Matching to Share Risk without Commitment

Johannes Gierlinger† and Sarolta Laczó‡

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Abstract

This paper reconsiders matching to share risk by requiring insurance transfers to be self-enforcing. We consider a marriage problem (i.e., two-sided, one-to-one matching without search frictions) with imperfectly transferable utility in a dynamic setting. With efficient risk sharing within the household, every stable matching is negative assortative with respect to spouses’ risk attitudes (Chiappori and Reny, 2006; Legros and Newman, 2007). We study the robustness of this result when risk sharing within the household is partial due to lack of commitment. We show that stable matchings might be positive assortative. More risk-averse agents can be more attractive risk-sharing partners, because they are able to credibly promise larger insurance transfers.

Keywords: assortative matching, risk sharing, limited commitment

JEL codes: D10, D86, C73, C78.

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†Universitat Autònoma de Barcelona and Barcelona GSE, Departament d’Economia i d’Història Econòmica, Edifici B, 08193 Bellaterra, Barcelona, Spain. Email: johannes.gierlinger@uab.es

‡Institut d’Anàlisi Econòmica (IAE-CSIC) and Barcelona GSE, Campus UAB, 08193 Bellaterra, Barcelona, Spain. Email: sarolta.laczo@iae.csic.es.
1 Introduction

If income shocks are imperfectly insurable on financial markets, family members can better smooth their consumption by making transfers to each other. This motive for household formation has been highlighted by several papers, including Kotlikoff and Spivak (1981), Rosenzweig and Stark (1989), Ogaki and Zhang (2001), and Hess (2004). Recent literature investigates whether risk sharing motives might help to explain the composition of households, see Schulhofer-Wohl (2006), Chiappori and Reny (2006), and Legros and Newman (2007). The risk attitudes of spouses are key to link individual preferences with decisions at the household level, such as consumption-saving (Mazzocco, 2004) and portfolio allocation.

Consider agents on two sides of a matching market (‘men’ and ‘women’) who may choose to form a union to share the exogenous risk each of them faces. Both men and women are heterogeneous with respect to their risk attitudes and can be ranked according to their risk aversion in the sense of Arrow and Pratt. Chiappori and Reny (2006) and Legros and Newman (2007) show that, under general conditions, any stable matching equilibrium is negative assortative. That is, the more risk-averse man is matched with the less risk-averse woman for any two couples. The intuition behind this result is that, since more risk-averse men are willing to give up a larger share of their income for insurance, they become attractive partners to less risk-averse women. Less risk-averse women have a comparative advantage in providing insurance to more risk-averse partners. The former are always willing and able to outbid more risk-averse women as they are willing to take on more risk in return for the same expected transfer. Note that differences in risk aversion in the model can be reinterpreted as differences in wealth, for example.\footnote{More precisely, as long as preferences are characterized my decreasing absolute risk aversion, an agent with higher wealth behaves in a less-risk-averse fashion toward a given lottery in the sense Arrow and Pratt.}

The aforementioned matching literature assumes risk to be shared efficiently within the household, as prescribed by Borch (1962) and Wilson (1968). However, Mazzocco (2007) finds that inter-temporal household decisions are subject to limited commitment, using data from the Consumer Expenditure Survey (CEX). Dercon and Krishnan (2000) provide evidence that perfect risk sharing is not achieved within households in rural Ethiopia.

In this paper we reconsider the marriage problem (i.e., two-sided, one-to-one matching without search frictions) with imperfectly transferable utility in a dynamic setting. We study the consequences of limited commitment on the properties of stable matchings. We build on the literature on dynamic risk sharing with self-enforcing transfers (Kocherlakota, 1996). Problems of commitment arise from a lack of formal institutions to enforce promises, since

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1 More precisely, as long as preferences are characterized my decreasing absolute risk aversion, an agent with higher wealth behaves in a less-risk-averse fashion toward a given lottery in the sense Arrow and Pratt.
agents with favorable current shocks may refuse to make agreed-upon transfers. In other words, under full commitment, each ‘spouse’ requires at least the utility he or she would get when single *ex ante*. In contrast, under limited commitment, any candidate sharing rule is required to be sustainable with voluntary transfers *ex post* as well, that is, given every possible history of endowment realizations.

The objects of interest in this paper are assortative matching predictions with respect to risk aversion. Both men and women are assumed heterogenous in their risk attitudes, but each man and each woman faces the same gender-specific, non-tradable risk. We study whether the negative assortative matching equilibria from static models (Chiappori and Reny, 2006; Legros and Newman, 2007) are robust to limited commitment. Compared to negative assortative matching, positive assortative matching results in less heterogeneity across partners but more heterogeneity across households.

First, we consider static contracts, as in Coate and Ravallion (1993), a natural first step in extending the literature, which only considers one-shot risks. We consider binary risks, as Legros and Newman (2007), in order to derive results analytically. We show that when the couple faces little aggregate risk, stable matchings are positive assortative, thereby we overturn the result of the perfect risk sharing case. We show that a more risk-averse agent is always willing and able to offer at least as much insurance as a less risk-averse agent. Hence, a more risk-averse agent is a more attractive risk sharing partner. Since this reasoning holds for both sides of the matching market, we obtain positive assortative matching.

We also show that if the correlation of men’s and women’s endowments is non-negative, then matching is negative assortative. To see this, consider the case where women face no endowment risk. Now, non-zero net transfers result in a risky consumption process for a woman, while her endowment is risk-free. Hence, a less risk-averse woman is willing and able to make at least as large transfers as a more risk-averse one. The reasoning on the other side of the market remains as before, since men’s consumption is less risky than their endowment. Hence the negative assortative matching result.

In these two cases it is possible to determine the overall effects of two forces. First, more risk-averse agents value insurance more and can credibly promise larger transfers as long as their consumption is less risky than their endowment. Second, the more the spouses differ in their risk aversion, the more scope there is to share aggregate risk. Without aggregate risk, the first force implies positive assortative matching and the second force is not present. When the correlation of endowments is non-negative, both forces imply negative assortative matching.
Second, we consider dynamic contracts, which provide the constrained-efficient solution of the risk sharing with limited commitment model (see Kocherlakota, 1996 and others). While the long-run allocations can still be characterized analytically, we have to consider the transition to that long-run equilibrium to study matching, which turns out to complicate the analysis significantly. First, we consider the case without aggregate risk. More risk-averse agents are more attractive risk sharing partners in the long run, however, they may be more constrained in transferring utility in the transition period. That is, there is a third force, which implies negative assortative matching. We provide examples for both negative assortative and positive assortative stable matchings. (Second, in ongoing work, we consider the case where the correlation of endowments is non-negative.)

We have emphasized matching to share risk between men and women, as Chiappori and Reny (2006). However, two-sided one-to-one matching with imperfectly transferable utility is relevant for a wide range of other applications, as Legros and Newman (2007) argue. One could think of agents working together on a project in a partnership, sharecropping agreements, or matching agents to projects. In all these cases, interaction is likely to occur repeatedly once a match is formed, and each time agents may decide to discontinue the partnership, hence limited commitment is relevant.

The next section introduces the economic environment and defines the objects of interests. Section 3 characterizes equilibria under full commitment, followed by Section 4 on dynamic risk sharing without commitment. We first consider static contracts (Section 4.1). We then extend our analysis to dynamic contracts (Section 4.2). Section 5 concludes.

2 The model

Consider an endowment economy with $2N$ infinitely-lived agents, each belonging to one of two disjoint sub-populations, ‘men’ and ‘women.’ Agents are exposed to an exogenous risk at each date $t = 1, 2, ..., \ldots$, which is independently and identically distributed (i.i.d.) over time. For simplicity, we assume that the support of the state of world in each period has only two elements, and that the endowment risk is gender-specific but does not vary across individuals of the same gender. Let $s_t \in \{\bar{s}, \bar{s}\}$ denote the state of the world at time $t$ that specifies the endowment of each woman and each man. Women face the i.i.d. risk $W = (\bar{w}, w)$, and men face $M = (\bar{m}, m)$. The state space $\{\bar{s}, \bar{s}\}$, state probabilities $\Pr(s_t = \bar{s}) = 1 - \pi$ and $\Pr(s_t = \bar{s}) = \pi$, and income vectors are common knowledge. Aggregate income of the couple is $Z = W + M = (\bar{z}, z)$. Without loss of generality we assume that $\bar{z} \geq z$. 
The instantaneous utility function of agent $i$ is $u(i, \cdot)$.\footnote{\(u(i, \cdot)\) could be interpreted as the indirect utility function for uninsurable risk in the presence of otherwise competitive financial markets. If it is statistically independent from tradable risk, then expected utility guarantees that the indirect utility function inherits monotonicity and risk aversion from the original utility function on total wealth (see Gollier, 2001).} We assume strict monotonicity and strict risk aversion, i.e., $u_c \equiv \frac{\partial u}{\partial c} > 0$ and $u_{cc} \equiv \frac{\partial^2 u}{\partial c^2} < 0$ everywhere. We assume further that $u$ satisfies the Inada conditions with respect to $c$ for all agents. In mathematical terms, $\lim_{c \to 0} u_c(i, c) = +\infty$, and $\lim_{c \to +\infty} u_c(i, c) = 0$. Define also

$$\mathbb{E}u(i, C) = (1 - \pi)u(i, \bar{\tau}) + \pi u(i, \bar{c}).$$

$\mathbb{E}u(i, C)$ is the expected per-period utility from consuming $\bar{c}$ in state $\bar{s}$ and $\bar{c}$ in state $\bar{\tau}$.

We assume that agents are heterogenous with respect to their risk preferences, and we index women and men by $i \in \{1, 2, \ldots, N\}$ and $j \in \{1, 2, \ldots, N\}$, respectively. We rank agents $\{1, \ldots, N\}$ in terms their risk aversion in the sense of Arrow and Pratt: for any $i'$ and $i''$ with $i'' > i'$, one can find an increasing and concave function $\phi$ such that $u(i'', \cdot) = \phi(u(i', \cdot))$. Analogously, $j'' > j'$ if Mr $j''$ is more risk averse than Mr $j'$ in the Arrow-Pratt sense.

The timing of agents’ interaction is adapted from Chiappori and Reny (2006). In an ex-ante stage ($t = 0$), agents of the two sides of the matching market meet and couples can be formed. Then at each date $t = 1, 2, \ldots$, nature draws a state, which is observed by all agents. If agents $i$ and $j$ are matched at the beginning of period $t$, each decides how much of the consumption good to transfer to his/her risk sharing partner. Unmatched (‘single’) agents simply consume their endowment. Unlike Chiappori and Reny (2006) and Legros and Newman (2007), we allow agents to (unilaterally) divorce. Each spouse would do so if he/she were better off single than staying ‘married’ and making the transfer prescribed by the risk sharing agreement. In other words, perfect risk sharing may not be achieved within the household, because commitment is limited, as in Kocherlakota (1996).

