price other assets (section 6.2), analyzes in detail the extent to which agents' forecasts could be rejected by the data or the equilibrium outcomes of the model (section 6.3), and finally offers a discussion of the rationality of agents' expectations about their own future choices (section 6.4).

#### 6.1 Different Parameters and Learning Specifications

We explored the robustness of the model along a number of dimensions. Performance turns out to be robust as long as agents are learning in some way about price growth using past price growth observations. For example,

\(^{38}\)For the learning model we choose the RE PD ratio as our starting value.
Adam, Beutel and Marcet (2013) use a model where agents learn directly about price growth (without risk-adjustment) using observations of past price growth; they document a very similar quantitative performance. Adam and Marcet (2010) considered learning about returns using past observations of returns, showing how this leads to asset price booms and busts. Furthermore, within the setting analyzed in the present paper, results are robust to relaxing Assumption 2. For example, the asset pricing moments are virtually unchanged when considering agents who also learn about risk-adjusted dividend growth, using the same weight $1/\alpha$ for the learning mechanism than for risk-adjusted price growth rates. Indeed, given the estimated gain parameter, adding also learning about risk-adjusted dividend growth contributes close nothing to replicating stock price volatility. We also explored a model of learning about risk-adjusted price growth that switches between ordinary least squares learning and constant gain learning, as in Marcet and Nicolini (2003). Again model performance turns out to be robust. Taken together, these findings suggests that the model continues to deliver an empirically appealing fit, as long as expected capital gains are positively affected by past observations of capital gains.

The model fails to deliver a good fit with the data, if one assumes that agents learn only about the relationship between prices and dividends, say about the coefficient in front of $D_t$ in the RE pricing equation (10), using the past observed relationship between prices and dividends, see Timmermann (1996). Stock price volatility then drops significantly below that observed in the data, illustrating that the asset pricing results are sensitive to the kind of learning introduced in the model. Our finding is that introducing uncertainty about the growth rate of prices is key for understand asset price volatility.

Similarly, for lower degrees of relative risk aversion around two, we find that the model continues to generate substantial volatility in stock prices, but not enough to quantitatively match the data.

At the same time, it is not difficult to obtain an even better fit than the one reported in section 5.2. For example, we imposed the restriction $\hat{\delta}_N \leq 1$ in the estimations reported in Table 3. Yet, in a setting with output growth and uncertainty, values above one are easily compatible with a well-defined model and positive real interest rates. Re-estimating Table 3 for $\gamma = 5$ without imposing the restriction on the discount factor, one obtains $\hat{\delta}_N = 1.0094$ and a p-value of 4.3% for the overall fit, instead of the 2.5% reported. The fit could similarly be improved by changing the parameters of the projection facility. Choosing $(\beta^L, \beta^U) = (200, 400)$ for the estimation
in Table 3 with $\gamma = 5$ instead of the baseline values $(\beta^L, \beta^U) = (250, 500)$ raises the p-value from 2.5% to 3.1%. \(^{39}\)

6.2 Pricing Other Assets

A natural question arising within our setting is to what extent the theory could be used to price also other assets, say the term structure of interest rates or the cross-section of stock returns. Since Assumption 1 implies that agents’ stochastic discount factor is largely identical to that emerging in the RE version of the model, the theory will generate pricing implications that differ from a RE model only to the extent that agents entertain also subjective beliefs about the future prices of these other assets, say the future spot prices of long-dated bonds. This suggests that by incorporating subjective beliefs about the intercept, slope and curvature components of the yield curve, the theory is likely to deliver interesting implications for bond price dynamics. Yet, even with objective bond price beliefs, the present theory can generate equilibrium bond prices that differ from RE equilibrium prices, provided one is not imposing Assumption 1. The presence of subjective stock price beliefs then gives rise to subjective consumption plans that differ from those implied in the RE equilibrium. These subjective consumption plans alter the pricing implications for other assets, even when assuming objective beliefs about the prices of these assets. Exploring these pricing implications appears to be an interesting avenue for further research.

6.3 Testing for the Rationality of Price Expectations

We have already discussed two ways in which the beliefs of agents constitute a small deviation from RE (see section 4.3). We now document to what extent agents make easily identifiable mistakes in their forecasts. We thereby focus on forecasts of variables that agents take as exogenous in their decision problem, i.e., $(P_t, Y_t, D_t)$.

In a first step, we check whether agents’ system of beliefs $\mathcal{P}$ is consistent with the behavior of actual data. Obviously, if agents’ belief system would be easily rejected by actual data we could not reasonably expect agents to hold such beliefs in the real world. In a second step, we explore in detail to what extent the belief system is consistent with the outcomes generated

\(^{39}\)Choosing $(\beta^L, \beta^U) = (300, 600)$ causes the p-value to decrease to 1.8%.
by the model. Since Assumption 2 in section 4.2 is consistent with agents knowing the objective processes for income and dividends, we concentrate on the question to what extent agents are making systematic mistakes in forecasting risk-adjusted price growth.

To this end we construct a test that exhausts all empirical implications of the subjective system of price beliefs (25). We then apply this test to actual data and to simulated data from the model. The test can be constructed by noticing that the price beliefs (25) imply that the first difference of risk-adjusted price growth has an MA(1) representation

$$\Delta \left( \frac{C_t}{C_{t-1}} \right)^{-\gamma} \frac{P_t}{P_{t-1}} = e_t,$$

(31)

where $e_t = \varepsilon_t - \varepsilon_{t-1} + \xi_t$ and $\Delta$ denotes the difference operator. Conversely, if $\Delta \left( \frac{C_t}{C_{t-1}} \right)^{-\gamma} \frac{P_t}{P_{t-1}}$ is normal and has an MA(1) representation, then it is possible to construct errors $\varepsilon$ and $\xi$ that are normal and independent such that $e_t = \varepsilon_t - \varepsilon_{t-1} + \xi_t$ holds, so that (25) holds. As a result, equation (31) exhausts the testable implications of agents’ price beliefs (25).

One might be tempted to test (31) using an augmented Dickey-Fuller (ADF) test, which involves running a regression with a certain number of lags and test if the residual is serially correlated. But this approach is problematic in our application: since the gain is small in the estimates in Tables 2 - 4 then $\sigma_{\xi}^2$ is small and the MA representation of $e_t$ has a near unit root. In this case the true autoregressive representation of $\Delta \left( \frac{C_t}{C_{t-1}} \right)^{-\gamma} \frac{P_t}{P_{t-1}}$ has coefficients that decay very slowly with the lag length. Then, the ADF test does not work because if we introduce only a few lags in the regression the error would be serially correlated and the test would be asymptotically invalid. Introducing many lags instead would cause the test to have little power for reasonable sample lengths.

A more attractive approach for testing (31) consists of observing that the MA(1) structure of $e_t$ implies that for any vector of variables $x_{t-2} \text{ dated } t - 2$

\text{Bray and Savin (1986) previously tested forecast rationality in a model of least squares learning using the Durbin and Watson test.}

\text{This can be seen by considering the limit case with } \sigma_{\xi} = 0 \text{ where } e_t = \varepsilon_t - \varepsilon_{t-1}, \text{ so that the AR coefficients fail to decay at all.}
or earlier, the regression

\[
\Delta \left( \frac{C_t}{C_{t-1}} \right)^{-\gamma} \frac{P_t}{P_{t-1}} = a' x_{t-2} + u_t
\]

(32)

for \( E(x_{t-2}u_t) = 0 \) should deliver \( a = 0 \). This provides directly testable implication of the belief system for a given choice of stationary regressors \( x \).\(^{42}\)

In particular, one can construct the test statistic

\[
\hat{Q}_T \equiv T \left( \frac{1}{T} \sum_{t=0}^{T} x_{t-2}u_t(0) \right) ' \tilde{S}_w^{-1} \left( \frac{1}{T} \sum_{t=0}^{T} x_{t-2}u_t(0) \right) \rightarrow \chi^2_n,
\]

where convergence is in distribution as \( T \to \infty \), \( n \) denotes the dimension of \( x_t \), \( S_w = \sum_{i=1}^{n} E_t (x_{t+i}u_{t+i}x'_tu_t) \), and \( u(0) \) denotes the residual under the null \( H_0 : a = 0 \), which is equal to \( \Delta \left( \frac{C_t}{C_{t-1}} \right)^{-\gamma} \frac{P_t}{P_{t-1}} \). This test is an off-the-shelf application of a differences in differences test proposed by Arellano and Bond (1991) in the panel data context.\(^{43}\) Unlike ADF, the test is asymptotically valid under the null for any stationary \( x_t \) and any fixed dimension \( n \).

