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This version: September 2012
(March 2011)

Barcelona GSE Working Paper Series

Working Paper nº 533

Endogenous Growth and Wave-Like Business Fluctuations*

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September 2012

Abstract

This paper argues that the observed long lags in innovation implementation rationalize Schumpeter's statement that "wave-like fluctuations in business ... are the form economic development takes in the era of capitalism." Adding implementation delays to an otherwise standard endogenous growth model with expanding product variety, the equilibrium path admits a Hopf bifurcation where consumption, R&D and output permanently fluctuate. This mechanism is quantitatively consistent with the medium-term movements of US aggregate output. An optimal allocation may be restored at equilibrium by the mean of a procyclical subsidy, needed to generate additional consumption smoothing. Finally, a procyclical R&D subsidy rate moving around 10% and designed to half consumption fluctuations increases the growth rate from 2.4% to 3.4% with a 9.6% (compensation equivalent) increase in welfare.

JEL Classification O3, E32

Keywords Endogenous growth; endogenous fluctuations; innovation implementation; time delays; medium-term cycles; Hopf bifurcation

*We thank Pascal Benassy, Jess Benhabib, Raouf Boucekkine, Diego Comin, and Boyan Jovanovic for their valuable comments. Omar Licandro acknowledges the financial support of the Spanish Ministry of Sciences and Technology (ECO2010-17943).

1 Introduction

The conjecture that in the modern era business fluctuations and economic growth are two faces of the same coin comes back to Schumpeter [54], who pointed out that "wave-like fluctuations in business ... are the form economic development takes in the era of capitalism." Starting from this premise, Schumpeter raised the key question of "why is it that economic development does not proceed evenly ..., but as it were jerkily; why does it display those characteristics ups and downs?" When searching for an answer, he drew attention to the critical fact that innovations "appear en masse at intervals", "discontinuously in groups or swarms," which "signifies a very substantial increase in purchasing power all over the business sphere."

Following the seminal work by Aghion and Howitt [3], Grossman and Helpman [39] and Romer [52], important developments have been undertaken in the last twenty years addressed to improve our understanding on the main channels through which innovations promote development and growth. Endogenous growth theory is in a fundamental sense Schumpeterian, since it stresses the critical role played by innovations in the observed growth of total factor productivity. However, little has been written since then on the relation between innovation and business fluctuations.

A natural candidate for the study of Schumpeterian wave-like business fluctuations is the observed long delay elapsed between the realization of R&D activities and the implementation and adoption of the associated innovations.[1] Schumpeter [54]'s description of the periodicity of business fluctuations is, in this sense, very appealing: "the boom ends and the depression begins after the passage of the time which must elapse before the products of the new enterprise can appear on the market." The argument in this paper is very close to Schumpeter's description: waves of innovations arrive en masse, moving the economy to a boom; the associated increase in productivity raises purchasing power all over the business sphere, inducing research activities to flourish; but, the new products will take a while to develop; when the new wave of innovations is eventually implemented, the new products enter the market producing a second boom, which will generate a third, then a fourth and so on and so for.

It is important to notice that Schumpeterian wave-like business fluctuations as described in the previous paragraph substantially differ from the type of fluctuations studied in modern business cycle theory. Inspired on Kydland and Prescott [46], it has focused on the study of high frequency movements, those between 4 and 40 quarters. Schumpeter, indeed, was more interested in medium (Juglar) and low (Kondratieff) frequency movements lasting around 10 and 50 years, respectively. A description of economic fluctuations more in accordance with the Schumpeterian's view was recently suggested by Comin and Gertler [23]. They estimate the

[1] Comin and Hobijn [25] study the pattern of technology diffusion around the globe and find that countries on average adopt technologies 47 years after their invention. Comin et al [26] find that, when compared to the US, lags in the use of technology are measured in decades for most countries. Adams [2] estimates that academic knowledge is a mayor contributor to productivity growth, but its effects lag roughly 20 years. Mansfield [47] estimates the mean adoption delay of twelve mayor 20th-century innovations in 8 years. Jovanovic and Lach [41] estimate at 8.1% the annual diffusion rate of new products.

medium-term movements of US per capita GDP growth by analyzing frequencies between 40 and 200 quarters, and find that it permanently undulates with a periodicity of around 11 years and an amplitude of around 8 percentage points from peak to valley. This paper focuses on Juglar cycles or, equivalently, on medium terms movements.

In this paper, wave-like fluctuations are modeled in a simple way by adding an implementation delay to an otherwise textbook endogenous growth model with expanding product variety. It shows that the equilibrium path admits a Hopf bifurcation where consumption, research and output permanently fluctuate. The main mechanism relating growth to wave-like fluctuations is based on the assumption that innovations being fundamental for economic growth require long implementation and adoption lags. The mechanics is the following. Let the economy initially react to a permanent positive shock by some concentration of research activities, which makes new ideas to appear en masse. This is the standard reaction of a dynamic general equilibrium model when the initial stock of (intangible) capital is relatively low. However, the economic effects of this wave of research activity will be delayed in time. When a swarm of new businesses will become eventually operative, the associated increase in productivity will inject additional resources to the economy –"a substantial increase in purchasing power" in Schumpeter's words. Consumption smoothing makes the rest by allocating the additional resources to create a second wave of innovation activities. This process will repeat again and again as time passes. In a simple quantitative exercise, where parameters are set to match some key aggregate features of the US economy, we show that the model is able to replicate medium-term movements of similar periodicity and amplitude to those observed by Comin and Gertler. In this sense, the suggested mechanism relating the sources of growth and business fluctuations is not only theoretically possible but quantitatively relevant.