We assume that people cannot remarry after divorce. That is, agents apply a trigger strategy: defectors are punished by exclusion from risk sharing arrangements forever by all agents. This assumption allows us to pin down the value of the outside option for the couples’ problem before solving it, which allows us to derive analytical results. It may be motivated by social or pecuniary limits on the frequency of marriage and divorce.

The aim of our paper is to characterize the sorting properties of stable matchings with respect to risk preferences in this environment. A matching, denoted $\mu$, is a one-to-one correspondence between men and women, and matching with none is also an option. That is, woman $i$ and man $j$ are a couple if $\mu(i) = j$ and $\mu(j) = i$. We will also use the notation
\langle i, j \rangle$ for a couple. If agent $i$ (or $j$) is single, then $\mu(i) = 0$ ($\mu(j) = 0$), and we also write $\langle i \rangle$ ($\langle j \rangle$).

In order to define a stable matching, our object of interest, we first introduce two concepts, individually rational couples and a blocking pair.

**Definition 1 (Stable matching).** A couple $\langle i, j \rangle$ is individually rational if $j \succeq_i 0$ and $i \succeq_j 0$, i.e., each agent is better off being in the couple than being single. A couple $\langle i, j \rangle$ is a blocking pair with respect to matching $\mu$ if $j \succ_i \mu(i)$ and $i \succ_j \mu(j)$. A matching is stable if all couples are individually rational and there are no blocking pairs.

Finally, we need to make our notions of assortative matchings precise.

**Definition 2 (Assortative matching).** A matching is positive assortative if for any two couples $j = \mu(i)$ and $j' = \mu(i')$, $j > j' \Rightarrow i > i'$. A matching is negative assortative if for any two couples $j = \mu(i)$ and $j' = \mu(i')$, $j > j' \Rightarrow i < i'$. We call two matchings payoff equivalent if the level of expected discounted life-time utility is the same in both matchings for all women and men. We say that the economy satisfies PAM (NAM) if each stable matching is payoff equivalent to a positive (negative) assortative matching.

In the next two sections we characterize stable matchings. First, we review the case where spouses can fully commit to the sharing rule once couples are formed. Second, we study the case where commitment is limited.

### 3 Full commitment

We first describe an existing couple’s problem in the one-period case, then in the multi-period case. Afterwards, we will consider the matching stage of our dynamic game.

Even when agents are able to commit to insurance transfers, the redistribution of the joint utility is complicated due to concave utility functions, i.e., we are in the imperfectly transferable utility (NTU) case. To illustrate this, consider a case with $N = 1$. Suppose further that both agents live for one period and that they are exposed to a one-shot risk, as in Chiappori and Reny (2006).

Call $v$ the (instantaneous) utility promised to the man. The optimal sharing rule for Ms $i$ and Mr $j$, denoted $C^*(i, j, v) = (\bar{c}^*(i, j, v), \underline{c}^*(i, j, v))$, is defined as

$$C^*(i, j, v) \equiv \arg \max_C \mathbb{E}u(i, C) \quad \text{s.t.} \quad \mathbb{E}u(j, Z - C) \geq v. \quad (1)$$
Changing $v$ we can find all Pareto-optimal sharing rules. Let $x(i, C)$ denote the marginal rate of substitution between the two states for Ms $i$, i.e.,

$$x(i, C) \equiv \frac{u_c(i, e)}{u_c(i, C)},$$

and similarly for men. Thanks to our regularity assumptions on the utility functions, the necessary and sufficient condition for efficient risk sharing,

$$x(i, C^*(i, j, v)) = x(j, Z - C^*(i, j, v)),$$

must be satisfied (Borch, 1962).³

Consider now infinitely-lived agents who face an i.i.d. risk each period. Given that preferences are separable over time, the well-known result that the ratio of marginal utilities should stay constant over time and across states holds. This also means that the sharing rule is stationary, that is, consumption shares do not depend on the past history, only on the current state, $s_t$. Therefore, a necessary and sufficient condition for optimality is that every optimal sharing rule in the infinite-horizon game must replicate a static efficient risk sharing arrangement each period. That is, any $C^*(i, j, v)$, understood now as the per-period sharing rule, which grants the instantaneous utility requirement $v$ to the man in each period, must also solve the dynamic problem in which Mr $j$ requires a discounted lifetime utility of $v/(1 - \beta)$. That is, the multi-period problem is the same as the one-period one, (1).

Consider now $N > 1$. We apply the results in Legros and Newman (2007) to characterize stable monotone matchings. A sufficient condition for our endowment economy with the sharing rule $C^*(i, j, v)$ to be NAM is that for all $i'' > i'$, $j'' > j'$, and all feasible utility levels $v' = \mathbb{E}u(j', Z - C)$ and $v'' = \mathbb{E}u(j'', Z - C)$,

$$C^*(i'', j'', v'') \succeq_v C^*(i'', j', v') \Rightarrow C^*(i', j'', v'') \succeq_v C^*(i', j', v'),$$

(GDD)

where the preference relation $\succeq_i$ ranks lotteries as a von Neumann-Morgenstern utility function, i.e., $C \succeq_i (\succeq_i)D \Leftrightarrow \mathbb{E}u(i, C) \geq (>)\mathbb{E}u(i, D)$. Condition (GDD) is a generalized decreasing difference condition, which says that if a more risk-averse woman $i''$ prefers to match with the more risk-averse man $j''$, so does a less risk-averse woman, and $i''$ cannot outbid $i'$. Importantly, (GDD) is not about greater willingness to bear risk. Instead, it requires less risk-averse women to have a comparative advantage in providing insurance to more risk-averse

³Note that the notation we have introduced in this paragraph implicitly assumes that payoffs in terms of utility only depend on the types and not on the identities of agents. Further, they are not influenced by what other households do, i.e., there are no externalities in this economy.
men. Similarly, the following generalized increasing difference (GID) condition guarantees PAM:

\[ C^*(i', j'', v'') \succeq_{v'} C^*(i', j', v') \Rightarrow C^*(i'', j'', v'') \succeq_{v''} C^*(i'', j', v'). \]  

Note that, unlike in the transferable utility case, stable matchings not only depend on how much surplus they generate, but also on the transferability of the surplus between matched agents. The above conditions require that the transferability of the surplus be monotone in type.

In the remainder of the paper we will make recurrent use of the sets \( C^+ \equiv \{(\bar{c}, \bar{c}) : \bar{c} \geq \bar{c} \} \) and \( C^- \equiv \{(\bar{c}, \bar{c}) : \bar{c} \leq \bar{c} \} \). \( C^+ \) contains all consumption rules which give at least as much in state \( \bar{c} \), i.e., when aggregate consumption is (weakly) higher, as in state \( \bar{c} \), and the opposite for \( C^- \). We will also use Jewitt (1986)’s order, which applies a single-crossing criterion to characterize agreements and disagreements over the ranking of lotteries.

**Definition 3 (Jewitt’s order).** Consider two random variables \( A \) and \( B \) with cumulative distribution functions \( F_A \) and \( F_B \), respectively. \( A \) dominates \( B \) in the sense of Jewitt, i.e., \( A \succeq_{SC} B \), whenever there exists \( x_0 \) such that for all \( x \)

\[ (x - x_0)(F_A(x) - F_B(x)) \geq 0. \]  

We also say that \( A \) is less risky than \( B \) in the sense of Jewitt if (2) holds. Jewitt (1986) shows that \( A \succeq_{SC} B \) if and only if for all \( i'' \) more risk-averse than \( i' \),

\[ A \succeq_{i'} B \Rightarrow A \succeq_{i''} B, \]

that is, whenever an agent prefers \( A \) over \( B \), then any more risk-averse agent does so too; and

\[ B \succeq_{i'} A \Rightarrow B \succeq_{i''} A, \]

that is, if an agent prefers \( B \) over \( A \), then any less risk-averse agent does so too. Note that the cumulative distribution functions (cdf’s) of two binary lotteries cross at most twice. If in addition the two lotteries, \( A \) and \( B \), are both in \( C^+ \) (or are both in \( C^- \)), then their cdf’s cross at most once. If they cross once, they can be ranked by Jewitt’s order from Definition 3. If they do not cross, they can be ranked by first-degree stochastic dominance.

Ranking the riskiness of lotteries according to Jewitt’s order is illustrated in Figure 1. The more risk-averse agent prefers the light-shaded area to the dark-shaded area, and the reverse is true for the less risk-averse agent. This means that lotteries in the light-shaded (dark-shaded) area are less (more) risky than \( X^+ \) (and \( X^- \)) in the sense of Jewitt. The two
agents agree on the ranking of other lotteries relative to $X^+$ (and $X^-$), and the preferred lotteries are said to be less risky.

We use Jewitt’s order to provide an alternative proof for the result in the literature (Legros and Newman, 2007; Chiappori and Reny, 2006) about the sorting of agents with heterogeneous risk preferences when the endowment risk is binary. Unlike the previous literature, we consider the absence of aggregate risk as well, i.e., the case where $\tau = \bar{z}$.

**Proposition 1.** Suppose full commitment. Then the economy satisfies NAM.

**Proof.** Efficient risk sharing requires consumptions to lie in $C^+$ for both risk sharing partners, $i$ and $j$. Define the crossing point $X(j, v) \in C^+$ for each $(j, v)$ pair as $C^*(i', j, v) \sim_v X(j, v) \sim_v C^*(i'', j, v)$. By revealed preference, $C^*(i', j, v) \succeq_v X(j, v) \succeq_v C^*(i'', j, v)$ and $C^*(i'', j, v) \succeq_v X(j, v) \succeq_v C^*(i', j, v)$. Hence,

$$C^*(i'', j, v) \succeq_{SC} X(j, v) \succeq_{SC} C^*(i', j, v), \forall (j, v),$$

since Ms $i''$ is more risk averse than Ms $i'$ by assumption. Suppose the premise of (GDD), i.e., $i''$ prefers to share risk with $j''$ rather than $j'$. By revealed preference over partners, $\mathbb{E}u(j', Z - C^*(i'', j'', v'')) \leq v'$. Otherwise, if $j'$’s utility constraint could be satisfied with the offer made to $j''$, then $j''$ would not be a preferred partner. Suppose first that $j''$ agrees with $j'$ on the ranking of the two offers, i.e., $\mathbb{E}u(j'', Z - C^*(i'', j', v')) \geq v''$. Then (3) together with $(Z - C) \in C^+$ implies $Z - C^*(i', j', v') \succeq_{SC} Z - C^*(i'', j', v')$. Therefore, $Z - C^*(i', j', v') \succeq_{SC} Z - C^*(i'', j', v') \succeq_{SC} Z - C^*(i'', j'', v'')$ and $\mathbb{E}u(j'', Z - C^*(i', j', v')) \geq v''$. A standard separating hyperplane argument implies that $i'$ can find a $C \succeq_v C^*(i', j', v')$. 