We test the null hypothesis below including in \( x_{t-2} \) a constant, as well as the variables \( \frac{D_{t-2}}{D_{t-3}} \), \( \frac{P_{t-2}}{P_{t-3}} \), \( \Delta \left( \frac{C_{t-2}}{C_{t-3}} \right)^{-\gamma} \frac{P_{t-2}}{P_{t-3}} \), \( \frac{P_{t-2}}{P_{t-3}} \), or \( \frac{P_{t-2}}{P_{t-3}} \) and their lags.\(^{44}\)

Table 5 reports the test statistic for the actual data. For each variable reported in the table, the test includes three lags of the variable and a constant (\( n = 4 \)). To compute risk-adjusted consumption growth in the data we assume \( \gamma = 5 \).\(^{45}\) The 5% critical value of the test statistic is 9.48, so that all the tests in Table 5 are accepted with a wide margin at this significance.

\(^{42}\)This test has previously been employed in section 6.2 in Adam, Kuang and Marce (2011).

\(^{43}\)In our setting differencing is useful to remove the random walk that is present under the agents’ null hypothesis, while in the panel context it is used to remove fixed effects, but the test statistic is the same. Furthermore, the approach of directly testing \( E(x_{t-2}u_t(0)) = 0 \) is more efficient than using first an inefficient estimator for \( a \), say OLS, and then testing in a second step for whether the null hypothesis holds based on this estimate.

\(^{44}\)The last two regressors fail to be stationary according to the agents beliefs \( \mathcal{P} \), so that an agent with beliefs \( \mathcal{P}' \) would modify the p-values to take into account non-stationarity. We have experimented with this and found small differences with the p-values from stationary \( x_t \).

\(^{45}\)We use the consumption data provided by Campbell and Cochrane (1999), which is available for the period 1947-1994.
level. This shows that agents’ belief system is a reasonable one to hold, given the way the data actually behaved.

<table>
<thead>
<tr>
<th>Regressor</th>
<th>$\hat{Q}_T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{P_{t-2}}{P_{t-3}}$</td>
<td>6.69</td>
</tr>
<tr>
<td>$\Delta \frac{P_{t-2}}{P_{t-3}}$</td>
<td>6.66</td>
</tr>
<tr>
<td>$\Delta \left( \frac{C_{t-2}}{C_{t-3}} \right)^{-\gamma} \frac{P_{t-2}}{P_{t-3}}$</td>
<td>6.97</td>
</tr>
<tr>
<td>$\frac{P_{t-2}}{P_{t-3}}$</td>
<td>6.33</td>
</tr>
<tr>
<td>$\frac{P_{t-2}}{P_{t-3}}$</td>
<td>4.68</td>
</tr>
</tbody>
</table>

Table 5: Empirical test of $\mathcal{P}$

Tables 6 and 7 report the rejection frequencies using simulated model data. The tables report the likelihood of rejecting the null at the 5% significance level, using the point estimates from Table 3 for $\gamma = 5$ and the point estimates from Table 4, respectively, when simulating the model.\(^{46}\)

Rejection frequencies are shown for different sample lengths comprising 15 years, 25 years, and 50 years of simulated quarterly data and using different numbers of lags in the test. Obviously, one cannot expect that this test is never rejected, even the correct model would be rejected due to Type I errors, i.e., about 5% of the times. One thus has to evaluate agents’ subjective belief model by checking whether the rejection frequencies exceed the 5% significance level.

Table 6 reports the rejection frequencies for the regressors that are stationary according to the agents’ beliefs $\mathcal{P}$. For 15 to 25 years of quarterly observations the rejection frequencies are at least as many times below the 5% as they are above this level. With 50 years of quarterly observations the rejection rates are higher for the dividend growth regressors for the model from Table 3 with $\gamma = 5$, and for risk-adjusted price growth for the model from Table 4. This indicates that these variables may help improving agents’ forecasts.

\(^{46}\)The rejection frequencies are obtained from simulating 1000 random samples of the specified length.
Table 6: Forecast rationality test consistent with $P$

Table 7 reports the rejection frequencies for the regressors $P_{t-2}/P_{t-3}$ and $P_{t-2}/D_{t-2}$. Table 7 shows that with regard to these additional variables there is a tendency to reject the null more often than 5%, when considering the model estimates from Table 3, but rejection frequencies are approximately in line with the significance level for the estimates from Table 4.

Overall, we conclude that it will not be easy for agents to reject their beliefs. While some tests reject too often relative to the significance level, others reject too little. Obviously, diagnosing a rejection may cause agents to reformulate their forecasting model, possibly by including additional regressors in the belief system (25). Investigating the implications of such belief changes is certainly of interest, but we leave this issue to future research.

Table 7: Forecast rationality test of an outside observer
6.4 Subjective versus Objective Plans

This section discusses to what extent agents’ expectations about their own future consumption and stockholding choices coincide with the objective expectations of future choices.

It is important to note that agents hold the correct perception regarding their own choices *conditional* on the realizations of the future values of the variables $P$, $Y$ and $D$. This is the case because agents choose an optimal plan $(C_t^i, S_t^i, B_t^i)$ satisfying (5) and take decisions according to this plan, so that $E^P [C_t^i | \omega'] = E [C_t^i | \omega']$ for all $t$ and $\omega'$. Nevertheless, the fact that agents hold expectations about $\omega'$ that are not exactly equal to those realized within the model means that expectations about $(C_t^i, S_t^i, B_t^i)$ that condition on less information may differ from the true expectations given such a reduced information set. This fact highlights that discrepancies between agents’ subjective expectations about own actions and objective expectations about these actions indicates the presence of subjective beliefs about contingencies, i.e., prices, as explored in the previous section.

We first show that the gap between subjective and objective consumption growth expectations is approximately zero. This gap can be expressed as\(^{47}\)

\[
E_t^P \left[ \frac{C_{t+1}^i}{C_t} \right] - E_t \left[ \frac{C_{t+1}^i}{C_t} \right] = E_t^P \left[ \frac{P_{t+1}(1 - S_{t+1}^i) - B_{t+1}^i}{D_t + Y_t} \right] - E_t \left[ \frac{C_{t+1}^i}{C_t} \right]
\]

Since the choices for $S_{t+1}^i$ and $B_{t+1}^i$ are bounded, we have that $P_{t+1}(1 - S_{t+1}^i) - B_{t+1}^i$ is bounded whenever beliefs about future values of the price $P_{t+1}$ are bounded. Assumption 1 in section 4.2 then insures that the subjective expectations in the last line of the preceding equation will approximately be equal to zero for every state, so that the gap between the subjective and the objective consumption expectations vanishes under the maintained assumptions.

\(^{47}\)The first equality uses the fact that under the objective beliefs $E \left[ C_{t+1}^i \right] = E \left[ C_t \right]$ in equilibrium, the second equality the investor’s budget constraint and the fact that $S_t^i = 1$ and $B_t^i = 0$ in equilibrium in $t$, and the last equality the fact agents have rational expectations about the processes for aggregate consumption.
The situation is different when considering subjective stockholding plans. Assuming an interior solution for stockholdings in period $t$, the agent’s first order condition satisfies in equilibrium

$$P_t = \delta E_t^P \left( \left( \frac{C_{t+1}^i}{C_t} \right)^{-\gamma} (P_{t+1} + D_{t+1}) \right).$$

With $C_{t+1}^i/C_t$ converging to $C_{t+1}/C_t$ under Assumption 1, this equation recovers our pricing equation. Yet, if this equation were to hold almost surely for each period and each contingency in the future under the agents’ subjective plan, then one could iterate forward on this equation and obtain (under a suitable transversality condition) that the equilibrium price must equal the present value of dividends. Since the equilibrium price is different, it must be the case that the agent expects that with positive probability either $S_t^i = \bar{S}$ or $S_t^i = \tilde{S}$ will hold in the future. These expectations, however, will not be fulfilled in equilibrium, as the agent will never buy or sell, as we have $S_t^i = 1$ for all $t$ along the equilibrium path. It would be relatively easy to remove this stark difference between agents’ expectations and the equilibrium outcomes by introducing agent heterogeneity in terms of preferences or beliefs. Agents would then occasionally hit their constraints in equilibrium, albeit they would presumably still hold inexact beliefs about the likelihood of such events occurring.