Additionally, the paper makes some welfare considerations resulting in a procyclical R&D policy. Firstly, it shows that detrended consumption is constant from the initial time in an optimal allocation, and both R&D and output converge by oscillations. Second, it proves that a procyclical subsidy/tax scheme would restore optimality. This results is due to the fact that consumption fluctuates less in the optimal allocation, implying that an optimal subsidy has to generate additional R&D investments during booms than during recessions. Interestingly, the policy does not affect output in the short run, since for a long while production is determined by past R&D investments; new investments will eventually become productive after a long delay. Finally, it quantitatively finds that a procyclical 10% subsidy rate halving consumption fluctuations will increase the growth rate from 2.4% to 3.4% with a 9.6% increase in welfare.

The model in this paper belongs to the literature on dynamic general equilibrium with time delays, including vintage capital, time-to-build and demographic theories. Firstly, fluctuations in the vintage capital literature are the result of machine replacement, as described in Benhabib and Rustichini [15], Boucekkine et al [19] and Caballero and Hammour [21].[2] Following the lumpy investment literature, initiated by Doms and Dunne [31], Cooper et al [27] find robust evidence on the existence of machine replacement, but little support for the contribution of

[2]See also Boucekkine and de la Croix [16].

machine replacement to the understanding of observed business fluctuations. Second, since the seminal paper by Kydland and Prescott [46], investment lags have been shown to make the business cycle highly persistent. Asea and Zak [5] and, more recently, Bambi [6] go further and prove that time-to-build may generate endogenous fluctuations. However, time-to-build delays are short relative to Junglar cycles, since they last some few quarters only. Finally, Boucekkine et al [17] find that in a demographic model with realistic survival probabilities and a vintage human capital structure, the adjustment of the economy may generate output movements of the order of a Kondratieff cycle. For all these reasons, implementation delays are more appealing for understanding Schumpeterian business fluctuations than vintage (human and physical) capital or time-to-build arguments.

There is an extensive literature on endogenous competitive equilibrium cycles, along the seminal contributions of Benhabib and Nishimura [13][14], where the raising of these persistent cycles arrive through an Hopf bifurcation in a multi-sector growth model. In our paper, endogenous cycles still emerge through an Hopf bifurcation but their main source is the presence of implementation delays instead of multiple sectors.

This paper is also related to Matsuyama [48] and Francois and Lloyd-Ellis [36], among the few exceptions connecting endogenous growth with cycles. Firstly, Matsuyama [48] shows that, under some conditions regarding the saving rate, endogenous cycles arrive in a discrete time Rivera-Batiz and Romer [51] endogenous growth model, where monopoly rents last only one period and implementing an innovation entails fixed costs. Along the cycle, the economy moves periodically from a Neoclassical regime to an AK regime. Research activities come en masse as in Schumpeter's theory, but they are countercyclical. In our theory, indeed, the economy is always in an AK regime and innovation activities are procyclical. Second, Francois and Lloyd-Ellis [36] link growth and cycles combining animal spirits, such as in Schleifer [53], to a Schumpeterian endogenous growth model. In their framework, a cyclical equilibrium exists because firms are interested in delaying implementation to the boom in order to maximize the expected length of incumbency. In our model, cycles are also related to implementation delays. However, differently from Francois and Lloyd-Ellis [36], business fluctuations are not the consequence of animal spirits but result from a Hopf bifurcation, and they materialize as medium-term movements instead of happening at high frequencies.

The idea that delayed gains in productivity may generate persistence has been deeply studied in the recent literature on "news shocks" –see Beaudry and Portier [10].[3] However, the main source of fluctuations in this literature remains exogenous. In our theory, indeed, current research activities and the associated future innovations may be seen as perfectly forecasted, endogenous news shocks. Endogenous news are at the basis of the the cyclical behavior of our economy, since more resources are allocated to produce current news when past news realize.

Implementation delays in this paper are indeed very different from the delay elapsing between the arrival of a general purpose technology (GPT) and its implementation. In fact, GPT

[3] More recently, Comin et al [24] stress the importance of endogenous adoption in the amplification of these shocks.

refers to major technology breakthroughs, whose implementation requires costly and very long restructuring. According to David [28], the implementation of a new GPT may generally take several decades: for example, the electric dynamos took three decades to attain a fifty percent diffusion level in the U.S.. Then the consequences of a discovery of a GPT may well reproduce the low (Kondratieff) frequency movements in the data but not the medium ones which are the objective of our analysis.