**Figure 1:** Jewitt’s order. Lotteries in the light-shaded (dark-shaded) area are less (more) risky than $X^+$ (and $X^-$) according to Jewitt’s order.
which provides exactly \( v'' \) to \( j'' \). That is, \( i' \) can always outbid \( i'' \) and match with \( j'' \). Suppose now that \( j'' \) disagrees with \( j' \) on the ranking of the two offers, i.e., now \( Z - C^*(i'', j'', v'') \geq j'' Z - C^*(i'', j', v') \). Then \( \exists \tilde{C} \in \mathbf{C}^+ \) such that \( \mathbb{E}u(j', Z - \tilde{C}) = v' \) and \( \mathbb{E}u(j'', Z - \tilde{C}) = v'' \). The disagreement implies

\[
C^*(i'', j', v') \geq_{\text{SC}} \tilde{C} \geq_{\text{SC}} C^*(i'', j'', v').
\]

If \( C^*(i', j', v') \geq_{\text{SC}} \tilde{C} \), then we know that \( X(j', v') \geq_{\text{SC}} \tilde{C} \geq_{\text{SC}} X(j'', v'') \). Then, given that the more risk-averse agent \( i'' \) prefers to match with \( j'' \) by assumption, i.e., she prefers the more risky lottery \( X(j'', v'') \) over the less risky lottery \( X(j', v') \), it must be that the less risk-averse agent \( i' \) also prefers \( X(j'', v'') \) over \( X(j', v') \). Further, \((3)\) implies that \( Z - C^*(i', j', v') \geq_{j''} Z - C^*(i'', j'', v'') \). If \( \tilde{C} \geq_{\text{SC}} C^*(i', j', v') \), then \( Z - C^*(i', j', v') \geq_{j''} Z - \tilde{C} \sim_{j''} Z - C^*(i'', j'', v'') \), and again there exists a contract \( C \) which is preferred by \( i' \) and grants \( j'' \) exactly \( v'' \), hence NAM.

The comparative statics result follows from Jewitt’s order and the fact that efficient risk sharing implies that both agents’ consumption is comonotonic with aggregate resources, i.e., all efficient sharing rules are such that both \( C^*(i, j, v) \) and \( Z - C^*(i, j, v) \) are in \( \mathbf{C}^+ \). On the one hand, less risk-averse women have a comparative advantage in reducing the spread in the consumption of their partner. On the other hand, more risk-averse men have a particular preference for reduced spreads. In the absence of aggregate risk, i.e., in the case where \( \pi = \bar{z} \), no woman has a comparative advantage in providing full insurance to any particular man, and vice versa. It is always possible to find alternative stable matchings by fixing the full insurance consumption while rearranging partners. Hence the payoff equivalence with the negative assortative equilibrium.

### 4 Limited commitment

Suppose now that commitment to risk sharing contracts is limited, i.e., insurance transfers must be self-enforcing in the sense of Thomas and Worrall (1988) and Kocherlakota (1996). This means that each spouse must be at least as well off as being single, i.e., forward-looking participation constraints (PCs) must be satisfied, at each time and given any history of endowment realizations. This implies that, in general, the consumption allocation does not only depend on current endowment realizations, but also on the whole history of shocks. Individual consumptions follow a Markov process in the long run whenever partial insurance occurs, even when endowments are i.i.d. (although consumption is i.i.d. in the long run with only two income states). Further, while the consumption allocation does not depend on
the initial Pareto weights in the long run with probability one whenever risk sharing is not perfect (Kocherlakota, 1996), these weights may matter for some periods, which are relevant for matching at time 0.

We first simplify the analysis by considering static contracts, as in Coate and Ravallion (1993). This allows analytical characterization and will provide the main intuition. We will then consider dynamic contracts.

4.1 Static contracts

Define $C(i, j, v)$ as the static consumption sharing rule which yields the highest utility to Ms $i$ such that it grants the expected per-period utility $v$ to Mr $j$ and satisfies the PCs. In mathematical terms,

$$ C(i, j, v) \equiv \arg \max_C \mathbb{E}u(i, C) $$

s. t. $\mathbb{E}u(j, Z - C) \geq v$, 

$$ u(i, \bar{c}) + \frac{\beta}{1 - \beta} \mathbb{E}u(i, C) \geq u(i, \overline{w}) + \frac{\beta}{1 - \beta} \mathbb{E}u(i, W), $$

$$ u(i, \underline{c}) + \frac{\beta}{1 - \beta} \mathbb{E}u(i, C) \geq u(i, w) + \frac{\beta}{1 - \beta} \mathbb{E}u(i, W), $$

$$ u(j, \bar{z} - \bar{c}) + \frac{\beta}{1 - \beta} \mathbb{E}u(j, Z - C) \geq u(j, \overline{m}) + \frac{\beta}{1 - \beta} \mathbb{E}u(j, M), $$

$$ u(j, \underline{z} - \underline{c}) + \frac{\beta}{1 - \beta} \mathbb{E}u(j, Z - C) \geq u(j, m) + \frac{\beta}{1 - \beta} \mathbb{E}u(j, M). $$

The first constraint, (4), is the promise keeping constraint. The other four constraints are the participation/enforcement constraints (PCs). They require that transfers be credible in the sense that each agent must be at least as well off respecting the terms of the contract (being 'married') as in autarky (being 'single') in both states of the world. Equations (5) and (6) are the woman’s PCs in states $\bar{s}$ and $\underline{s}$, respectively. Similarly, equations (7) and (8) are the man’s PCs in states $\bar{s}$ and $\underline{s}$, respectively. We say that $C$ is feasible for $(i, j, v)$ if it satisfies all five constraints above. We assume that $v$ is such that the set of feasible sharing rules is non-empty.

Note that the right-hand side of constraints (5)-(8) is the value of autarky. That is, we assume that agents apply a trigger strategy. This means that if one spouse fails to respect the terms of the risk sharing contract at time $t$, his/her partner will transfer 0 in all periods $\tau > t$, i.e., the couple splits. Further, we implicitly assume that remarrying is not possible. This is the most severe subgame-perfect punishment for defection in this environment (Kocherlakota, 1996). In other words, it is an optimal penal code in the sense of Abreu (1988).
Note that promising strictly positive transfers in all states can never be preferred to autarky. Therefore, it must be that one spouse makes a positive net transfer in one of the two states, and the other spouse in the other state. This implies that we can simplify our problem by eliminating two of the four PCs. Further, a positive insurance transfer must flow from the relatively lucky spouse towards the relatively unlucky one. In mathematical terms, if \( x(i, W) > x(j, M) \), then the woman makes a positive transfer in state \( s \), hence (5) and (8) are slack; while if \( x(i, W) < x(j, M) \), then the woman makes a positive transfer in state \( s \), hence (6) and (7) are slack.

Recall that the full commitment result, Proposition 1, relies on comparative static properties of an interior solution. Instead, in the case where commitment is limited and partial insurance occurs, the consumption allocation in each state is at the boundary of the constraint set.

We first provide a counterexample to show that NAM ceases to be an equilibrium property when commitment is limited.

**Example 1.** Consider an economy with two women, Ms \( i' \) and Ms \( i'' \), and two men, Mr \( j' \) and Mr \( j'' \). The preferences of each individual are characterized by constant relative risk aversion. In particular, \( u(i', y) = u(j', y) = \frac{y^{1-1.5}}{1.5} = \frac{-2}{\sqrt{y}} \) and \( u(i'', y) = u(j'', y) = \frac{y^{1-2}}{1-2} = \frac{-1}{y} \). That is, Ms \( i'' \) (Mr \( j'' \)) is more risk averse than Ms \( i' \) (Mr \( j' \)). Indeed, \( u(i'', \cdot) = \phi(u(i', \cdot)) \) (\( u(j'', \cdot) = \phi(u(j', \cdot)) \)), with \( \phi(y) = -0.25y^2 \) increasing and concave on the nonpositive domain. Further, suppose that \( \beta = 2/3 \), \( W = (2, 4) \), and \( M = (4, 2) \), and the two states occur with equal probabilities. Consider the couple \( \langle i'', j'' \rangle \). It is easy to see that the PCs (6) and (7) are satisfied with equality for the full insurance sharing rule \( C = (3, 3) \), which yields the expected per-period utility \(-1/3\) to both Ms \( i'' \) and Mr \( j'' \). Now, it is similarly easy to see that this sharing rule violates (6) and (7) for Ms \( i' \) and Mr \( j' \), respectively. Further, Ms \( i' \) (Mr \( j' \)) cannot promise an expected utility higher than \(-0.339\) to Mr \( j'' \) (Ms \( i'' \)) without violating her (his) own PC, (6) ((7)).

Hence \( \langle i'', j'' \rangle \) is a blocking pair to \( \langle i', j'' \rangle \) and \( \langle i'', i' \rangle \), and it must be part of any stable matching. Finally, there exist self-enforcing sharing rules with strictly positive transfers for \( \langle i', j' \rangle \) which are preferred to autarky by both Ms \( i' \) and Mr \( j' \). For instance, \( C = (2.2, 3.8) \) is such a sharing rule. Therefore, this economy satisfies PAM.

This example shows that the more risk-averse woman, Ms \( i'' \), is able to make transfers which

\[\footnote{We have computed the sharing rule numerically here.}\]
cannot be credibly promised by Ms $i'$. This makes Ms $i''$ an attractive partner. In particular for Mr $j''$, whom she prefers over Mr $j'$ for just the same qualities. He can be asked to make a larger transfer in state $\pi$ thanks to being relatively less constrained by (7). When the two more risk-averse agents form a couple, they are able to reach the Pareto-optimal allocation $(3, 3)$. The less risk-averse agents can only promise partial insurance.

Next, we consider two cases where we can derive results analytically with respect to monotone assortative matchings. We first study the case where the couple faces no aggregate risk. Second, we consider the case where the correlation between men’s and women’s endowments is non-negative.

4.1.1 No aggregate risk

Consider a situation in which the aggregate income for each couple does not vary across states, i.e., $\overline{z} = \underline{z} = z$. Without loss of generality, suppose that women receive more in state $s$, i.e., $W \in C^-$. In this case $x(i, W) > x(j, M)$, and we can disregard the constraints (5) and (8).