7 Conclusions and Outlook

A simple consumption-based asset pricing model is able to quantitatively replicate a number of important asset pricing facts, provided one slightly relaxes the assumption that agents perfectly know how stock prices are formed in the market. We assume that agents are internally rational, in the sense that they formulate their doubts about market outcomes using a consistent set of subjective beliefs about prices and maximize expected utility given this set of beliefs. The system of beliefs is internally consistent in the sense that it specifies a proper joint distribution of prices and fundamental shocks at all dates. Furthermore, the perceived distribution of price behavior, while different from the true distribution, is nevertheless close to it and the discrepancies are hard to detect.

In such a setting, optimal behavior dictates that agents learn about the equilibrium price process from past price observations. This gives rise to a
self-referential model of learning about prices that imparts momentum and mean reversion behavior into the price dividend ratio. As a result, sustained departures of asset prices from their fundamental (RE) value emerge, even though all agents act rationally in the light of their beliefs.

We submit our consumption based asset pricing model also to a formal econometric test based on the method of simulated moments. The model performs remarkably well, despite its simplicity. While the model gives rise to a significant equity premium, it fails to match the empirical premium found for reasonable degrees of relative risk aversion. When risk aversion is as high as in some of the previous work, the model can also replicate the equity premium, but we leave a full treatment of this issue to future research.

Given the difficulties documented in the empirical asset pricing literature in accounting for stock price volatility in a setting with time-separable preferences and RE, our results suggest that learning about price behavior may be a crucial ingredient to understanding stock price volatility. Indeed, the most convincing case for models of learning can be made by explaining facts that appear ‘puzzling’ from the RE viewpoint, as we attempt to do in this paper.

The finding that large asset price fluctuations can result from optimizing agents with subjective beliefs is also relevant from a policy perspective. The desirability of policy responding to asset price fluctuations will depend to a large extent on whether or not asset price fluctuations are fundamentally justified.
References


A  Appendix

A.1  Data Sources


In the estimation part of the paper we use moments that are based on the same number of observations as we have data points. Since we seek to match the return predictability evidence at the five year horizon (c₅ and R₅) we can only use data points up to 2007:1. For consistency the effective sample end for all other moments reported in Table 1 has been shortened by five years to 2007:1. In addition, due to the seasonal adjustment procedure for dividends described below and the way we compute the standard errors for the moments described in appendix A.6, the effective starting date was 1927:2. The names of the data series used are reported below.

To obtain real values, nominal variables have been deflated using the ‘USA BLS Consumer Price Index’ (Global Fin code ‘CPUSAM’). The monthly price series has been transformed into a quarterly series by taking the index value of the last month of the considered quarter.

The nominal stock price series is the ‘SP 500 Composite Price Index (w/GFD extension)’ (Global Fin code ‘SPXD’). The weekly (up to the end of 1927) and daily series has been transformed into quarterly data by taking the index value of the last week/day of the considered quarter. Moreover, the series has been normalized to 100 in 1925:4.

As nominal interest rate we use the ‘90 Days T-Bills Secondary Market’ (Global Fin code ‘ITUSA3SD’). The monthly (up to the end of 1933), weekly (1934-end of 1953), and daily series has been transformed into a quarterly series using the interest rate corresponding to the last month/week/day of the considered quarter and is expressed in quarterly rates, i.e., not annualized.

Nominal dividends have been computed as follows

\[ D_t = \left( \frac{I^D(t)/I^D(t-1)}{I^{ND}(t)/I^{ND}(t-1)} - 1 \right) I^{ND}(t) \]

where \( I^{ND} \) denotes the ‘SP 500 Composite Price Index (w/GFD extension)’ described above and \( I^D \) is the ‘SP 500 Total Return Index (w/GFD exten-

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sion)’ (Global Fin code ‘SPXTRD’). We first computed monthly dividends and then quarterly dividends by adding up the monthly series. Following Campbell (2003), dividends have been deseasonalized by taking averages of the actual dividend payments over the current and preceding three quarters.

A.2 Details on the phase diagram

The second order difference equation (23) describes the evolution of beliefs over time and allows to construct the directional dynamics in the \((\beta_t, \beta_{t-1})\) plane, as shown in Figure 2 for the case \((\varepsilon^c_t)^{-\gamma} \varepsilon^d_t = 1\). Here we show the algebra leading to the arrows displayed in this figure as well the effects of realizations \((\varepsilon^c_t)^{-\gamma} \varepsilon^d_t \lesssim 1\). Define \(x'_t \equiv (x_{1,t}, x_{2,t}) \equiv (\beta_t, \beta_{t-1})\). The dynamics can then be described by

\[
x_{t+1} = \left( x_{1,t} + f_{t+1} \left( \frac{a^{1-\gamma} + \frac{a^{1-\gamma}(x_{1,t}-x_{2,t})}{1-\delta x_{1,t}}}{x_{1,t}} \right) (\varepsilon^c_t)^{-\gamma} \varepsilon^d_t - x_{1,t}, x_{1,t} \right)
\]

The points in Figure 2 where there is no change in each of the elements of \(x\) are the following: we have \(\Delta x_2 = 0\) at points \(x_1 = x_2\), so that the 45° line gives the point of no change in \(x_2\), and \(\Delta x_2 < 0\) above this line. We have \(\Delta x_1 = 0\) for \(x_2 = \frac{1}{\delta} - \frac{x_1(1-\beta x_{11})}{1-\delta x_{1,t}}\). For \((\varepsilon^c)^{-\gamma} \varepsilon^d = \rho_x\) this is the curve labelled "\(\beta_{t+1} = \beta_t\)" in Figure 2, and we have \(\Delta x_1 > 0\) below this curve. So for \((\varepsilon^c)^{-\gamma} \varepsilon^d = \rho_x\), the zeroes for \(\Delta x_1\) and \(\Delta x_2\) intersect are at \(x_1 = x_2 = a^{1-\gamma} \rho_x\), which is the REE value and also at \(x_1 = x_2 = \delta^{-1}\), which is the limit of rational bubble equilibria. These results give rise to the directional dynamics shown in figure 2. Finally, for \((\varepsilon^c)^{-\gamma} \varepsilon^d > \rho_x\) \((\varepsilon^c)^{-\gamma} \varepsilon^d < \rho_x\) the curve "\(\beta_{t+1} = \beta_t\)" in Figure 2 is shifted upwards (downwards) as indicated by the function \(x_2 = \frac{1}{\delta} - \frac{x_1(1-\delta x_{11})}{a^{1-\gamma}(\varepsilon^c)^{-\gamma} \varepsilon^d} \).

A.3 Proof of mean reversion

To prove mean reversion for the general learning scheme (18) we need the following additional technical assumption on the updating function \(f_t\):

**Assumption A1** There is a \(\eta > 0\) such that \(f_t(\cdot, \beta)\) is differentiable in the interval \((-\eta, \eta)\) for all \(t\) and all \(\beta\).
Furthermore, letting

\[ D_t \equiv \inf_{\Delta \in (-\eta, \eta), \beta \in (0, \beta^U)} \frac{\partial f_t(\Delta, \beta)}{\partial \Delta}, \]

we have

\[ \sum_{t=0}^{\infty} D_t = \infty \]

This is satisfied by all the updating rules considered in this paper and by most algorithms used in the stochastic control literature. For example, it is guaranteed in the OLS case where \( D_t = 1/(t + \alpha_1) \) and in the constant gain where \( D_t = 1/\alpha \) for all \( t, \beta \). The assumption would fail and \( \sum D_t < \infty \), for example, if the weight given to the error in the updating scheme is \( 1/t^2 \). In that case beliefs could get ‘stuck’ away from the fundamental value simply because updating of beliefs ceases to incorporate new information for \( t \) large enough. In this case, the growth rate would be a certain constant but agents would forever believe that the growth rate is another constant, different from the truth. Hence in this case agents would make systematic mistakes forever. Therefore, Assumption A1 is likely to be satisfied by any system of beliefs that puts a "grain of truth" on the RE equilibrium.