Finally, this paper shares with Comin and Gertler [23] the view that lags of technology adoption do generate medium-term movements in models of endogenous productivity growth. In Comin and Gertler's words, medium-term movements "reflect a persistent response of economic activity to the high-frequency fluctuations normally associated with the cycle." They are the endogenous reaction to a sequence of exogenous shocks. In our theory, indeed, medium-term movements are endogenous and self-sustained, and they could emerge independently of the existence of any high frequency exogenous productivity process.

The paper is organized as follows. Sections 2 and 3 describe the economy and define the decentralized equilibrium, respectively. The main dynamic properties are studied in Section 4. In particular, it proves the existence of a persistent cycle. Section 5 quantitatively studies its empirical relevance. Section 6 analyzes the optimal allocation and suggests a countercyclical R&D subsidy as a Pareto improving policy. A counterfactual exercise is performed showing that a 10% R&D countercyclical subsidy halving consumption fluctuations generates first order welfare gains. Finally, Section 7 concludes.

2 The Economy

The economy is populated by a continuum of infinitely lived, identical households of unit measure, holding a constant flow labor endowment of one unit. There is a sole final good, used for consumption purposes only. Household *preferences* are represented by:

$$U = \int_0^\infty \log(c_t)\, \mathrm{e}^{-\rho t} \mathrm{d}t, \tag{1}$$

where c_t is per capita consumption and $\rho > 0$ represents the subjective discount rate.

In line with the literature on expanding product variety, the final consumption good is produced by the mean of a CES technology defined on a continuum of intermediary inputs in the support $[0, n]$. Differently from the existing literature, we assume that adopting new technologies requires a time delay $d > 0$, meaning that varieties discovered at time t become operative at time $t + d$. It can be interpreted as an *implementation delay* which elapses from the discovery of a new variety to its economic implementation. As usual, the extent of product variety n is assumed to represent also the aggregate state of knowledge. Knowledge is assumed to positively affect the productivity of the consumption sector as an externality, i.e., n has a positive effect on the production of the consumption good. Knowledge produced at time t is assumed to become public information at the time $t + d$, when for the first time the innovation

is produced and sold. Under the previous assumptions, the consumption good technology is

$$c_t = n_{t-d}^{\nu+1-\frac{1}{\alpha}} \left(\int_0^{n_{t-d}} x_t(j)^{\alpha} \, \mathrm{d}j \right)^{\frac{1}{\alpha}}, \qquad 0 < \alpha < 1 \tag{2}$$

where n_{t-d} represents the extent of operative varieties at time t, and $x_t(j)$ is the amount of the intermediary input j used at time t in the production of c_t. This consumption good technology implies a constant (and equal) elasticity of substitution between every pair of varieties, $\theta = \frac{1}{1-\alpha} > 1$. The parameter ν is the elasticity of the externality n, but also the return to specialization as explained extensively in Ethier [34] and Benassy [12]; from now on we assume $\nu = 1$ to simplify our analysis and at the same time to distinguish between the markup charged by the monopolistic firms producing $x(j)$ and the degree of returns to specialization.[4] The assumption that the externality operates only through the measure of operative varieties n_{t-d} is consistent with the *love of variety* argument as suggested by Dixit and Stiglitz [30].

Technology in the intermediary sector is assumed to be symmetric across varieties

$$x_t(j) = l_t(j), \tag{3}$$

where $l_t(j)$ is labor allocated to the production of variety j. Total labor L allocated to the production of the intermediary sector is given by

$$\int_0^{n_{t-d}} x_t(j) \, \mathrm{d}j = L_t. \tag{4}$$

Notice that $L_t \in [0, 1]$, since the total labor endowment is one.

An efficient allocation of labor to the production of the consumption good, spreading through the intermediary sector, results from maximizing (2) subject to (4). It is easy to see that an efficient allocation is symmetric, meaning $x_t(j) = x_t$ for all j, which implies

$$c_t = n_{t-d} L_t \quad \textit{and} \quad n_{t-d} x_t = L_t. \tag{5}$$

As stated above, labor allocated to the production of the consumption good benefits from a knowledge externality, n, which comes linearly in the reduced form of the consumption goods technology (5). In the following sections, we show that optimal and equilibrium allocations are both efficient in the sense defined above –see Koeninger and Licandro [43].

Finally, *R&D* activities are also assumed to be linear on labor and addressed to the creation of new intermediary inputs. The innovation technology creating these new varieties is assumed to be:

$$\dot{n}_t = A n_{t-d} (1 - L_t), \tag{6}$$

where $1 - L_t$ is labor assigned to R&D production, its marginal productivity depending on parameter A, $A > 0$. It is also assumed that the R&D sector benefits from a positive externality depending linearly on the extent of operative varieties.

[4] In the case of zero delay, the main qualitative properties of the model do not depend on the elasticity being unity –see Benassy [12]. However, when the implementation delay is strictly positive, a unit elasticity allows for a mathematical study of the main properties of the model, which would not be the case otherwise.