Note that the promise keeping constraint, (4), may be slack. Denote by $v(i, j)$ the threshold utility that Ms $i$ grants to Mr $j$ below which (4) is slack. That is, $C(i, j, v) = C(i, j, v(i, j))$ for all $v \leq v(i, j)$. The following lemma states a useful comparative statics property of $v(i, j)$.

**Lemma 1.** Consider static contracts. Suppose no aggregate risk. Then $v(i'', j) \geq v(i', j), \forall j$.

**Proof.** Suppose, without loss of generality, that $W \in C^-$. First, note that if $\beta$ is sufficiently low and autarky is the only self-enforcing outcome for both $i'$ and $i''$, then $v(i'', j) = v(i', j)$. Second, for low $v$, the corresponding $C(i, j, v)$ must satisfy (7) with equality. Then, given that the right-hand side of (7), autarky utility, does not depend on the risk sharing partner, if the first term on the left-hand side, $u(j, \overline{z} - \overline{z})$, increases, the second term, $\propto \mathbb{E}u(j, Z - C)$, must decrease, and vice versa. Then, among all $C \in C^- \text{satisfying (7) with equality}, \mathbb{E}u(j, Z - C)$ increases with $\overline{z}$ and decreases with $\underline{z}$. Third, if such $C$ satisfies (6) for $i'$, then the same is true for $i''$ as well. To see this, let us rewrite (6) as

$$\frac{1 - \beta \Pr (\overline{z})}{\beta \Pr (\underline{z})} \leq \frac{u(i, \overline{z}) - u(i, \underline{z})}{u(i, \underline{z}) - u(i, \overline{z})}.$$

Since $i''$ is more risk averse than $i'$ by assumption, we can write $u(i'', \cdot) = \phi (u(i', \cdot))$, where $\phi()$ is a strictly increasing and strictly concave function. Then we have

$$\frac{1 - \beta \Pr (\overline{z})}{\beta \Pr (\underline{z})} \leq \frac{u(i', \overline{z}) - u(i', \underline{z})}{u(i', \underline{z}) - u(i', \overline{z})} \leq \frac{\phi(u(i', \overline{z})) - \phi(u(i', \underline{z}))}{\phi(u(i', \underline{z})) - \phi(u(i', \overline{z}))}.$$

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At the same time \( x(i'', C) \geq x(i', C) \) for all \( C \in C^- \). This implies that \( i'' \) can and will choose a lower \( \xi \) and a higher \( \overline{z} \) compared to \( i' \), which means better insurance. This means that \( \mathbb{E}u(j, Z - C) \) increases with Ms \( i \)'s risk aversion, and the result follows.

We are now ready to state the main result of this section.

**Proposition 2.** Consider static contracts. Under limited commitment and no aggregate risk, the economy satisfies PAM.

**Proof.** With \( W \in C^- \), optimal contracts which satisfy the participation constraints must lie in \( C^- \) for any \( i \). Assume the premise of GID, i.e., \( C(i', j'', v'') \geq_C C(i', j', v') \). To exhaust all cases, we organize the proof along which of the three constraints, (4), (6), and (7), bind for couple \( \langle i', j'' \rangle \).

Consider first a slack promise keeping constraint, i.e., \( v'' \leq v(i', j') \). By Lemma 1, \( v'' \leq v(i'', j'') \) too. By definition, \( i'' \) obtains her maximal utility among all contracts which satisfy the participation constraint (7) of \( j'' \). Since constraint (7) of \( j' \) is more stringent, \( i'' \) prefers \( j'' \), irrespective of \( v' \), hence GID.

In the remaining cases (4) must be binding for \( \langle i', j'' \rangle \). First, consider the case in which no participation constraint binds for \( \langle i', j'' \rangle \). Then they share risk perfectly and reach full insurance. Since (6) is weaker for \( i'' \), also \( \langle i'', j'' \rangle \) features full insurance. Moreover, being more risk averse, \( i'' \) prefers \( j'' \), irrespective of \( v' \), hence GID.

Second, consider the case in which (7) binds for the couple \( \langle i', j'' \rangle \). Then the same constraints must be binding for \( \langle i'', j'' \rangle \) and consumption is pinned down by the intersection of (4) and (7), hence, \( C(i'', j'', v'') = C(i', j'', v'') \). Suppose indirectly that there exists some \( C \in C^- \) which satisfies (7) for the less risk-averse man \( j' \), but which is preferred to the contract \( C(i, j'', v'') \) by some woman. By revealed preference, this \( C \) must have either violated (7) or (4) for \( j'' \). However, any promise keeping constraint crosses an indifference curve of a woman at most once on \( C^- \), where the gains from trade relative to the crossing point arise by reducing spreads. Moreover, since (7) corresponds to a twisted constraint (4) which overweights the man’s preferred state \( \overline{s} \), also (7) and (4) cross at most once.
In particular, any reduction of spreads relative to the crossing point which satisfies (7) is automatically preferred to the crossing point, i.e., (4) is slack. Therefore, all $C$ which are preferred to $C(i', j', v')$ by $i'$ and which satisfy (7) for $j''$ are also preferred by $j''$. Since (7) is less stringent for $j''$ than for $j'$, any such $C$ satisfies (4) and (7) for $j''$, a contradiction to $C(i'', j', v') \succeq v C(i'', j'', v'')$, hence GID.

Third, we assume that (6) binds for the couple $\langle i', j'' \rangle$. Depending on the binding constraints for $\langle i', j' \rangle$, three subcases have to be distinguished.

- Suppose that (4) and (6) bind for the couple $\langle i', j' \rangle$. For any two distinct points in $C^-$ which satisfy (6) with equality, the lottery with higher $\xi$ yields strictly higher expected utility, due to the excess weight (6) puts on state $\bar{s}$. Therefore, $i'$ is indifferent between $j'$ and $j''$ if and only if $C(i', j', v') = C(i', j'', v'')$. Constraint (6) must be slack for $i''$ at $C(i', j', v')$. Her offer, $C(i'', j', v')$, therefore still satisfies (4) with equality, but reduces the spread compared to $C(i', j', v')$. The promise keeping constraints of $j'$ and $j''$ cross at most once on $C^-$. Since they cross at $C(i', j', v') = C(i', j'', v'')$, every reduction in spreads which satisfies (4) for $j'$ leaves (4) for $j''$ slack and grants her utility in excess of $v''$. Therefore, $i''$ can increase her utility matching with $j''$, and GID holds.

- Suppose that (4) and (7) bind for $\langle i', j' \rangle$. Then $C(i', j', v') = C(i'', j', v')$. Moreover, since (6) overweights the favorable state of the woman $\bar{s}$, while (7) overweights $s$, any point on (6) which delivers the same expected utility to a woman as a point on (7) must be a reduced spread of the latter. Therefore, $C(i', j'', v'')$ is less risky than $C(i', j', v')$ in the sense of Jewitt. Hence Ms $i''$ also prefers $C(i', j'', v'')$, which is feasible for the couple $\langle i'', j'' \rangle$, to $C(i', j', v') = C(i'', j', v')$, hence GID.

- Finally, suppose that (7) binds, while (4) is slack for $\langle i', j' \rangle$, i.e., $v' \leq v(i', j') \leq v(i'', j')$. Call $\tilde{C} \in C^-$ the corresponding crossing point of indifference curves, which satisfies $C(i, j', v(i, j')) \sim_i \tilde{C}$ for both women. Further, call $\tilde{C} \in C^-$ the contract for $\langle i', j'' \rangle$ such that $C(i', j', v(i', j')) \sim_{i'} \tilde{C}$ and where (6) for $i'$ holds as equality. Since they lie on the same indifference curve of $i'$, either $\tilde{C} \succeq_{SC} \tilde{C}$, $\tilde{C}$ being a reduced spread of $\tilde{C}$ here, or the reverse relation holds. In the first case, GID follows from Jewitt’s order and the fact that $i''$ can only improve upon $\tilde{C}$ by being with $j''$. Now consider the alternative case. Then $C(i'', v', j') \succeq_{SC} \tilde{C} \succeq_{SC} \tilde{C}$, where the first relation follows from the single crossing of the indifference curves of $i'$ and $i''$. Moreover, the promise keeping constraint (4) is slack for $j''$ at $\tilde{C}$ due to the single crossing property of an indifference curve of $i'$ with any promise keeping constraint of $j''$. For the same reason, (4) must also be slack for $j''$ at $C(i'', j', v')$. Therefore, a more stringent set of constraints is
satisfied at $C(i'', j', v')$, while (4) of $j''$ is slack. Hence GID.

It is clear from the proof of Proposition 2 that if we allow for some aggregate risk, we have to make an assumption on the discount factor $\beta$ to ensure PAM. This is because the well-known folk theorem applies here: if $\beta$ is sufficiently high, perfect risk sharing occurs in the long run (Kimball, 1988). Hence, for $\beta$ high enough we are back to the full commitment case and NAM.

The intuition behind Proposition 2 is the following. Absent aggregate risk, all candidate rules $C$ are such that $C \in C^-$ and $Z - C \in C^+$, since both the Pareto frontier of full insurance and women’s initial endowment, $W$, lie in $C^-$. Moreover, for every proposal coming from Ms $i'$, Ms $i''$ can provide at least as much insurance without violating her PC. In other words, more risk-averse women are always able to outbid less risk-averse women. Similarly, as compared to any credible transfers from Mr $j'$, larger transfers are credible from Mr $j''$. That is, the greater the level of risk aversion, the more insurance the couple can achieve.

Figure 2: Two women compete for a man. The areas enclosed by the curves $J$ and $I'$ or $I''$ mark the sets of self-enforcing sharing rules. For this specification only the more risk-averse woman, Ms $i''$, can promise full insurance, as in Example 1.

Figure 2 illustrates how the PCs favor the more risk-averse agent. Intuitively, the disciplining device of being sent into autarky poses less of a threat to less risk-averse agents. Importantly, the slopes are different from the slopes of the indifference curves (not represented). In particular, (7) overweighs the consumption of Mr $j$ in state $\bar{s}$, while (6) overweighs the
consumption of Ms $i$ in state $s$.

### 4.1.2 Non-negative correlation between endowments

Assume that both $W$ and $M$ are in $C^+$, i.e., the correlation between men’s and women’s endowments is non-negative. Assume that $x(i,W) < x(j,M)$, for all $(i,j)$. In this case a necessary condition for sharing rules to be preferred to autarky is that women make a transfer in state $s$, as before. However, in contrast to the previous section, agreeing on non-zero transfers means that women take on more risk, i.e., $W \succeq_{SC} C$. Note that a special case which satisfies these assumptions is where women are endowed with a sure income $\overline{w} = w \equiv w$. Under this assumption $W$ and $M$ are uncorrelated. In fact, a necessary condition for uncorrelated endowments in the two-states set-up is that either $W$ or $M$ be constant.