The statement about limsup is equivalent to saying that if \( \beta_t > a \) in some period \( t \), then for any \( \delta > 0 \) sufficiently small, there is a finite period \( t'' > t \) such that \( \beta_{t''} < a + \delta \).

Fix \( \delta > 0 \) such that \( \delta < \min(\eta, (\beta_t - a)/2) \) where \( \eta \) is as in Assumption A1.

We first prove that there exists a finite \( t' \geq t \) such that

\[ \Delta \beta_{t''} \geq 0 \quad \text{for all } \tilde{t} \text{ such that } t < \tilde{t} < t', \quad \text{and} \]
\[ \Delta \beta_{t'} < 0 \quad \text{(33)} \]
\[ \Delta \beta_{t''} < 0 \quad \text{(34)} \]

To prove this, choose \( \delta = \eta \left( 1 - \delta \beta^U \right) \). Since \( \beta_t < \beta^U \) and \( \delta > 0 \) it is impossible that \( \Delta \beta_{\tilde{t}} \geq \epsilon \) for all \( \tilde{t} > t \). Let \( \tilde{t} \geq t \) to be the first period where \( \Delta \beta_{\tilde{t}} < \epsilon \).

There are two possible cases: either \( i) \Delta \beta_{\tilde{t}} < 0 \) or \( ii) \Delta \beta_{\tilde{t}} \geq 0 \).

In case \( i) \) we have that (33) and (34) hold if we take \( t' = \tilde{t} \).

In case \( ii) \) \( \beta_t \) can not decrease between \( t \) and \( \tilde{t} \) so that

\[ \beta_{\tilde{t}} \geq \beta_t > a + \eta \]
Furthermore, we have
\[
    T(\beta_t, \Delta \beta_t) = a + \frac{\Delta \beta_t}{1 - \delta \beta_t} < a + \frac{\epsilon}{1 - \delta \beta_t}
\]
where the first equality follows from the definition of \(T\) in the main text. The previous two relations imply
\[
    \beta_t > T(\beta_t, \Delta \beta_t)
\]
Therefore
\[
    \Delta \beta_{t+1} = f_{t+1} (T(\beta_t, \Delta \beta_t) - \beta_t) \beta_t < 0
\]
and in case ii) we have that (33) and (34) hold for \(t' = \bar{t} + 1\).
This shows that (33) and (34) hold for a finite \(t'\) as in the first part of the statement of Mean Reversion in the text. Now we need to show that beliefs eventually fall below \(a + \eta\) and do decrease monotonically.
Consider \(\eta\) as defined above. First, notice that given any \(j \geq 0\), if
\[
    \Delta \beta_{t+j} < 0 \quad \text{and} \quad \beta_{t+j} > a + \eta \tag{35}
\]
then
\[
    \Delta \beta_{t+j+1} = f_{t+j+1} \left( a + \frac{\Delta \beta_{t+j}}{1 - \delta \beta_{t+j}} \right) - \beta_{t+j}, \beta_{t+j} < f_{t+j+1} (a - \beta_{t+j}, \beta_{t+j}) \tag{37}
\]
\[
    < f_{t+j+1} (-\eta, \beta_{t+j}) \leq -\eta D_{t+j+1} \leq 0 \tag{38}
\]
where the first inequality follows from (35), the second inequality from (36) and the third from the mean value theorem, \(\eta > 0\) and \(D_{t+j+1} \geq 0\). Assume, towards a contradiction, that (36) holds for all \(j \geq 0\). Since (35) holds for \(j = 0\), it follows by induction that \(\Delta \beta_{t+j} \leq 0\) for all \(j \geq 0\) and, therefore, that (38) would hold for all \(j \geq 0\) hence
\[
    \beta_{t+j} = \sum_{i=1}^{j} \Delta \beta_{t+i} + \beta_t \leq -\eta \sum_{i=1}^{j} D_{t+i} + \beta_t
\]
for all \(j > 0\). Assumption A1 above would then imply \(\beta_t \to -\infty\) showing that (36) can not hold for all \(j\). Therefore there is a finite \(j\) such that \(\beta_{t+j}\) will go below \(a + \eta\) and \(\beta\) is decreasing from \(t'\) until it goes below \(a + \eta\).

For the case \(\beta_t < a - \eta\), choosing \(\epsilon = \eta\) one can use a symmetric argument to construct the proof.
A.4 Proof of Geometric Ergodicity

Defining $\eta_t \equiv (\varepsilon_t^c)^{-\gamma} \varepsilon_t^d$ and using (22) and (27) we can write the learning algorithm that gives the dynamics of $\beta_t$ as

$$
\begin{bmatrix}
\beta_t \\
\Delta \beta_t
\end{bmatrix} = F
\begin{bmatrix}
\beta_{t-1} \\
\Delta \beta_{t-1}
\end{bmatrix}, \eta_{t-1}
$$

where the first element of $F$, denoted $F_1$, is given by the right side of (27) and $F_2(\beta, \Delta \beta, \eta_{t-1}) \equiv F_1(\beta, \Delta \beta, \eta_{t-1}) - \beta$. Therefore,

$$
F'_t \equiv \frac{\partial F (\beta_t, \eta_{t-1})}{\partial \beta_t} = w'_t \cdot \begin{bmatrix}
A_t, & 1 - \frac{1}{\alpha} + B_t \\
A_t, & -\frac{1}{\alpha} + \frac{1}{\alpha} B_t
\end{bmatrix}
$$

for $A_t = \frac{1}{\alpha} \frac{\delta \eta_{t-1}}{1 - \delta \beta_{t-1}}$, $B_t = \frac{1}{\alpha} \frac{\delta \Delta \beta_{t-1} \eta_{t-1}}{(1 - \delta \beta_{t-1})^2}$, with $w'_t$ denoting the derivative of $w$ at period $t$. The eigenvalues of the matrix in brackets are

$$
\lambda^+_t, \lambda^-_t = \frac{A_t + 1 - \frac{1}{\alpha} + B_t \pm \sqrt{(A_t + 1 - \frac{1}{\alpha} + B_t)^2 - 4A_t}}{2}
$$

Since $A_t, B_t \to 0$ for large $\alpha$ we have that $\lambda^+_t$ is the larger eigenvalue in modulus and that the radicand is positive. We wish to find a uniform bound for $\lambda^+_t$, because given that $|w'_t| < 1$ this will be a uniform bound for the largest eigenvalue of $F'_t$. Such a bound will play the role of $\rho_\theta(\varepsilon_t)$ in the definition of the "L2 unit circle condition" on p. 942 in Duffie and Singleton (1993) (henceforth DS).