Note that consumption and R&D technologies, (5) and (6) respectively, collapse to

$$\dot{n}_t = A\left(n_{t-d} - c_t\right). \tag{7}$$

The AK structure of the model, see Rebelo [50], can be easily seen if the extent of product variety n_{t-d} is interpreted as (intangible) capital. In the following, we will refer to (7) as the feasibility constraint.[5]

3 Decentralized Equilibrium

The economy is decentralized in the standard way. The market for the final consumption good is supposed to be perfectly competitive, so that individuals and firms take the consumption price, normalized to unity, as given. Innovations are protected by an infinitely lived patent and the market for intermediary inputs is monopolistically competitive. The R&D sector is perfectly competitive, implying that research firms make zero profits. Finally, the labor market is also assumed to be perfectly competitive.

Final Good Firms – A representative firm produces the consumption good by the mean of technology (2). It takes intermediary prices as given and maximizes profits by choosing $x_t(j)$ for $j \in [0, n_{t-d}]$:

$$\max \quad c_t - \int_0^{n_{t-d}} p_t(j) x_t(j) dj \tag{8}$$

subject to the consumption good technology (2), while $p(j)$ indicates the relative price of the intermediate good j. From the first order condition associated to this problem the inverse demand function

$$p_t(j) = n_{t-d}^{2\alpha-1} \left(\frac{c_t}{x_t(j)}\right)^{1-\alpha} \tag{9}$$

can be easily derived.

Intermediary Good Firms – Firms producing intermediary goods operate under monopolistic competition. They maximize profits subject to the inverse demand function (9) and the technology constraint (3), which collapses to

$$\pi_t = \max_{p_t} \quad p_t^{\frac{1}{\alpha-1}} (p_t - w_t).$$

The optimal price rule is

$$p_t = \frac{1}{\alpha} w_t, \tag{10}$$

where the real wage w is equal to the marginal cost of production (technology is linear in labor), and $\frac{1}{\alpha}$ represents the markup over marginal costs, which depends inversely on the elasticity of substitution across varieties.

[5] Equivalently, it can be assumed that labor is only used to the production of goods, and output is assigned to both consumption and R&D, with L representing the consumption to output ratio and A the rate at which the consumption good is transformed into innovations.

R&D – Successful researchers receive a patent of infinite life. At equilibrium, the patent value v_t is equal to the present value of the associated flow of monopolistic profits. The corresponding Hamilton-Jacobi-Bellman equation (e.g. Acemoglu [1] page 436) can be written as

$$r_t = \frac{\pi_t}{v_t} + \frac{\dot{v}_t}{v_t}. \tag{11}$$

The value of the patent has at least to cover the innovation cost,

$$v_t \geq \frac{1}{An_{t-d}} = \frac{w_t(1-L_t)}{\dot{n}_t}. \tag{12}$$

Households – The household's intertemporal maximization problem is

$$\max \int_0^\infty \log(c_t)\, \mathrm{e}^{-\rho t} \mathrm{d}t \tag{13}$$

subject to the instantaneous budget constraint

$$\dot{a}_t = r_t a_{t-d} + w_t - c_t \tag{14}$$

and the initial condition $a_t = \bar{a}_t$, for $t \in [-d, 0]$, where $\bar{a}_t$ is an exogenously given continuous positive function defined on the t domain, formally $\bar{a}_t \in C([-d,0];\mathbb{R}_+)$. a_{t-d} represents the value at t of patents produced up to time $t-d$, which refer to variety already implemented in the economy; the patents are also assumed to be owned by households. Non consumed income is then saved in the form of new patents. The households problem is an optimal control problem with delay. The positivity constraints $c_t \geq 0$ and $a_t \geq 0$ are implicitly assumed. It is possible, using the concavity of utility functional and the linearity of the state equation (in line with e.g. in Freni et al. [37]) to prove existence and uniqueness of the optimal solution for such a problem.

Following existing theory (see e.g. Kolmanovskii and Myshkis [44] for the finite horizon case and Agram et al. [4] Theorem 3.1, for the infinite horizon case), a given state-control pair (a_t, c_t) is optimal if there exists an absolutely continuous costate function μ_t such that

$$\frac{e^{-\rho t}}{c_t} = \mu_t \tag{15}$$

$$-\dot{\mu}_t = r_{t+d}\mu_{t+d}, \tag{16}$$

$$\lim_{t\to\infty} a_t \mu_t = 0, \quad \text{or, equivalently,} \quad \lim_{t\to\infty} a_t {c_t}^{-1}\, \mathrm{e}^{-\rho t} = 0. \tag{17}$$

The optimality conditions (15)-(16) collapse into the following Euler-type equation

$$\frac{\dot{c}_t}{c_t} = \underbrace{r_{t+d}}_{R\&D\ returns} \cdot \underbrace{\frac{c_t}{c_{t+d}} e^{-\rho d}}_{discount\ factor} - \rho. \tag{18}$$

The representative household faces then the following trade-off, consuming at time t or buying new patents which will become operative at time $t+d$.