As in the previous section, the promise keeping constraint, \(4\), may be slack, i.e., $C(i,j,v) = C(i,j,\underline{v}(i,j))$ for all $v \leq \underline{v}(i,j)$. Now the following comparative statics property holds.

**Lemma 2.** Consider static contracts. Assume a non-negative correlation between endowments and $x(i,W) < x(j,M)$, $\forall (i,j)$. Then $\underline{v}(i'',j) \leq \underline{v}(i',j)$, $\forall j$.

**Proof.** The proof is similar to the proof of Lemma 1, we only highlight the differences. Among all $C \in C^+$ satisfying \(7\) with equality, $\mathbb{E}u(j, Z - C)$ increases with $\overline{c}$ and decreases with $\underline{c}$. If such $C$ satisfies \(6\) for $i''$, then the same is true for $i'$ as well. To see this, let us rewrite \(6\) as

$$1 - \beta \Pr (\overline{s}) \leq \frac{u(i, \overline{c}) - u(i, \overline{w})}{\beta \Pr (\overline{s})} \leq \frac{u(i'', \overline{c}) - u(i'', \overline{w})}{\beta \Pr (\overline{s})} \leq \frac{\phi^{-1}(u(i'', \overline{c})) - \phi^{-1}(u(i'', \overline{w}))}{\phi^{-1}(u(i'', \overline{c})) - \phi^{-1}(u(i'', \overline{w}))} = \frac{u(i', \overline{c}) - u(i', \overline{w})}{u(i', \overline{w}) - u(i', \overline{c})}.$$

Since $i''$ is more risk averse than $i'$ by assumption, we can write $\phi^{-1}(u(i'', \cdot)) = (u(i', \cdot))$, where $\phi^{-1}()$ is a strictly increasing and strictly convex function. Then we have

$$1 - \beta \Pr (\overline{s}) \leq \frac{u(i'', \overline{c}) - u(i'', \overline{w})}{u(i'', \overline{w}) - u(i'', \overline{c})} \leq \frac{\phi^{-1}(u(i'', \overline{c})) - \phi^{-1}(u(i'', \overline{w}))}{\phi^{-1}(u(i'', \overline{c})) - \phi^{-1}(u(i'', \overline{w}))} \leq \frac{u(i', \overline{c}) - u(i', \overline{w})}{u(i', \overline{w}) - u(i', \overline{c})}.$$

At the same time, $x(i'', C) \leq x(i', C)$ for all $C \in C^+$. This implies that $\mathbb{E}u(j, Z - C)$ increases with Ms $i$’s risk tolerance, and the result follows.

If there is aggregate risk and $(M + W) \in C^+$, then all candidate rules, $C$, must be such that both $C$ and $Z - C$ lie in $C^+$, since both the unconstrained-efficient rules, $C^*$, and women’s endowment, $W$, lie in $C^+$. Moreover, PCs are more stringent for more risk-averse women, since $W \succeq_{SC} C$. Therefore, intuitively, for every proposal coming from Ms $i''$, Ms $i'$ is willing and able to outbid her. Hence the following.
**Proposition 3.** Consider static contracts. Under limited commitment, non-negative correlation between endowments, and \(x(i,W) < x(j,M), \forall\langle i,j \rangle\), the economy satisfies NAM.

*Proof.* The proof follows the logic of the proof of Proposition 2. □

In the case of a risk-free endowment for women, all the comparative statics work in the direction of NAM. The PCs of more risk-averse women are harder to satisfy since autarky means a less risky consumption in all periods. Therefore, the less risk-averse woman willingly and credibly promises larger transfers. In addition, the more risk-averse man is willing and able to give up a larger share of his consumption in state \(s\) for any promise in state \(\bar{s}\).

### 4.1.3 Discussion

The previous subsections covered two cases where it is possible to determine the overall effects of two forces, namely, (i) more risk-averse women (men) value insurance more and can credibly promise larger transfers as long as \(C \succeq_{SC} W (Z - C \succeq_{SC} M)\), and (ii) there is more scope to share aggregate risk the more the spouses differ in their risk aversion. Without aggregate risk, as in our first special case, only the first effect is present. More risk-averse agents can credibly promise larger insurance transfers, hence all agents prefer to share risk with a more risk-averse partner, which implies PAM. When the correlation between endowments is non-negative and, loosely speaking, women’s endowment is less risky than men’s, as in our second special case, less risk-averse women can credibly promise larger transfers, since larger transfers imply less insurance for women in this case, i.e., \(W \succeq_{SC} C\). Among men still the more risk averse are the more attractive risk sharing partners, because \(Z - C \succeq_{SC} M\) still. This means that the first effect is in the direction of NAM. The second effect is always in the direction of NAM. Therefore, in this case the economy is characterized by NAM. Note also that with full commitment only the second effect is present.

As we have already discussed, the direction of the first effect depends on whether agents’ consumption is less or more risky than their endowment. This, however, may change across potential couples. To see this, suppose that there exists a potential couple \(\langle i, j'' \rangle\) with \(x(i,W) > x(j'',M)\), and that this couple cannot achieve partial insurance. There can be a more risk-averse man \(j''\) such that for the couple \(\langle i, j'' \rangle\) \(x(i,W) > x(j'',M)\) as well, but for whom the gains from trade are large enough so that non-zero self-enforcing insurance transfers are possible. At the same time, there may exist a less risk-averse man \(j'\) such that for the couple \(\langle i, j' \rangle\) \(x(i,W) < x(j',M)\). For this couple too, the gains from trade might be large enough to achieve partial insurance. Hence, for Ms \(i\) the self-enforcing risk sharing
arrangements with \( j'' \) and \( j' \) achieve partial insurance, while no enforceable transfers could be made between \( i \) and \( j'' \).

### 4.2 Dynamic contracts

Static contracts are not constrained-efficient in a dynamic risk sharing model with limited commitment (e.g. Kocherlakota, 1996). That is, Ms \( i \) could provide the expected per-period utility \( v \) to Mr \( j \) and increase her own utility by proposing a self-enforcing dynamic contract rather than \( C(i, j, v) \). Such contracts may specify the consumption allocation at time \( t \) conditional on the whole history of shocks up to \( t \).

To deal with dynamic contracts, we have to introduce additional notation. Remember that \( s_t \in \{\bar{s}, \underline{s}\} \) denotes the state of the world at time \( t \). Let \( s^t = (s_1, s_2, ..., s_t) \) denote the history of state realizations up to and including time \( t \), and \( \pi(s^t) \) its probability. Let \( c(s^t) \) \((z(s_t) - c(s^t))\) denote consumption by the woman (the man) at time \( t \) given that history \( s^t \) has occurred. Finally, we define \( C \) as the dynamic contract which specifies consumptions for any history of state realizations.

\[
C(i, j, v) \equiv \arg\max_{\{c(s^t)\}} \sum_{t=1}^{\infty} \sum_{s^t} \pi(s^t) \beta^{t-1} u(i, c(s^t))
\]

s. t.

\[
\sum_{t=1}^{\infty} \sum_{s^t} \pi(s^t) \beta^{t-1} u(j, c(s^t)) \geq \frac{1}{1-\beta} v,
\]

\[
u(i, c(s^{t-1}, \bar{s})) + \sum_{r=t+1}^{\infty} \sum_{s^r \supseteq s^t} \pi(s^r | s^t) \beta^{r-t} u(i, c(s^r)) \geq u(i, \bar{w}) + \frac{\beta}{1-\beta} E u(i, W),
\]

\[
u(i, c(s^{t-1}, \underline{s})) + \sum_{r=t+1}^{\infty} \sum_{s^r \supseteq s^t} \pi(s^r | s^t) \beta^{r-t} u(i, c(s^r)) \geq u(i, w) + \frac{\beta}{1-\beta} E u(i, W),
\]

\[
u(j, z(\bar{s}) - c(s^{t-1}, \bar{s})) + \sum_{r=t+1}^{\infty} \sum_{s^r \supseteq s^t} \pi(s^r | s^t) \beta^{r-t} u(j, z(s_r) - c(s^t)) \geq u(j, \bar{w}) + \frac{\beta}{1-\beta} E u(j, M),
\]

\[
u(j, z(\underline{s}) - c(s^{t-1}, \underline{s})) + \sum_{r=t+1}^{\infty} \sum_{s^r \supseteq s^t} \pi(s^r | s^t) \beta^{r-t} u(j, z(s_r) - c(s^t)) \geq u(j, m) + \frac{\beta}{1-\beta} E u(j, M),
\]

\( \forall s^t \), where \( \pi(s^r | s^t) \) is the probability of history \( s^r \supseteq s^t \) occurring given that history \( s^t \) has occurred up to time \( t \). Note that future decision variables enter into today's PCs, therefore the problem is not recursive without a co-state variable. Let \( v_t \) denote the promised per-period utility at time \( t \), which is a function of the current state and last period's promised utility, i.e., \( v_t(v_{t-1}, s_t) \). We can write the problem in the following recursive form:
\[
\max_{\{c(t_{t-1}, s_t)\}} \mathbb{E} V_i(v_{t-1}, s_t) = \mathbb{E}[u(i, c(v_{t-1}, s_t)) + \beta \sum_{s_{t+1}} \pi(s_{t+1}) V_i(v_t(v_{t-1}, s_t), s_{t+1})]
\]

s.t. \( \mathbb{E} V_j(v_{t-1}, s_t) \geq \frac{1}{1 - \beta} v_{t-1} \),

\( u(i, c(v_{t-1}, \bar{\sigma})) + \beta \sum_{s_{t+1}} \pi(s_{t+1}) V_i(v_t(v_{t-1}, s_t), s_{t+1}) \geq u(i, \bar{\sigma}) + \frac{\beta}{1 - \beta} \mathbb{E} u(i, W) \),

\( u(j, c(v_{t-1}, \underline{\sigma})) + \beta \sum_{s_{t+1}} \pi(s_{t+1}) V_j(v_t(v_{t-1}, s_t), s_{t+1}) \geq u(j, \underline{\sigma}) + \frac{\beta}{1 - \beta} \mathbb{E} u(j, W) \),

\( u(j, z(\bar{\sigma}) - c(v_{t-1}, \bar{\sigma})) + \beta \sum_{s_{t+1}} \pi(s_{t+1}) V_j(v_t(v_{t-1}, s_t), s_{t+1}) \geq u(j, \bar{\sigma}) + \frac{\beta}{1 - \beta} \mathbb{E} u(j, W) \),

\( u(j, z(\underline{\sigma}) - c(v_{t-1}, \underline{\sigma})) + \beta \sum_{s_{t+1}} \pi(s_{t+1}) V_j(v_t(v_{t-1}, s_t), s_{t+1}) \geq u(j, \underline{\sigma}) + \frac{\beta}{1 - \beta} \mathbb{E} u(j, W) \),

\( \forall v_{t-1}, \forall t \), where \( v_t \), the per-period utility promised to Mr \( j \), serves as a co-state variable (Abreu, Pearce, and Stacchetti, 1990), and \( v_0 = v \).