Consider the function $f_a(x) = x + a + \sqrt{(x + a)^2 - 4a}$ for some constant $a > 0$ and $x$ large enough for the radicand to be positive. For $\varepsilon > 0$ the mean value theorem implies

$$
f_a(x + \varepsilon) \leq \left(1 + \frac{x}{\sqrt{(x + a)^2 - 4a}}\right) \varepsilon + f_a(x)
$$

Evaluating this expression at $a = A_t$, $\varepsilon = B_t$ and $x = 1 - 1/\alpha$ we have

$$
\lambda^+_t \leq B_t + \frac{f_{A_t}(1 - \frac{1}{\alpha})}{2} = B_t + 1 - \frac{1}{\alpha} \quad \text{for } \Delta \beta_{t-1} \geq 0 \quad (39)
$$
where we used
\[ f_{A_t}(1 - \frac{1}{\alpha}) < A_t + 1 - \frac{1}{\alpha} + \sqrt{(A_t + 1 - \frac{1}{\alpha})^2 - 4A_t(1 - \frac{1}{\alpha})} = 2 \left( 1 - \frac{1}{\alpha} \right) \]

Since \( f_{A_t}(\cdot) \) is monotonic, using the expression for \( B_t \) we have
\[ \lambda_t^+ \leq \frac{1}{2} f_{A_t}(1 - \frac{1}{\alpha} + B_t) \leq \frac{f_{A_t}(1 - \frac{1}{\alpha})}{2} < 1 - \frac{1}{\alpha} \quad \text{for } \Delta \beta_{t-1} < 0 \quad (40) \]

From (27) we have
\[ \Delta \beta_t \leq \frac{1}{\alpha} \left( \frac{\eta_{t-1} a^{1-\gamma}}{\eta_{t-2}} \left( 1 + \frac{\Delta \beta_{t-1}}{1 - \delta \beta_{t-1}} \right) - \beta_{t-1} \right) \quad (41) \]

So, if \( \Delta \beta_{t-1} \geq 0 \), using \( \beta_{t-1} > 0 \)
\[ \Delta \beta_t \leq \frac{1}{\alpha} \eta_{t-1} a^{1-\gamma} \left( 1 + \frac{\Delta \beta_{t-1}}{1 - \delta \beta_{t-1}} \right) \]

Therefore, adding the right side of this inequality to (40), and using the inequality for (39) we have that for all \( \Delta \beta_{t-1} \)
\[ \lambda_t^+ \leq \frac{1}{\alpha} \frac{a^{1-\gamma} \delta}{(1 - \delta \beta_{t-1})^2} \left( \frac{a^{1-\gamma}}{\alpha} \eta_{t-2} \left( 1 + \frac{\Delta \beta_{t-2}}{1 - \delta \beta_{t-2}} \right) \right) \eta_{t-1} + 1 - \frac{1}{\alpha} \]
\[ \leq \frac{1}{\alpha^2} \tilde{K} \eta_{t-2} \eta_{t-1} + 1 - \frac{1}{\alpha} \]

for a constant \( 0 < \tilde{K} < \infty \), where we used \( |\Delta \beta_{t-1}|, \beta_{t-1}, \beta_{t-2} < \beta^U \).

Since \( w' \leq 1 \), it is clear from the mean value theorem that \( \frac{\tilde{K}}{\alpha^2} \eta_{t-2} \eta_{t-1} + 1 - \frac{1}{\alpha} \) plays the role of \( \rho_0(\varepsilon_t) \) in the Definition of "L^2 unit circle condition" of DS, where our \( \alpha \) plays the role of \( \theta \), and \( \eta_{t-1} \eta_{t-2} \) the role of \( \varepsilon_t \) in DS. Therefore, we need to check that \( E \left( \frac{\tilde{K}}{\alpha^2} \eta_t \eta_{t-1} + 1 - \frac{1}{\alpha} \right)^2 < 1 \) for \( \alpha \) large enough. A routine calculation shows that
\[ E \left( \frac{\tilde{K}}{\alpha^2} \eta_t \eta_{t-1} + 1 - \frac{1}{\alpha} \right)^2 = 1 - \frac{1}{\alpha} - \frac{1}{\alpha} \left[ 1 - \frac{1}{\alpha} - 2(1 - \frac{1}{\alpha}) \frac{\tilde{K}}{\alpha^2} E \left( \eta_t \eta_{t-1} \right) - \frac{\tilde{K}^2}{\alpha^3} E \left( \eta_t^2 \eta_{t-1} \right) \right] \]

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which is smaller than one for large enough \( \alpha \).

This proves that for large \( \alpha \) the variable \( \beta_t \) satisfies DS’ \( L^2 \) unit-circle condition, hence it satisfies DS’ AUC condition, and Lemma 3 in DS guarantees that \( \beta_t \) is geometrically ergodic.

Now, adding \( a^{1-\gamma} \eta_{t-1} \) on both sides of (41) and taking expectations at the ergodic distribution we have

\[
E \left( \beta_{t-1} - \eta_{t-1}a^{1-\gamma} \right) \leq E \left( \frac{\Delta \beta_{t-1}}{1 - \delta \beta_{t-1}} \eta_{t-1}a^{1-\gamma} \right)
\]

(42)

Our previous argument shows that the right side is arbitrarily small for \( \alpha \) large, therefore \( E\beta_{t-1} \leq E\eta_{t-1}a^{1-\gamma} \). A similar argument shows that \( \text{var} \beta_t \) goes to zero as \( \alpha \to \infty \). Therefore for \( \alpha \) large \( \beta_t \leq \beta^L \) with arbitrarily large probability so that (41) holds as equality with arbitrarily large probability. Taking expectations on both sides for the realizations where this holds as equality, we have that \( E\beta_t \to E\eta_{t-1}a^{1-\gamma} = \beta^{RE} \) as \( \alpha \to \infty \) which completes the proof.

A.5 Differentiable projection facility

The function \( w \) used in the differentiable projection facility is

\[
w(x) = U \left\{ \begin{array}{ll}
\beta^L & \text{if } x \leq \beta^L \\
\beta^L + \frac{x-\beta^L}{x+\beta^U-2\beta^L}(\beta^U - \beta^L) & \text{if } \beta^L < x \leq \beta^U \\
\beta^L + \frac{1}{2}(\beta^U - \beta^L) & \text{if } x \geq \beta^U
\end{array} \right.
\]

(43)

Clearly \( w \) is continuous, the only point where continuity is questionable is at \( x = \beta^L \), but it is easy to check that

\[
\lim_{x \to \beta^L} w(x) = \lim_{x \to \beta^L} w(x) = \beta^L
\]

\[
\lim_{x \to \beta^L} w'(x) = \lim_{x \to \beta^U} w'(x) = 1
\]

\[
\lim_{x \to \infty} w(x) = \beta^U
\]

In our numerical applications we choose \( \beta^U \) so that the implied PD ratio never exceeds \( U^{PD} = 500 \) and \( \beta^L = \delta^{-1} - 2(\delta^{-1} - \beta^U) \), which implies that the dampening effect of the projection facility starts to come into effect for
values of the PD ratio above 250. Therefore, this dampening is applied in few observations. While the projection facility might suggest that profitable trading rules could be devised, this is true only if one assumes that the parameters $\beta^U$ and $\beta^L$ are fixed and unchanging over time, as we do here for simplicity. In a slightly more realistic model it would be difficult for agents to time stock purchases and stock sales to exploit the projection facility.

### A.6 Details of the MSM procedure

The estimation method and the proofs adapt results from standard MSM estimation. Online appendix B to this paper contains a much more detailed account of these results. We use the definitions introduced at the beginning of section 5.

Let $N$ be the sample size, $(y_1, ..., y_N)$ the observed data sample, with $y_t$ containing $m$ variables. Define sample moments $\tilde{M}_N \equiv \frac{1}{N} \sum_{t=1}^{N} h(y_t)$ for a given moment function $h : R^m \rightarrow R^n$. Sample statistics $\tilde{S}_N$ shown in (28) are not exactly moments of the data, for example $\tilde{M}_N$ contains the sample moments $\tilde{\alpha}_r(P_t/D_t)$ and $\tilde{\alpha}_{0}(P_t/D_t, P_{t-1}/D_{t-1})$, while $\tilde{S}_N$ contains the serial correlation of $P_t/D_t$ which is a function of these moments. The sample statistics can be written as $\tilde{S}_N \equiv S(\tilde{M}_N)$ for a statistics function $S : R^n \rightarrow R^s$. The $h$, $S$ mappings in our application are written explicitly in the online appendix B.