Decentralized Equilibrium – The decentralized equilibrium is symmetric, meaning that (5) holds, and equation (9) becomes

$$p_t = n_{t-d}. \tag{19}$$

Recall that the consumption good is the numeraire, which implies that p_t is the price of the intermediary input relative to the price of consumption. An expansion in product variety improves productivity in the consumption sector, inducing an increase in the relative price of the intermediary input as reflected by (19). From (10) and (19), the wage rate at equilibrium is

$$w_t = \alpha n_{t-d}. \tag{20}$$

Market power makes the equilibrium real wage equal to a fraction α, the inverse of the markup, of labor marginal productivity – aggregate technology is in (5). Consequently, the real wage maps the behavior of current technology.

From (5), (10) and (19), intermediary profits can be written as

$$\pi_t = (1-\alpha)\,\frac{c_t}{n_{t-d}} > 0. \tag{21}$$

Profits are proportional to total sales per firm, the proportionally factor being directly related to the markup rate.

Combining the R&D technology (6), the price rule (10), equations (12), (19), and the free entry condition, we find that the patent value is constant when there is positive research:

$$v_t = \frac{\alpha}{A}, \qquad with \qquad \dot{n}_t \geq 0 \tag{22}$$

Moreover, R&D returns are equal to

$$r_{t+d} = \frac{1-\alpha}{\alpha}\frac{c_{t+d}}{n_t}A. \tag{23}$$

Combining equation (18) with (23) leads to the equilibrium Euler equation

$$\frac{\dot{c}_t}{c_t} = \underbrace{\frac{1-\alpha}{\alpha}\frac{c_{t+d}}{n_t}\,A}_{R\&D\ return}\ \underbrace{\mathrm{e}^{-\rho d}\left(\frac{c_t}{c_{t+d}}\right)}_{discount\ factor} - \rho = \frac{1-\alpha}{\alpha}\,A\,\mathrm{e}^{-\rho d}\,\frac{c_t}{n_t} - \rho. \tag{24}$$

The private return to R&D arrives after a period of length d. For this reason, it has to be discounted using the appropriate ratio of marginal utilities. Moreover, the private return to R&D is different from the social return, which is equal to A. Under log utility, the term in c_{t+d} cancels and the Euler-type equation does not depend on it, on the consumption over number of varieties ratio $\frac{c_t}{n_t}$.

On the other hand the other equilibrium equation can be found from the instantaneous budget constraint taking into account that the assets' market clearing condition is

$$a_t = vn_t \qquad \Rightarrow \qquad \dot{a}_t = v\dot{n}_t$$

since v_t is the constant found in equation (22). We then have the following definition.

Definition 1 Decentralized Equilibrium. *A decentralized equilibrium is a path (c_t, n_t), for $t \geq 0$, verifying the feasibility condition*

$$\dot{n}_t = A(n_{t-d} - c_t), \quad t \geq 0, \tag{25}$$

the Euler-type equation

$$\frac{\dot{c}_t}{c_t} = \frac{1-\alpha}{\alpha} A \, \mathrm{e}^{-\rho d} \frac{c_t}{n_t} - \rho, \quad t \geq 0, \tag{26}$$

the initial condition $n_t = \bar{n}_t$, $\bar{n}_t \in C([-d, 0]; \mathbb{R}_+)$, the transversality condition

$$\lim_{t \to \infty} n_t {c_t}^{-1} \, \mathrm{e}^{-\rho t} = 0, \tag{27}$$

and the irreversibility constraint $\dot{n}_t \geq 0$ for $t \geq 0$.

4 Dynamics of the Decentralized Equilibrium

The study of the DDE system (25)-(26) cannot be done by using standard ODE tools (e.g. the Hartmann-Grossmann Theorem) due to its inherent infinite dimension. The main contribution of this section is to generalize standard ODE tools to an infinite dimensional case. An intuitive argument on the raising of endogenous cycles is also provided at the end of the section.

4.1 Balanced Growth Paths

A balanced growth path (BGP) is defined as a solution of the system (25)-(26) such that, for a suitable $g \in \mathbb{R}_+$,

$$c_t = c_0 e^{gt}; \qquad n_t = n_0 e^{gt}$$

for any $t \geq 0$. At a balanced growth path, the transversality condition (17) is automatically satisfied since $\rho > 0$, while the positivity of g guarantees that the irreversibility constraint is satisfied. So, a BGP is an equilibrium according to Definition 1.

From equation (24), for all $t \geq 0$, the consumption to knowledge ratio is constant

$$\frac{c_t}{n_t} = \frac{\alpha(g+\rho)e^{\rho d}}{(1-\alpha)A}, \tag{28}$$

while the growth rate g is the unique real solution of the transcendental equation

$$Ae^{-gd} - g = \frac{\alpha(g+\rho)e^{\rho d}}{1-\alpha}, \tag{29}$$

which is found by substituting (28) into (25). Let denote it by g_e. Such solution is positive under the following parametric condition:

$$A \geq \frac{\alpha \rho e^{\rho d}}{1-\alpha} =: A^e_{\min}, \tag{30}$$

which we will assume from now on. Note that the strict inequality in (30) implies strict positivity of g_e. Moreover a straightforward application of the implicit function theorem on (29) shows that $\frac{\partial g_e}{\partial A} > 0$ and $\frac{\partial g_e}{\partial \alpha} < 0$, implying that both more productive economies and economies with larger markups grow faster at steady state. We summarize the discussion in the following proposition.