The characteristics of constrained-efficient allocations in endowment economies with limited commitment have been established in the literature, see Kocherlakota (1996) and Ligon, Thomas, and Worrall (2002). The constrained-efficient solution can be fully characterized by a set of state-dependent optimal intervals on the co-state variable, \([\bar{v}(s), \bar{\sigma}(s)]\), or, equivalently, on consumption, \([\bar{c}(s), \bar{\sigma}(s)]\). The bounds of these intervals give the minimum and maximum utility/consumption values that satisfy the PCs in each states. The literature typically focuses on the long-run allocations. The probability that the economy reaches this allocation converges to 1 as \( t \) converges to infinity.

With only two income states, we know that in the long run each agent’s consumption may take at most two values. Consumptions are pinned down by (11) and (12) if \( x(i, W) > x(j, M) \), and by (10) and (13) if \( x(i, W) < x(j, M) \). Hence, in the long run, history no longer matters.

However, to study matching at time 0, it is essential to take into account the transition to that long-run equilibrium. Four cases are possible with non-zero insurance transfers: \( v \) (along with the other parameters) is such that (i) a PC binds in both states at time 1, hence partial risk sharing occurs in the long run, (ii) a PC binds at time 1 only in one of the two states, and one will bind in the other state as well afterwards, hence partial risk sharing occurs in the long run, (iii) a PC binds at time 1 only in one of the two states, but no constraint will ever bind in the other state, hence perfect risk sharing occurs in the long run, or (iv) no PC ever binds, hence perfect risk sharing occurs in the long run. The contract is dynamic only in cases (ii) and (iii). Case (i) is equivalent to the case where \( v \leq v(i, j) \), discussed in
Section 4.1. It is obvious that in case (iv) as well the consumption allocation only depends on the current state.

To analyze cases (ii) and (iii), remember that $\pi$ denotes the probability of state $s$ occurring. Abusing notation slightly, let $s$ and $c$ now denote the long-run consumptions by Ms $i$ in state $s$ and $c$, respectively. Let $\hat{c}$ denote her consumption before any PC binds. Assume, without loss of generality, that $x(i, W) > x(j, M)$. Consider case (ii). Suppose that $v$ is such that at time 1 the PC of Ms $i$ does not bind in state $s$ (and as long as state $\overline{s}$ has not occurred), while Mr $j$’s PC binds if state $\overline{s}$ occurs. This means that consumption by Ms $i$ is $\hat{c}$ as long as state $s$ does not occur, is $c$ in all periods when $s$ occurs, and will be $\overline{c}$ in state $\overline{s}$ if state $\overline{s}$ has occurred at least once. Simple algebra yields that the per-period utility for Ms $i$ is

$$
\frac{\pi - \pi \beta}{1 - \pi \beta} u(i, \hat{c}) + \frac{\pi \beta - \pi^2 \beta}{1 - \pi \beta} u(i, \overline{c})
$$

in this case. If $v$ is such that at time 1 the PC of Mr $j$ does not bind in state $\overline{s}$ (and as long as state $s$ has not occurred), while Ms $i$’s PC binds if state $\overline{s}$ occurs, then the per-period utility for Ms $i$ is

$$
\frac{(1 - \pi) - (1 - \pi) \beta}{1 - (1 - \pi) \beta} u(i, \hat{c}) + \frac{(1 - \pi) \beta - (1 - \pi)^2 \beta}{1 - (1 - \pi) \beta} u(i, \overline{c}) + \pi u(i, \overline{c}).
$$

Importantly, whenever partial risk sharing occurs in the long run, as in cases (i) and (ii), the long-run consumption values, $\overline{c}$ and $\overline{c}$, do not depend on $v$. In case (i) consumptions at each date are independent of $v$. In case (ii) $\hat{c}$ depends on $v$.

Consider case (iii) now. Suppose that $v$ is such that at time 1 the PC of Ms $i$ does not bind in state $s$, while Mr $j$’s PC binds if state $\overline{s}$ occurs. Ms $i$’s PC will not bind after $\overline{s}$ has occurred either. This means that consumption by Ms $i$ is $\hat{c}$ as long as state $\overline{s}$ does not occur, and is $\overline{c}$ in all periods if $\overline{s}$ has occurred at least once. Simple algebra yields that the per-period utility for Ms $i$ is

$$
\frac{\pi - \pi \beta}{1 - \pi \beta} u(i, \hat{c}) + \frac{1 - \pi}{1 - \pi \beta} u(i, \overline{c})
$$

in this case. Note that $\overline{c} = \overline{c}$, and this value is given by the binding PC of Mr $j$. If $v$ is such that at time 1 the PC of Mr $j$ does not bind in state $\overline{s}$, while Ms $i$’s PC binds if state $\overline{s}$ occurs, and Mr $j$’s PC will not bind after $\overline{s}$ has occurred either, the per-period utility for Ms $i$ is

$$
\frac{(1 - \pi) - (1 - \pi) \beta}{1 - (1 - \pi) \beta} u(i, \hat{c}) + \frac{\pi}{1 - (1 - \pi) \beta} u(i, \overline{c}).
$$
Note that \( c = \bar{c} \) still, but now this value is given by the binding PC of Ms \( i \). With perfect risk sharing in the long run, \( c = \bar{c} \) depend on whether \( v \) is high (hence Ms \( i \)'s PC binds) or low (hence Mr \( j \)'s PC binds). As before, \( \bar{c} \) depends on \( v \).

As with static contracts, we study two cases: (i) where the couple faces no aggregate risk and (ii) where the correlation between men’s and women’s endowments is non-negative.

### 4.2.1 No aggregate risk

Remember that \( z \) denotes aggregate income, constant across the two states of the world. We assume, without loss of generality, that \( W \in C^- \). Then \( x(i,W) > x(j,M) \), and hence we can disregard the constraints (10) and (13) in the long run.

Let \( C^{lr}(i, j, v) = (\bar{c}(i, j, v), c(i, j, v)) \) denote the vector of long-run consumptions for Ms \( i \) sharing risk with Mr \( j \) who receives \( v \). We first make comparative statics statements about the long run. Then we will take into account the transition. First, we suppose that partial risk sharing occurs in the long run, i.e., \( \bar{c}(i, j, v) < c(i, j, v) \). These consumption values are independent of \( v \). This is relevant for case (i) and case (ii). Second, we consider the case where one agent’s PC is binding in one period, but no PC is binding in any other period, i.e., case (iii).

**Lemma 3.** Suppose no aggregate risk, and that a PC binds in both states in the long run for all potential couples. Then \( C^{lr}(i, j'', v'') \geq C^{lr}(i, j', v') \), \( \forall i \). That is, all women weakly prefer to share risk with a more risk-averse man in the long run. Further, \( \bar{c}(i, j'', v'') \geq \bar{c}(i, j', v') \), and the inequality is strict if some insurance is self-enforcing.

**Proof.** If \( \beta \) is sufficiently low so that autarky is the only self-enforcing outcome for both the couple \( (i, j'') \) and the couple \( (i, j') \), then \( C^{lr}(i, j'', v'') = C^{lr}(i, j', v') \). We show that \( C^{lr}(i, j', v') \) is feasible for the couple \( (i, j'') \). For this we have to show that Mr \( j'' \)'s PC is satisfied at \( C^{lr}(i, j', v') \). Let us rewrite (12) in the long-run as

\[
1 - \frac{\beta \pi}{\beta \pi} < \frac{u(j, z - \bar{c}(i, j, v)) - u(j, m)}{u(j, m) - u(j, z - \bar{c}(i, j, v))}.
\]

Then

\[
1 - \frac{\beta \pi}{\beta \pi} < \frac{u(j', z - \bar{c}(i, j', v')) - u(j', m)}{u(j', m) - u(j', z - \bar{c}(i, j', v'))} < \frac{\phi(u(j', z - \bar{c}(i, j', v')) - \phi(u(j', m))}{\phi(u(j', m)) - \phi(u(j', z - \bar{c}(i, j', v')))} = \frac{u(j'', z - \bar{c}(i, j', v')) - u(j'', m)}{u(j'', m) - u(j'', z - \bar{c}(i, j', v'))},
\]

where \( \phi() \) is strictly increasing and strictly concave. Then, the statement holds by revealed preference. Since (12) is slack for \( j'' \) at \( C^{lr}(i, j', v') \), consumption dispersion can be reduced
by increasing $\bar{c}$, which increases per-period utility in the long run. Hence, $\bar{c}(i, j''', v''') > \bar{c}(i, j', v')$.

**Lemma 4.** Suppose no aggregate risk, and that a PC binds once and perfect risk sharing occurs afterwards for all potential couples. Then $C^{lr}(i, j'', v'') \geq i C^{lr}(i, j', v')$, $\forall i$. That is, all women weakly prefer to share risk with a more risk-averse man in the long run. Further, $\bar{c}(i, j'', v''') \geq \bar{c}(i, j', v')$, and the inequality is strict if Mr $j$’s PC determines the long-run consumption.

**Proof.** In this case $\bar{c}(i, j'', v''') = \zeta(i, j'', v'')$ and $\bar{c}(i, j', v') = \zeta(i, j', v')$. However, whether this value is pinned down by (11) or (12) depends on $v$. Let us consider the two cases in turn. If (11) binds once, then $C^{lr}(i, j'', v''') = C^{lr}(i, j', v')$. If (12) binds, then we can show that $C(i, j')$ is feasible for the couple $(i, j'')$, as in the proof of Lemma 3. It follows by revealed preference that all women prefer to share risk with the more risk-averse man in the long run in this case as well. It also follows that $\bar{c}(i, j'', v''') > \bar{c}(i, j', v')$, as in Lemma 3. $$

**Proposition 4.** Suppose no aggregate risk. GID holds as long as $v'$ is such that we are in case (i), (iii), or (iv) for all potential couples. That is, for such $v'$ the stable matching is positive assortative.

**Proof.** Lemma 3 and Lemma 4 say that all women always weakly prefer to share risk with a more risk-averse man from the moment an enforcement constraint binds. Till that moment, they consume $\hat{c}(i, j, v)$.

In case (i), when $v$, along with the other parameters, is such that the relevant PC binds in each state at time 1, PAM follows from Lemma 3. In case (iv), where no PC is ever binding, any matching is payoff equivalent with the positive assortative equilibrium, as with static contracts. Next, we consider case (iii), where a PC binds once and perfect risk sharing occurs in the log run.