Let $y_t(\theta)$ be the series generated by our model of learning for parameter values $\theta$ and some realization of the underlying shocks. Denote the true parameter value $\theta_0$. Let $M(\theta) \equiv E \left[h(y_t(\theta))\right]$ be the moments for parameter values $\theta$ at the stationary distribution of $y_t$, let $M_0 \equiv M(\theta_0)$ be the true moments, and $\tilde{S}(\theta) \equiv S(M(\theta))$ the statistics for parameter $\theta$. Denote by $M_0^j$ the true $j$-th autocovariance

$$M_0^j \equiv E \left[h(y_t(\theta_0)) - M_0 \right] [h(y_{t-j}(\theta_0)) - M_0]^j$$

Define $S_w \equiv \sum_{j=-\infty}^{\infty} M_0^j$. A consistent estimator $\tilde{S}_{w,N} \rightarrow S_w$ is found using standard Newey-West estimators. The variance for the sample statistics $\tilde{S}_N$ reported in the second column of Table 1 are given by

$$\Sigma_{S,N} \equiv \frac{\partial S(M_N)}{\partial M} \tilde{S}_{w,N} \frac{\partial S(M_N)'}{\partial M}$$

Note that the model is not needed for this estimator, we use only observed data. The exact form of $\frac{\partial S(M)}{\partial M}$ is in online appendix B.
Duffie and Singleton (1993) show that to apply standard MSM asymptotics one needs geometric ergodicity, we showed that this holds in our model in section 4.3. Note that the smooth bounding function $\omega$ in equation (27) guarantees that a Monte-Carlo approximation to $\tilde{S}$ is differentiable as is required for MSM asymptotic distribution.

Letting $\Sigma_S$ be the asymptotic var-cov matrix of the sample statistics, under standard assumptions it can be shown that

$$\tilde{\Sigma}_{S,N} \rightarrow \Sigma_S \text{ and } \hat{\theta}_N \rightarrow \theta_0 \text{ a.s. as } N \rightarrow \infty. \quad (44)$$

Also, letting $B_0=\frac{\partial M'(\theta_0)}{\partial \theta} \frac{\partial S(M_0)}{\partial M}$ it can be shown that

$$\sqrt{N} \left[ \hat{\theta}_N - \theta_0 \right] \rightarrow \mathcal{N}(0, (B_0 \Sigma_S^{-1} B_0)^{-1}) \quad (45)$$

$$\sqrt{N} \left[ \hat{S}_N - S(M(\hat{\theta}_N)) \right] \rightarrow \mathcal{N}(0, \Sigma_S - B'_0 (B_0 \Sigma_S^{-1} B_0)^{-1} B_0) \quad (46)$$

$$\hat{W}_N \equiv N \left[ \hat{S}_N - \tilde{S}(\hat{\theta}) \right]' \hat{\Sigma}_{S,N}^{-1} \left[ \hat{S}_N - \tilde{S}(\hat{\theta}) \right] \rightarrow \chi^2_s(47)$$

in distribution as $N \rightarrow \infty$. Also, the weighting matrix $\hat{\Sigma}_{S,N}^{-1}$ is optimal among all weighting matrices of the statistics. The t-statistics in Tables 2-3-4 use variances from (46), the p-values for $\hat{W}_N$ are based on (47).

As can be seen from the above formula we need to invert $\hat{\Sigma}_{S,N}$ and its limit $\Sigma_S$. A nearly singular $\hat{\Sigma}_{S,N}$ presents problems as the distribution of $\hat{W}_N$ is nearly ill-defined and the distribution in short-samples is not close to a $\chi^2$. This singularity occurs because one of the statistics in $\hat{S}_N$ is nearly perfectly correlated with all the others. This only means that this is a redundant statistic so we can drop it from $\hat{S}_N$ in the estimation. To select the redundant statistic we predict each element of $\hat{S}_N$ with all the others according to $\hat{\Sigma}_{S,N}$ and drop the statistic for which the $R^2$ is less than 1%. As it turns out this occurs only for $\hat{c}_2$ with an $R^2 = 0.006$. 

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B Online Appendix - Not for Publication

B.1 MSM estimation and testing

Here we provide an estimator for the covariance matrix of the statistics $\hat{S}_{S,N}$, we give some more details about consistency of the estimator defined in (29), and derive the needed asymptotic distribution results.

Let $N$ be the sample size, $(y_1, ..., y_N)$ the observed data sample, with $y_t$ containing $m$ variables. The standard version of the method of simulated moments (MSM) is to find parameter values that make the moments of the structural model close to sample moments $\hat{M}_N \equiv \frac{1}{N} \sum_{t=1}^{N} h(y_t)$ for a given moment function $h : \mathbb{R}^m \rightarrow \mathbb{R}^q$. However, many of the statistics that we wish to match in Table 1 are not of this form, but they are functions of moments. Formally, the statistics in Table 1 can be written as $\hat{S}_N \equiv \mathcal{S}(\hat{M}_N)$ for a statistic function $\mathcal{S} : \mathbb{R}^q \rightarrow \mathbb{R}^r$ mapping sample moments $\hat{M}_N$ into the considered statistics. Explicit expressions for $h(\cdot)$ and $\mathcal{S}(\cdot)$ in our particular application are stated in section B.2. In the text we talked about "moments" as describing all statistics to be matched in (28). In this appendix we properly use the term "statistic" as possibly different from "moment".

We propose to base our MSM estimates and tests on matching the statistics $\hat{S}_N$. Since this deviates from standard MSM we need to adapt standard proofs and asymptotic distribution results. The proofs follow standard steps so we provide an outline of the argument and the derivations. The statistics to be matched $\hat{S}_N$ can be all or a subset of the statistics in (28), the reason we sometimes consider a subset will be discussed in detail after result "Asymptotic Distribution of MSM" below. Since we have endogenous state variables in the model (namely past beliefs $\beta_{t-1, \beta_{t-2}}$) the asymptotic theory result needed is from Duffie and Singleton (1993) (DS).

Let $y_t(\theta)$ be the series generated by the structural model at hand for parameter values $\theta$ and some realization of the underlying shocks. All the results below are derived under the null hypothesis that the model is true, more specifically, that the observed data is generated by the structural model at hand for a true parameter value $\theta_0$. Let $M(\theta) \equiv E[ h(y_t(\theta)) ]$ be the true moments for parameter values $\theta$ at the stationary distribution of $y_t(\theta)$, hence $M_0 \equiv M(\theta_0)$ are the true moments, and let $\hat{S}(\theta) \equiv \mathcal{S}(M(\theta))$ be the true statistics when the model parameter is $\theta$. Denote by $M_0^j$ the $j$-th
autocovariance of the moment function at the true parameter, that is

\[ M_0^j \equiv E \left[ h(y_t(\theta_0)) - M_0 \right] \left[ h(y_{t-j}(\theta_0)) - M_0 \right]' \]

Define

\[ S_w \equiv \sum_{j=-\infty}^{\infty} M_0^j \]  (48)

We use the following estimate of the variance for the sample statistics \( \hat{S}_N \)

\[ \hat{\Sigma}_{S,N} \equiv \frac{\partial S(M_N)}{\partial M'} \hat{S}_{w,N} \frac{\partial S(M_N)'}{\partial M} \]

The asymptotic properties of this estimate are given in the following result:

**Variance of \( \hat{S}_N \)** Suppose that

a) \( S_w < \infty \), and we have consistent estimators of this matrix \( \hat{S}_{w,N} \rightarrow S_w \) a.s. as \( N \rightarrow \infty \)

b) \( S \) is continuously differentiable at \( M_0 \).

c) the observed process \( \{y_t\} \) is stationary and ergodic

Then we have that

\[ \hat{\Sigma}_{S,N} \rightarrow \Sigma_S \equiv \frac{\partial S(M_0)}{\partial M'} S_w \frac{\partial S(M_0)'}{\partial M} \]  (49)

and that \( \Sigma_S \) is the asymptotic covariance matrix of \( \hat{S}_N \):

\[ E \left[ \hat{S}_N - S(M_0) \right] \left[ \hat{S}_N - S(M_0) \right]' \rightarrow \Sigma_S \]  (50)

both limits occurring a.s. as \( N \rightarrow \infty \).

Therefore, \( \hat{\Sigma}_{S,N} \) is a consistent estimator of the asymptotic variance of the sample statistics.