Proposition 1 *The solution of the system (25)-(26) is a BGP if and only if the conditions below are satisfied:*

i) inequality (30) holds;

ii) the growth rate is given by the unique positive solution g_e of (29);

iii) the initial condition $\bar{n}_t$ has the form $\bar{n}_t = n_0 e^{g_e t}$ with $n_0 > 0$ and $t \in [-d, 0]$;

iv) given n_0, c_0 is the solution of (28) for $t = 0$.

For any given $n_0 > 0$ a BGP exists and is unique.

4.2 Linearized System

This section studies the dynamic behavior of the decentralized equilibrium around the BGP when $\bar{n}_t$ is sufficiently close but different from its specification in Proposition 1, point *iii)*.

To accomplish this task we use a linearization procedure which exploits the results of Diekmann et al. [29], Chapters VII-VIII-IX-X, in particular Theorem 6.1, p. 257, Theorem 5.3, p.266 and Theorem 2.7 p.291. More precisely, we consider the detrended and linearized version of the system (25)-(26) around a BGP and use the fact that the linearized system preserves the same stability properties of the non linear one under some conditions which will be explicitly stated.

Let us define $\tilde{x}_t = x_t e^{-g_e t}$, $x_t = \{c_t, n_t\}$, with $\tilde{c}_t, \tilde{n}_t$ representing detrended consumption and detrended knowledge capital, respectively. Equations (25)-(26) then become

$$\dot{\tilde{n}}_t = A(\tilde{n}_{t-d} e^{-g_e d} - \tilde{c}_t) - g_e \tilde{n}_t \tag{31}$$

$$\frac{\dot{\tilde{c}}_t}{\tilde{c}_t} = \frac{1-\alpha}{\alpha} A\, e^{-\rho d} \frac{\tilde{c}_t}{\tilde{n}_t} - (\rho + g_e). \tag{32}$$

By linearizing the Euler-type equation (32) around the BGP and using (28), we get

$$\dot{\tilde{c}}_t = (g_e + \rho)\tilde{c}_t - \frac{(g_e+\rho)^2 \alpha e^{\rho d}}{A(1-\alpha)} \tilde{n}_t. \tag{33}$$

Existence and uniqueness of a continuous solution for the linearized system of delay differential equations (31)-(33) is guaranteed, for example, by Theorem 6.2 page 171 in Bellman and Cooke [11]. Moreover, the characteristic equation associated to such linearized system is (see e.g. Kolmanovskii and Nosov [45], p.50, or Hale and Lunel [40], p.198)

$$h(\lambda) := \lambda^2 - \rho\lambda - \lambda A e^{-(g_e+\lambda)d} + A(g_e+\rho)e^{-(g_e+\lambda)d} - A(g_e+\rho)e^{-g_e d} = 0. \tag{34}$$

Proposition 2 *The series expansion of the Laplace transform solution of the system (31)-(33), given the initial conditions $\tilde{n}_t = \bar{n}_t e^{-g_e t}$, $\bar{n}_t \in C([-d,0];\mathbb{R}_+])$, and $\tilde{c}_0 = c_0$, is*

$$\tilde{n}_t = \sum_{r=0}^{+\infty} p_r(t) e^{\lambda_r t} \tag{35}$$

$$\tilde{c}_t = \frac{1}{A}\sum_{r=0}^{+\infty}\left(Ae^{-(g_e+\lambda_r)d}p_r(t-d) - (g_e+\lambda_r)p_r(t) - p_r'(t)\right)e^{\lambda_r t} \tag{36}$$

where $\{\lambda_r\}_{r=0}^{+\infty}$ are the roots of the characteristic equation (34) and $\{p_r(t)\}_{r=0}^{+\infty}$ are polynomials of degree $k-1$ where k is the multiplicity of λ_r. When $k=1$ we have

$$p_r = \frac{\phi(\lambda_r)}{h'(\lambda_r)} \tag{37}$$

where

$$\phi(\lambda) = -Ac_0 + (\lambda - \rho - g_e)\left[\bar{n}_0 + Ae^{-(g_e+\lambda)d}\int_{-d}^{0}\bar{n}_t e^{-(g_e+\lambda)t}dt\right].$$

Proof. See Appendix. ■

From the above proposition, the solution of the linearized system (31)-(33) depends on the roots of the characteristic equation (34), the so-called *spectrum* of the system. The next subsection investigates its properties. It is indeed worth noting that, consistently with the initial conditions being defined on a time interval, the characteristic equation (34) admits an infinite (countable) number of complex roots. So, differently from the ODE case, the system has an infinite spectrum. Before moving to the analysis of the spectrum, it is also worth noting that the initial condition $\tilde{n}_t$ in $t \in [-d,0]$ is now a generic positive function while it had to be a constant in order to the economy be on its balanced growth path.