First, suppose that Mr $j$’s constraint binds once and perfect risk sharing occurs afterwards. Then $\hat{c}(i, j, v)$ is implicitly given by

$$\frac{\pi - \pi \beta}{1 - \pi \beta} u(j, z - \hat{c}(i, j, v)) + \frac{1 - \pi}{1 - \pi \beta} u(j, z - \hat{c}(i, j, v)) = v,$$

where $\bar{c}(i, j, v)$ is independent of the risk preferences if Ms $i$, hence $\hat{c}(i, j, v)$ is as well. That is, $\bar{c}(i'', j, v) = \bar{c}(i', j, v)$ and $\hat{c}(i'', j, v) = \hat{c}(i', j, v)$, $\forall (j, v)$. $\exists \bar{v}'$ such that

$$\frac{\pi - \pi \beta}{1 - \pi \beta} u(i', \hat{c}(i', j', v')) + \frac{1 - \pi}{1 - \pi \beta} u(i', \bar{c}(i', j', v'))$$

$$= \frac{\pi - \pi \beta}{1 - \pi \beta} u(i', \hat{c}(i', j'', v'')) + \frac{1 - \pi}{1 - \pi \beta} u(i', \bar{c}(i', j'', v'')).$$
From Lemma 4 we know that \( \bar{c}(i, j'', \bar{v}'') > \bar{c}(i, j', v') \), \( \forall i \). We also know that \( \hat{c}(i, j, v) > \bar{c}(i, j, v) \), \( \forall (i, j, v) \) here. Further, the above equality implies \( \hat{c}(i', j'', \bar{v}'') < \hat{c}(i', j', v') \). Hence the consumption process for Ms \( i' \) is less risky in the sense of Jewitt if she shares risk with Mr \( j'' \) rather than Mr \( j' \), given that she is indifferent between the two potential spouses. Then, the more risky woman, Ms \( i'' \), prefers the less risky process as well, hence GID.

Second, suppose that Ms \( i' \)'s constraint binds once and perfect risk sharing occurs afterwards. Then \( \hat{c}(i, j, v) \) is implicitly given by

\[
\frac{(1 - \pi) - (1 - \pi)\beta}{1 - (1 - \pi)\beta} u(j, z - \hat{c}(i, j, v)) + \frac{\pi}{1 - (1 - \pi)\beta} u(j, z - \hat{c}(i, j, v)) = v,
\]

where \( \hat{c}(i, j, v) \) is independent of the risk preferences of Mr \( j \), and \( \hat{c}(i, j, v) < \hat{c}(i, j, v) \). As in the previous case, \( \exists \bar{v}'' \) such that \( i' \) is indifferent between the two potential spouses. We know that \( \hat{c}(i', j'', \bar{v}'') = \hat{c}(i', j', v') \). The above equality then implies that \( \hat{c}(i', j'', \bar{v}'') = \hat{c}(i', j', v') \).

We also know that \( \hat{c}(i'', j'', \bar{v}'') = \hat{c}(i', j', v') \). Hence, we have

\[
\hat{c}(i', j'', \bar{v}'') = \hat{c}(i', j', v') < \hat{c}(i'', j'', \bar{v}'') = \hat{c}(i'', j', v') < \hat{c}(i', j', v') = \hat{c}(i', j', v')
\]

and the consumption process that Mr \( j' \) is getting sharing risk with Ms \( i'' \) is less risky in the sense of Jewitt than the one he is getting with \( i' \). Then, the more risk-averse man, Mr \( j'' \), gets a higher utility from the process \( (\hat{c}(i'', j', v'), \hat{c}(i'', j', v')) \) than the process \( (\hat{c}(i', j', v'), \hat{c}(i', j', v')) \). Hence, Ms \( i'' \) can consume \( \hat{c} > \hat{c}(i'', j', v') \) in the transition periods while still giving \( \bar{v}'' \) to Mr \( j'' \). This means that Ms \( i'' \) prefers to share risk with Mr \( j'' \) rather than Mr \( j' \), hence GID.

\[ \square \]

In case (ii) each agent’s consumption may take three different values, two in the long run and one during the transition. It turns out that this makes the analysis significantly more complicated. More importantly, a new effect is relevant: the transferability of utility during the transition. In particular, agents may make a positive net transfer in their unfavorable state during the transition. A less risk-averse agent has a comparative advantage in making such a transfer.

We provide an algorithm to verify whether GID or GDD holds in numerical examples. We consider cases where partial risk sharing occurs in the long run at least for the couple \( (i', j') \), otherwise we are back to case (iii), and partial or perfect risk sharing occurs in the long run for the other three potential couples. We show how to find stable matchings in such cases. This is in order to provide two examples: a stable matching which is negative assortative and one which is positive assortative.
Algorithm.

Step 0. Specify endowment processes, utility functions, and $\beta$.

Step 1. Compute the long-run allocation for each potential couple. If partial risk sharing occurs, this involves solving a system of two equations, (11) and (12), with two unknowns, consumption by Ms $i$ in state $\pi$ and in state $\overline{\pi}$. If perfect risk sharing is self-enforcing in the long run, compute both the lower limit and the upper limit of consumptions, determined by Ms $i$’s and Mr $j$’s binding PC.

Step 2. Choose $v'$ (making sure that the feasible set is non-empty).

Step 3. Use

$$\frac{\pi - \pi \beta}{1 - \pi \beta} u (j, z - \hat{c}(i', j', v')) + (1 - \pi) u (j, z - \bar{c}(i', j', v')) + \frac{\pi \beta - \pi^2 \beta}{1 - \pi \beta} u (j, z - c(i, j', v')) = v',$$

$i = \{i', i''\}$, to compute $\hat{c}(i', j', v')$ and $\bar{c}(i'', j', v')$. If perfect risk sharing occurs in the long run for the couple $(i'', j')$, then $\bar{c}(i'', j', v') = c(i'', j', v')$ is determined by Ms $i''$’s (Mr $j''$’s) binding PC if $v'$ is high (low). Compute the per-period utility for Ms $i''$ sharing risk with Mr $j''$ as

$$\mathcal{E}u(i'', j', v') = \frac{\pi - \pi \beta}{1 - \pi \beta} u (i'', \hat{c}(i'', j', v')) + (1 - \pi) u (i'', \bar{c}(i'', j', v')) + \frac{\pi \beta - \pi^2 \beta}{1 - \pi \beta} u (i'', c(i'', j', v')).$$

Step 4. Compute $v''$ such that Ms $i'$ is indifferent between the two potential partners. That is, $v''$ is implicitly given by

$$\frac{\pi - \pi \beta}{1 - \pi \beta} u (i', \hat{c}(i', j', v')) + (1 - \pi) u (i', \bar{c}(i', j', v')) + \frac{\pi \beta - \pi^2 \beta}{1 - \pi \beta} u (i', c(i', j', v__)) = v'',$

$$= \frac{\pi - \pi \beta}{1 - \pi \beta} u (i', \hat{c}(i', j'', v'')) + (1 - \pi) u (i', \bar{c}(i', j'', v'')) + \frac{\pi \beta - \pi^2 \beta}{1 - \pi \beta} u (i', c(i', j'', v'')).$$

Step 5. Use

$$\frac{\pi - \pi \beta}{1 - \pi \beta} u (j'', z - \hat{c}(i'', j'', v'')) + (1 - \pi) u (j'', z - \bar{c}(i'', j'', v'')) + \frac{\pi \beta - \pi^2 \beta}{1 - \pi \beta} u (j'', z - c(i'', j'', v'')) = v''$$

to compute $\hat{c}(i'', j'', v'')$. The per-period utility for Ms $i''$ sharing risk with Mr $j''$ is then given by

$$\mathcal{E}u(i'', j'', v'') = \frac{\pi - \pi \beta}{1 - \pi \beta} u (i'', \hat{c}(i'', j'', v'')) + (1 - \pi) u (i'', \bar{c}(i'', j'', v'')) + \frac{\pi \beta - \pi^2 \beta}{1 - \pi \beta} u (i'', c(i'', j'', v'')).$$
Step 6. Compare $\mathbb{E}u(i'', j'', v'')$ and $\mathbb{E}u(i'', j', v')$. If $\mathbb{E}u(i'', j'', v'') > \mathbb{E}u(i'', j', v')$, then we have found a stable matching which is positive assortative. If $\mathbb{E}u(i'', j'', v'') < \mathbb{E}u(i'', j', v')$, then we have found a stable matching which is negative assortative.

Note that three agents, Ms $i'$, Mr $j'$, and Mr $j''$, are indifferent between their respective potential partners, only Ms $i''$ may strictly prefer one or the other partner. Hence, it is easy to see that the matching is stable.

Next, we show that there exist both negative and positive assortative stable matchings with dynamic contracts, even when there is no aggregate risk. Our first example is the same economy as in Example 1 above. We set $v' = -1.1$. We find that the stable matching is negative assortative. For our second example we increase $\beta$ from $2/3$ to 0.7. We set $v' = -1.1$ again. We find that the stable matching is positive assortative in this alternative economy.

**Example 1 with dynamic contracts.** Recall that $u(i', y) = u(j', y) = -2/\sqrt{y}$, $u(i'', y) = u(j'', y) = -1/y$, $\beta = 2/3$, $W = (2, 4)$, $M = (4, 2)$, and the two states occur with equal probabilities. Consider $v' = -1.1$. Applying the above algorithm, we find $v'' = -0.3059$ and $\mathbb{E}u(i'', j'', v'') = -0.4139 < -0.4072 = \mathbb{E}u(i'', j', v')$. This means that we have found a stable matching which is negative assortative.

Table 1 presents the three consumption values, $\hat{c}$, $\bar{c}$, and $\underline{c}$, for all four potential couples.

<table>
<thead>
<tr>
<th>Couple</th>
<th>$\hat{c}$</th>
<th>$\bar{c}$</th>
<th>$\underline{c}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(i', j')$</td>
<td>1.207</td>
<td>2.609</td>
<td>3.391</td>
</tr>
<tr>
<td>$(i', j'')$</td>
<td>1.151</td>
<td>2.891</td>
<td>3.195</td>
</tr>
<tr>
<td>$(i'', j')$</td>
<td>1.635</td>
<td>2.805</td>
<td>3.109</td>
</tr>
<tr>
<td>$(i'', j'')$</td>
<td>1.525</td>
<td>3.000</td>
<td>3.000</td>
</tr>
</tbody>
</table>

$\hat{c}$ is consumption by Ms $i$ as long as state $\bar{s}$ occurs, i.e., Mr $j$ gets the favorable shock, initially. $\bar{c}$ ($\underline{c}$) is her consumption in state $\bar{s}$ ($\underline{s}$) in the long run.