**Proof.** Assumptions a), c) imply

\[ \hat{M}_N \rightarrow M_0 \text{ a.s. as } N \rightarrow \infty \]

and assumption b) gives (49).
Assumptions a), c) imply

\[ E [M_N - M_0] [M_N - M_0]' \rightarrow S_w \text{ a.s. as } N \rightarrow \infty \]

The mean value theorem implies that

\[ S(M_N) - S(M_0) = \frac{\partial S(\tilde{M}_N)}{\partial M'} [M_N - M_0] \quad (51) \]

for a \( \tilde{M}_N \rightarrow M_0 \) a.s. as \( N \rightarrow \infty \). Taking expectations of \( [S(M_N) - S(M_0)] [S(M_N) - S(M_0)]' \) we have (50). \( \blacksquare \)

Conditions a), c) are standard minimal assumptions used in time series asymptotic results, condition b) is clearly satisfied in our application, see the expression for \( S \) in section B.2. We choose consistent estimates \( \hat{S}_{w,N} \) applying the Newey West estimator using only the data. Hence the estimator \( \hat{\Sigma}_{S,N} \) can be found purely from data, without using the model or its parameter estimates. We now turn to

**Consistency** Let \( \hat{\theta}_N \) be the estimator defined in (29), where the maximization is over a set \( \Theta \subset R^n \). Assume

a) \( \Theta \) is compact, the process \( \{y_t(\theta)\} \) is well defined for all \( \theta \in \Theta \), \( \tilde{S} \) is continuous in \( \Theta \), and \( \theta_0 \in \Theta \).

b) \( \{y_t(\theta)\} \) is geometrically ergodic for all \( \theta \in \Theta \)

c) \( \Sigma_S \) is invertible

d)

\[ \left[ \tilde{S}(\theta_0) - \tilde{S}(\theta) \right]' \Sigma_S^{-1} \left[ \tilde{S}(\theta_0) - \tilde{S}(\theta) \right] > 0 \quad \text{for all } \theta \neq \theta_0 \]

Then

\[ \hat{\theta}_N \rightarrow \theta_0 \text{ a.s. as } N \rightarrow \infty. \]

The proof is easily obtained by adapting the consistency result from DS.

Condition a) is standard in GMM applications, the set \( \Theta \) should be large enough to insure that it contains admissible values of the true parameter values. DS emphasize that a strong form of ergodicity is needed as in condition b), we showed in appendix A.4 that this holds for \( \alpha \) large enough, therefore
b) is guaranteed if \( \Theta \) is restricted to large \( \alpha \). Conditions c) and d) are standard identification requirements that the statistics selected are sufficient to identify the true parameter values. A necessary condition for d) is that the number of parameters is less than the number of statistics \( s \).

Let

\[
B_0 = \frac{\partial M'(\theta_0)}{\partial \theta} \frac{\partial S'(M_0)}{\partial M}
\]

**Asymptotic Distribution** In addition to all the assumptions in the above results, assume that \( B_0 \Sigma_S^{-1} B_0' \) is invertible. Then

\[
\sqrt{N} \left[ \hat{\theta}_N - \theta_0 \right] \to N(0, (B_0 \Sigma_S^{-1} B_0)^{-1}) \quad \text{(52)}
\]

\[
\sqrt{N} \left[ \hat{S}_N - S(M(\hat{\theta})) \right] \to N(0, \Sigma_S - B_0' (B_0 \Sigma_S^{-1} B_0)^{-1} B_0) \quad \text{(53)}
\]

\[
\tilde{W}_N \to \chi^2_{n-s} \quad \text{(54)}
\]

in distribution as \( N \to \infty \), where \( \tilde{W}_N \equiv N \left[ \hat{S}_N - \hat{S}(\hat{\theta}) \right]' \Sigma^{-1}_{S,N} \left[ \hat{S}_N - \hat{S}(\hat{\theta}) \right] \)

**Proof.** The central limit theorem and (51) imply

\[
\sqrt{N} \left[ S(M_N) - S(M_0) \right] = \frac{\partial S(M_N)}{\partial M'} \sqrt{N} \left[ M_N - M_0 \right] \to N(0, \Sigma_S) \quad \text{(55)}
\]
in distribution. Letting

\[
B(\theta, M) = \frac{\partial M'(\theta)}{\partial \theta} \frac{\partial S'(M)}{\partial M}
\]

The asymptotic distribution of the parameters is derived as

\[
S(M(\hat{\theta}_N)) - S(M(\theta_0)) = B(\hat{\theta}, \hat{M}) \left[ \hat{\theta}_N - \theta_0 \right] \quad \text{(56)}
\]

\[
B(\hat{\theta}_N, \hat{M}_N) \Sigma_{S,N}^{-1} \left[ S(M(\hat{\theta})) - S(M(\theta_0)) \right] = B(\hat{\theta}_N, \hat{M}_N) \Sigma_{S,N}^{-1} B(\hat{\theta}, \hat{M})' \left[ \hat{\theta} - \theta_0 \right]
\]

where the last equality follows because \( B(\hat{\theta}_N, \hat{M}_N) \Sigma_{S}^{-1} \left[ S(M_N) - S(\hat{\theta}_N) \right] = 0 \) at the maximum of (29). This implies (52).
To obtain (53) we use mean value theorem and (56) to conclude
\[
S(\widetilde{M}_N) - S(M(\tilde{\theta}_N)) = S(\widetilde{M}_N) - S(M(\theta_0)) + B(\tilde{\theta}, \widetilde{M}) \left[ \theta_0 - \tilde{\theta} \right] = \\
\left( I - \left[ B(\tilde{\theta}_N, \widetilde{M}_N)\Sigma_S^{-1}B(\tilde{\theta}, \widetilde{M}) \right]^{-1} B(\tilde{\theta}_N, \widetilde{M}_N)\Sigma_S^{-1} \right) \left[ S(\widetilde{M}_N) - S(M(\theta_0)) \right]
\]
This gives (53).

(54) follows from (53) and that \( \left( \Sigma_S - B_0' \left[ B_0\Sigma_S^{-1}B_0' \right]^{-1} B_0 \right) \) is an idempotent matrix. □

We mentioned in the main text and earlier in this appendix that we drop some statistics from \( \widehat{S}_N \) when \( \widehat{\Sigma}_{S,N} \) is nearly singular. The reason is that, as stated above, we need invertibility of \( \Sigma_S \) both for consistency and asymptotic distribution. In practice, a nearly singular \( \widehat{\Sigma}_{S,N} \) creates many problems. First, the results for the test \( W \) change very much with small changes in the model or testing procedure and the maximization algorithm is nearly unstable, making it difficult to find a maximum numerically. This happens because in this case the formula for \( \widehat{W}_N \) nearly divides zero by zero, hence the objective function is nearly undefined, and the asymptotic distribution is not necessarily a good approximation to the true distribution of the test statistic. But this is not a bad situation for an econometrician, it just means that one of the statistics is redundant, so it makes sense to simply drop one statistic from the estimation.

To decide which statistic to drop we compute the variability of each statistic that can not be explained by a linear combination of the remaining statistics. This is analogous to the \( R^2 \) coefficient of running a regression of each statistic on all the other statistics when the regression coefficients are computed from \( \widehat{\Sigma}_{S,N} \). We drop the statistics for which this \( R^2 \) is less than 1%, as it turns out this only occurs for \( \widehat{c}_2^5 \) with an \( R^2 \) equal to 0.006. After we drop \( \widehat{c}_2^5 \) the estimation results become sufficiently stable.\(^{48}\) As we explain in our discussion around Table 2, the model is in any case able to match \( \widehat{c}_2^5 \) even when we drop it from the statistics used in the estimation.