4.3 Spectrum of Roots of the Characteristic Equation

We start with the following result.

Proposition 3 *Assume that (30) holds. Then the characteristic equation (34) has*

(i) two real roots: $\lambda_0 > g_e + \rho > 0$ and $\lambda_1 = 0$; such roots are simple;

(ii) all the other roots are given by $\lambda_r = \mu_r + i\nu_r$ $(r \geq 2)$ where $\mu_r \in \mathbb{R}$, $\nu_r \in \mathbb{R}$, $\nu_r \neq 0$ for every $r \geq 2$. Moreover, for any root $\lambda_r = \mu_r + i\nu_r$ also its conjugate $\mu_r - i\nu_r$ is a root. Finally, if there exist purely imaginary roots (i.e. with $\nu_r = 0$) then they are simple.

Proof. See Appendix. ■

Now we proceed to study the complex roots of the spectrum under the assumption (30). What is important for our purposes is to know (depending on the parameters of the problem) how many complex roots have positive or zero real part, since this fact will influence the existence and the stability of equilibrium.

The parameters of the problem are ρ, α, A and d with the restrictions $\rho > 0$, $\alpha \in (0,1)$, $A > 0$, $d > 0$ and (30). Forgetting for a while such restrictions, and following, for example, Kolmanovskii and Nosov [45], p.55, we define the D-Subdivision D_j as the set in the space $(\rho, \alpha, A, d) \in \mathbb{R}^4$, such that the characteristic equation (34) has j and only j roots with strictly positive real part. We have the following result:

Proposition 4 *Assume that (30) holds and $\rho > 0$, $\alpha \in (0,1)$, $A > 0$, and $d > 0$. Then subdivisions D_1 and D_3 are non empty open regions in $\mathbb{R}^4_+$.*

Proof. See Appendix ■

From now on, we denote with $\overline{D}_1$ the closure of D_1, meaning D_1 and its boundary.

In Figure 1, the continuous line represents the locus separating D_1 from D_3 in the space (ρ, α), which corresponds to the parameter configuration where the spectrum has a pair of purely imaginary roots. The figure is produced for given d and g_e. Then, for continuous variation of the two parameters (ρ, α) crossing this curve the number of roots with positive real part changes from one to three since a couple of conjugate roots passes through the imaginary axis. This feature will be critical for the rising of permanent cycles as we will see later in Sections 4.4 and 4.5. Note finally that, as it comes from Proposition 4, in Figure 1 the subdivisions D_1 and D_3 are not empty.

Figure 2 shows how the curve separating D_1 and D_3, moves when the delay d increases. As it can be seen, it moves to the left making permanent cycles more plausible for smaller values of ρ and α. In the limit, when d goes to zero, D_3 becomes empty as expected.

4.4 Equilibrium Path of the Linearized System

To study the existence and the behavior of the equilibrium, the first step is to redefine the concept of equilibrium for the linearized system: from now on we call it the "linearized" equilibrium. This can be easily done by a slight modification of Definition 1, namely replacing the Euler-type equation with its linearized counterpart (33). In the following, we propose such definition for the original, non detrended, variables.

Definition 2 *A "linearized" equilibrium associated to the exogenously given nondecreasing initial condition $\bar{n}_t \in C([-d,0];\mathbb{R}_+)$, is a path (c_t, n_t), for $t \geq 0$, verifying the feasibility condition*

$$\dot{n}_t = A(n_{t-d} - c_t), \quad t \geq 0, \tag{38}$$

the linearized Euler-type equation

$$\dot{c}_t = (\rho + 2g_e)c_t - \frac{(g_e+\rho)^2 \alpha \mathrm{e}^{-\rho d}}{A(1-\alpha)} n_t, \quad t \geq 0, \tag{39}$$

the initial condition $n_t = \bar{n}_t$, $\forall t \in [-d, 0]$, the transversality condition

$$\lim_{t\to\infty} n_t {c_t}^{-1} \, \mathrm{e}^{-\rho t} = 0, \tag{40}$$

and the irreversibility constraint $\dot{n}_t \geq 0$ for $t \geq 0$.

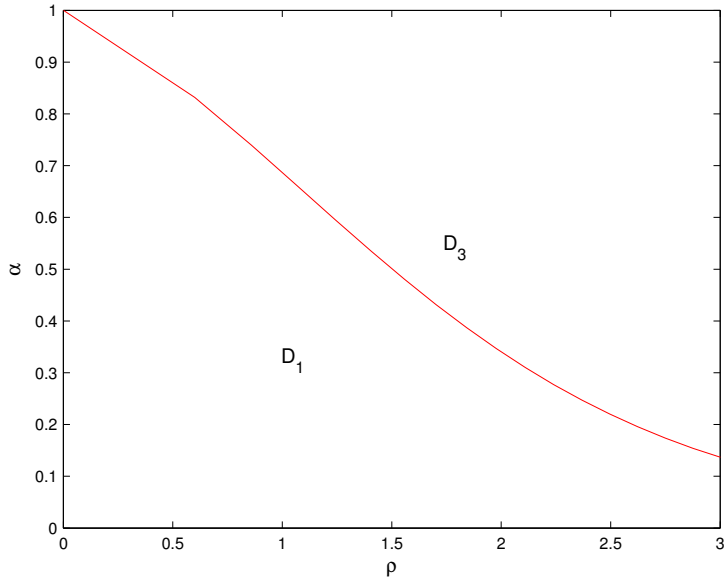

Figure 1: D-Subdivision of $h(\lambda)$ in the parameters space (ρ, α) when $A \simeq A_e$.