These numbers make it clear that the long-run allocation yields higher utility when sharing risk with a more risk-averse agent. However, consumption during the transition is lower for Ms $i$ when sharing risk with a more risk-averse agent. Further, the lower initial consumption is more costly, the more risk averse Ms $i$ is. Hence the negative assortative matching result. Note that for the couple $(i'', j'')$ perfect risk sharing is self-enforcing in the long run, however, only if total consumption is shared equally.
Example 2. Set $\beta = 0.7$ now. All other parameters are as in Example 1. Consider $v' = -1.1$ again. Applying the above algorithm, we find $v'' = -0.3065$ and $E u(i'', j'', v'') = -0.3986 > -0.4204 = E u(i'', j', v')$. This means that we have found a stable matching which is positive assortative. Table 2 presents the three consumption values, $\hat{c}$, $\bar{c}$, and $\bar{c}$, for all four potential couples.

<table>
<thead>
<tr>
<th>Table 2: Consumption by Ms $i$, $\beta = 0.7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\langle i', j' \rangle$</td>
</tr>
<tr>
<td>$\langle i', j'' \rangle$</td>
</tr>
<tr>
<td>$\langle i'', j' \rangle$</td>
</tr>
<tr>
<td>$\langle i'', j'' \rangle$</td>
</tr>
</tbody>
</table>

$\hat{c}$ is consumption by Ms $i$ as long as state $\bar{s}$ occurs, i.e., Mr $j$ gets the favorable shock, initially. $\bar{c}$ ($\bar{c}$) is her consumption in state $\bar{s}$ ($\bar{s}$) in the long run.

Unlike in the economy of Example 1, now perfect risk sharing is self-enforcing in the long run for any consumption level between 2.963 and 3.037 for the couple $\langle i'', j'' \rangle$. Ms $i''$ optimally chooses 2.963 for herself and gives 3.037 to Mr $j''$. This allows Ms $i''$ to increase her consumption in the transition period. As a result she is better off sharing risk with $j''$ rather than $j'$.

Figure 3 illustrates how both GID and GDD may occur. CE$'(i, j)$ (CE$''(i, j)$) denotes the certainty equivalent of the long-run allocation for the couple $\langle i, j \rangle$ computed using the preferences of the less (more) risk-averse agent. Choose a utility level for Mr $j'$, $v'$. Remember that $\hat{c}$ is consumption by Ms $i$ during the transition, hence the indifference curves of Mr $j$ are upward sloping. The point where $v'$ crosses CE$'(i', j')$ determines $\hat{c}(i', j', v')$. Then we can draw an indifference curve for Ms $i'$ which crosses the point $(\hat{c}(i', j', v'), CE'(i', j'))$. The point where this indifference curve crosses CE$'(i', j'')$ determines $\hat{c}(i', j'', v'')$. However, the certainty equivalent for Mr $j''$ of $\langle i', j'' \rangle$'s long-run allocation is lower, because he is more risk averse. In particular, it is on the CE$''(i', j'')$ line. Through this point we can then draw the relevant indifference curve of Mr $j''$, where his utility is $v''$. The point where $v'$ crosses CE$'(i'', j'')$ determines $\hat{c}(i'', j', v')$. The square on the CE$''(i', j'')$ line then represents the allocation for Ms $i''$ sharing risk with Mr $j'$. Finally, the point where $v''$ crosses CE$''(i'', j'')$ determines $\hat{c}(i'', j'', v'')$. The square on the CE$''(i'', j'')$ line represents the allocation for Ms $i''$ sharing risk with Mr $j''$. It is clear that we could draw indifference curves for Ms $i''$ so that she prefers the latter to the former allocation, and also so that the opposite holds. That is,
Figure 3: Assortative matching with dynamic contracts. The horizontal axis represents consumption by Ms $i$ during the transition. The vertical axis represents the certainty equivalent of the long-run allocation. By assumption, Ms $i'$ and Mr $j'$ (as well as Ms $i''$ and Mr $j''$) have the same preferences. Hence, $CE'(i',j') = CE'(i'',j')$ and $CE''(i',j'') = CE''(i'',j')$. The squares represent the two options for Ms $i''$. These two points cannot be unambiguously ranked.

The two squares cannot be unambiguously ranked for any utility function more concave than that of Ms $i'$.

4.2.2 Non-negative correlation between endowments

Now both $M$ and $W$ are in $C^+$. We assume that $x(i, W) > x(j, M)$, $\forall (i, j)$, hence we can disregard the constraints (10) and (13) in the long run.

Recall that $C^{lr}(i, j, v) = (\bar{c}(i, j, v), c(i, j, v))$ denotes the vector of the long-run consumptions for Ms $i$ sharing risk with Mr $j$. We first assume that partial risk sharing occurs in the long run, which is relevant for cases (i) and (ii). We first make comparative static statements about the long run. Then we will take into account the transition, as in the previous section.

Lemma 5. Assume a non-negative correlation between endowments, and that a PC binds in both states in long run for all potential couples. Then $C^{lr}(i, j'', v'') \succeq_i C^{lr}(i, j', v')$, $\forall i$, and $C^{lr}(i', j, v) \succeq_j C^{lr}(i'', j, v)$, $\forall j$. That is, all women (men) weakly prefer to share risk with a more (less) risk-averse man (woman) in the long run. Further, $\bar{c}(i, j'', v'') \geq \bar{c}(i, j', v')$ and $c(i', j, v) \leq c(i'', j, v)$, and the inequalities are strict if some insurance is self-enforcing.
We omit the proof, because it is similar to the proof of Lemma 3.

Lemma 5 says that all women weakly prefer to share risk with a more risk-averse man, while all men weakly prefer to share risk with a less risk-averse woman, from the moment an enforcement constraint binds. In case (i), when \( v \), along with the other parameters, is such that the relevant PC binds in each state at time 1, NAM follows from Lemma 5.

In case (ii), women consume \( \hat{c}(i, j, v) \), and men consume \( \hat{z} - \hat{c}(i, j, v) \) or \( \hat{z} - \hat{c}(i, j, v) \) as long as no PC binds initially. If no PC binds as long as state \( \pi \) occurs initially, Ms \( i \) can provide additional utility to Mr \( j \) by making a positive transfer. Unlike in the case without aggregate risk, this transfer is made in the state where her consumption is higher in the long run. However, without any transfer, her consumption in the transition period is in between the long-run values. Unlike in the case without aggregate risk, it is not clear whether a more or a less risk-averse agent has a comparative advantage in making this transfer. Hence, the direction of the third force, as well as the overall effect, is ambiguous, and both negative and positive assortative matchings can be stable.

However, for any parametrized economy and given any \( v' \) such that the set of feasible allocations is non-empty for both \( \langle i', j' \rangle \) and \( \langle i'', j'' \rangle \), we can determine the assortativeness of the stable matching, as in the case without aggregate risk. We only have to modify the algorithm described in the previous section to take into account that aggregate consumption varies across states. In the following example the stable matching is negative assortative.

**Example 3.** Utility functions and men’s income process are as in Example 1. Suppose that women get 3 in both states, and \( \beta = 0.75 \). Consider \( v' = -1.17 \). Applying the above algorithm, we find \( v'' = -0.3414 \), and \( \mathbb{E}(v'', j'', v'') = -0.33317 < -0.3271 = \mathbb{E}(v'', j', v') \). This means that we have found a stable matching which is negative assortative. Table 3 presents the three consumption values, \( \hat{c}, \hat{\bar{c}}, \text{and} \hat{\zeta} \), for all four potential couples.

<table>
<thead>
<tr>
<th></th>
<th>( \hat{c} )</th>
<th>( \hat{\bar{c}} )</th>
<th>( \hat{\zeta} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \langle i', j' \rangle )</td>
<td>3.214</td>
<td>3.038</td>
<td>2.978</td>
</tr>
<tr>
<td>( \langle i', j'' \rangle )</td>
<td>2.933</td>
<td>3.413</td>
<td>2.787</td>
</tr>
<tr>
<td>( \langle i'', j' \rangle )</td>
<td>3.251</td>
<td>3.068</td>
<td>2.959</td>
</tr>
<tr>
<td>( \langle i'', j'' \rangle )</td>
<td>2.830</td>
<td>3.350</td>
<td>2.825</td>
</tr>
</tbody>
</table>

\( \hat{c} \) is consumption by Ms \( i \) as long as state \( \pi \) occurs, i.e., Mr \( j \) gets the favorable shock, initially. \( \hat{\bar{c}} \) (\( \hat{\zeta} \)) is her consumption in state \( \pi \) (\( \hat{\bar{c}} \)) in the long run.
We now consider the two cases where perfect risk sharing occurs in the long run. In case (iv), where no PC is ever binding for any couple, any matching is negative assortative, as in the full commitment case. Case (iii) with aggregate risk turns out to be significantly more complicated than without aggregate risk. First of all, each spouse’s consumption varies across the two states, hence overall there may be three consumption values, just as in the case without aggregate risk and with partial risk sharing in the long run. Further, even when constraints (9) and (12) bind, the consumption allocation still depends on the preferences of Ms $i$ as well, because how aggregate risk is shared optimally depends on the risk preferences of both partners. However, in this case as well, we can use the algorithm described in the previous section to determine the assortativeness of the stable matching given any feasible $v'$ for any parametrized economy.

5 Concluding remarks

Recent theoretical literature on the marriage problem with imperfectly transferable utility predicts that matching is negative assortative with respect to spouses’ risk attitudes. This paper shows that this prediction is not robust to relaxing the assumption of perfect risk sharing once couples are formed. We have shown that when commitment is limited, positive assortative matching may occur if the correlation between men’s and women’s incomes is negative. This implies less heterogeneity in risk attitudes within households and more heterogeneity across households compared to the full commitment case.

The reason behind positive assortativeness is that more risk-averse agents can be more attractive risk sharing partners as long as their consumption is less risky than their endowment, given that the value of their outside option, hence their decision power, is lower. In other words, less risk sharing is sustainable with voluntary transfers between less risk-averse agents, making both sides prefer the more risk-averse agents on the other side of the matching market.

Our main result that lack of commitment provides a motive for negative assortative matching could be best tested provided an exogenous change in the enforcement technology. For example, the widespread adoption of unilateral divorce laws in the United States in the 1970s could be looked at from this perspective. We leave this to future work.

Limited commitment is relevant whenever matched agents are likely to interact repeatedly. Hence, the results of this paper may shed light on assortative matching not just of spouses, but also of agents working together on a project in a partnership, of sharecroppers, and of firms in a joint venture.
References