There are various ways to compute the moments of the model \( \widehat{S}(\theta) \) for a given \( \theta \in \mathbb{R}^n \). We use the following Monte-Carlo procedure. Let \( \omega^* \) denote

\(^{48}\) An alternative to dropping one moment would be to match the linear combinations corresponding to the principal components of \( \hat{S}_w \). While this has some advantages from the point of view of asymptotic theory, we find it less attractive from the economic point of view, since the economic meaning of these principal components would be unclear, while each individual moment does have clear economic interpretation.
a realization of shocks drawn randomly from the known distribution that the underlying shocks are assumed to have and \((y_1(\theta, \omega^i), \ldots, y_N(\theta, \omega^i))\) the random variables corresponding to a history of length \(N\) generated by the model for shock realization \(\omega^i\) and parameter values \(\theta\). Furthermore, let

\[
M_N(\theta, \omega^i) \equiv \frac{1}{N} \sum_{t=1}^{N} h(y_t(\theta, \omega^i))
\]

denote the model moment for realization \(\omega^i\). We set the model statistics \(\tilde{S}(\theta)\) equal to

\[
\frac{1}{K} \sum_{i=1}^{K} S(M_N(\theta, \omega^i))
\]

for large \(K\). In other words, \(\tilde{S}(\theta)\) is an average across a large number of simulations of length \(N\) of the statistics \(S(M_N(\theta, \omega^i))\) implied by each simulation. We use \(K\) of the order of 1000, therefore the model moments are computed with \(KN\) observations. These are the averages reported as model moments in tables 2 to 4 of the main text.

Many papers on MSM emphasize the dependence of the estimates on the ratio of number of observations in simulations to \(N\). Since this is 1000 in our application this adds a negligible factor to the asymptotic variance-covariance matrices computed and we entirely ignore it in our results.

### B.2 The statistic and moment functions

This section gives explicit expressions for the statistic function \(S(\cdot)\) and the moment functions \(h(\cdot)\) that map our estimates into the framework just discussed in this appendix.

The underlying sample moments needed to construct the statistics of interest are

\[
M_N \equiv \frac{1}{N} \sum_{t=1}^{N} h(y_t)
\]
\[ h(\mathbf{y}_t) = \begin{bmatrix}
    r^*_t \\
    PD_t \\
    (r^*_t)^2 \\
    (PD_t)^2 \\
    PD_t PD_{t-1} \\
    r^*_t r^*_t \\
    (r^*_t)^2 r^*_t \\
    r^*_t PD_{t-20} \\
    r^*_t r^*_t \\
    D_t/D_{t-1} \\
    D_t/D_{t-1}^2
\end{bmatrix}, \quad \mathbf{y}_t = \begin{bmatrix}
    PD_t \\
    D_t/D_{t-1} \\
    PD_{t-1} \\
    D_{t-1}/D_{t-2} \\
    \vdots \\
    PD_{t-19} \\
    D_{t-19}/D_{t-20} \\
    PD_{t-20} \\
    r^b_t \\
    \vdots \\
    r^b_{t-19}
\end{bmatrix}\]

where \(r^*_t\) denotes the excess stock return over 20 quarters, which can be computed using from \(\mathbf{y}_t\) using \((PD_t, D_t/D_{t-1}, r^b_t, \ldots, PD_{t-19}, D_{t-19}/D_{t-20}, r^b_{t-19}, PD_{t-20})\). \(^{49}\)

The ten statistics we consider can be expressed as function of the moments as follows:

\[ S(M) \equiv \begin{bmatrix}
    E(r^*_t) \\
    E(PD_t) \\
    \sigma_{r^*_t} \\
    \sigma_{PD_t} \\
    \rho_{PD_{t-1}} \\
    \sigma^2_2 \\
    R_2^2 \\
    E(r^b_t) \\
    E(\Delta D_t/D_{t-1}) \\
    \sigma_{D_t/D_{t-1}}
\end{bmatrix}, \quad \begin{bmatrix}
    M_1 \\
    M_2 \\
    \sqrt{M_3 - (M_1)^2} \\
    \sqrt{M_4 - (M_2)^2} \\
    \sqrt{M_5 - (M_3)^2} \\
    \sqrt{M_6 - (M_4)^2} \\
    c_2^2(M) \\
    R_2^2(M) \\
    M_9 \\
    M_{10} \\
    \sqrt{M_{11} - (M_{10})^2}
\end{bmatrix}\]

where \(M_i\) denotes the \(i\)-th element of \(M\) and the functions \(c_2^2(M)\) and \(R_2^2(M)\) define the OLS and \(R^2\) coefficients of the excess returns regressions, more precisely

\(^{49}\)The excess return is defined as \(1 + r^*_t = \prod_{j=1}^{20} (1 + r^*_t) / \prod_{j=1}^{20} (1 + r^b_t)\)

where \(1 + r^*_t = \frac{P_t + D_t}{P_{t-1}} = \frac{PD_{t+1} + D_t}{PD_{t-1} + D_t-1}\) is the one-period quarterly stock return and a function of \(\mathbf{y}_t\).
\[c_5^\delta(M) \equiv \begin{bmatrix} 1 & M_2 \\ M_2 & M_4 \end{bmatrix}^{-1} \begin{bmatrix} M_6 \\ M_8 \end{bmatrix}\]

\[R_6^5(M) \equiv 1 - \frac{M_7 - [M_6, M_8]c_5^\delta(M)}{M_7 - (M_6)^2}\]

### B.3 Derivatives of the statistic function

This appendix gives explicit expressions for \(\partial S/\partial M\)' using the statistic function stated in appendix B.2. Straightforward but tedious algebra shows

\[
\frac{\partial S_i}{\partial M_j} = \begin{cases} 
1 & \text{for } (i, j) = (1, 1), (2, 2), (8, 9), (9, 10) \\
\frac{1}{2s_i(M)} & \text{for } (i, j) = (3, 3), (4, 4), (10, 11) \\
\frac{-M_j}{S_i(M)} & \text{for } (i, j) = (3, 1), (4, 2), (10, 10) \\
\end{cases}
\]

\[
\frac{\partial S_5}{\partial M_2} = \frac{2M_2(M_5 - M_4)}{(M_4 - M_2^2)^2}, \quad \frac{\partial S_5}{\partial M_5} = \frac{1}{M_4 - M_2^2}, \quad \frac{\partial S_5}{\partial M_4} = -\frac{M_5 - M_2^2}{(M_4 - M_2^2)^2}
\]

\[
\frac{\partial S_2}{\partial M_j} = \frac{\partial c_2^\delta(M)}{\partial M_j} \quad \text{for } i = 2, 4, 6, 8
\]

\[
\frac{\partial S_7}{\partial M_j} = \frac{[M_6, M_8] \frac{\partial c_2^\delta(M)}{\partial M_j}}{M_7 - M_6^2} \quad \text{for } j = 2, 4
\]

\[
\frac{\partial S_7}{\partial M_6} = \frac{c_2^\delta(M) + [M_6, M_8] \frac{\partial c_2^\delta(M)}{\partial M_6}}{M_7 - M_6^2} (M_7 - M_6^2) - 2M_6 [M_6, M_8]c_5^\delta(M)
\]

\[
\frac{\partial S_7}{\partial M_7} = \frac{M_6^2 - [M_6, M_8]c_5^\delta(M)}{(M_7 - M_6^2)^2}
\]

\[
\frac{\partial S_7}{\partial M_8} = \frac{c_2^\delta(M) + [M_6, M_8] \frac{\partial c_2^\delta(M)}{\partial M_8}}{M_7 - M_6^2}
\]

Using the formula for the inverse of a 2x2 matrix

\[c_5^\delta(M) = \frac{1}{M_4 - M_2^2} \begin{bmatrix} M_4M_6 - M_2M_8 \\ M_8 - M_2M_6 \end{bmatrix}\]
we have

\[
\frac{\partial c^5(M)}{\partial M_2} = \frac{1}{M_4 - M_2^2} \left( 2M_2c^5(M) - \begin{bmatrix} M_8 \\ M_6 \end{bmatrix} \right)
\]

\[
\frac{\partial c^5(M)}{\partial M_4} = \frac{1}{M_4 - M_2^2} \left( -c^5(M) + \begin{bmatrix} M_6 \\ 0 \end{bmatrix} \right)
\]

\[
\frac{\partial c^5(M)}{\partial M_6} \equiv \frac{1}{M_4 - M_2^2} \begin{bmatrix} M_4 \\ -M_2 \end{bmatrix}
\]

\[
\frac{\partial c^5(M)}{\partial M_8} \equiv \frac{1}{M_4 - M_2^2} \begin{bmatrix} -M_2 \\ 1 \end{bmatrix}
\]

All remaining terms $\partial S_i/\partial M_j$ not listed above are equal to zero.