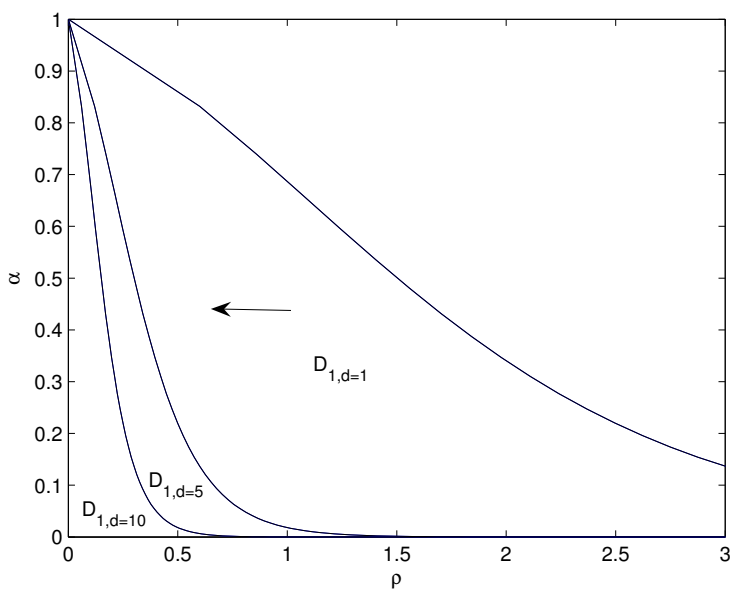

Figure 2: D_1 regions for different values of the parameter d, with $d_1 < d_2 < d_3$.

We have the following result.[6]

Proposition 5 *Let assume the parameters belong to the nonempty region $\overline{D}_1$. Then for any exogenously given nondecreasing initial condition $\bar{n}_t \in C([-d,0];\mathbb{R}_+)$ there exists a unique $c_0 > 0$ such that the solution path (n_t, c_t) of the system (38)-(39) satisfies the transversality condition (40). Such c_0 is given by*

$$c_0 = (\lambda_0 - g_e - \rho)\left[\frac{\bar{n}_0}{A} + e^{-(g_e+\lambda_0)d}\int_{-d}^{0}\bar{n}_t e^{-(g_e+\lambda_0)t}dt\right] > 0. \tag{41}$$

Moreover the path (n_t, c_t) is

$$\begin{aligned} n_t &= p_1 e^{g_e t} + \sum_{r=2}^{+\infty} p_r(t) e^{(g_e+\lambda_r)t} \qquad (42)\\ c_t &= \frac{1}{A}\left[\left(Ae^{-g_e d} - g_e\right)p_1 e^{g_e t} + \sum_{r=2}^{+\infty}\left(Ae^{-(g_e+\lambda_r)d}p_r(t-d) - (g_e-\lambda_r)p_r(t) - p_r'(t)\right)e^{\lambda_r t}\right] \end{aligned}$$

Proof. See Appendix. ■

In D_3, the existence of three roots with positive real part does not allow us to characterize the solution using the same proof strategy as in Proposition 5, since c_0 can only rule out one of the three roots. Consequently, the system will at some point violate the positiveness of c_t and/or n_t. To overcome this problem, the equilibrium conditions should be written taking into account the positivity restrictions explicitly, which makes the problem extremely challenging since the mathematical theory in such case is still in development and then we cannot use results from that literature. The problem of characterizing the dynamics in all the D-Subdivisions emerges also in a time to build framework as documented in Bambi [6]. In that context Bambi et al [7] show that the solution of a time-to-build economy can be characterized in the D_j regions, with $j \geq 2$, by properly restricting the set of the initial conditions, $\bar{n}_t$, in order to rule out the j roots with positive real part. Unfortunately, Bambi et al [7] methodology cannot be applied to our problem.

4.5 Equilibrium Path of the Original Nonlinear System

Now we want to apply the theory developed in Diekmann et al. [29] (Chapters VII-VIII-IX-X) to get information on the equilibrium for initial conditions close to those specified for the BGP. We apply such theory to the detrended system (31)-(32). We have the following result

Proposition 6 *Let assume the parameters belong to the nonempty region $\overline{D}_1$. Then for any exogenously given nondecreasing initial condition $\bar{n}_t \in C([-d,0];\mathbb{R}_+)$ close enough to the BGP there exists a $c_0 > 0$ such that the solution path (n_t, c_t) of the system (25)-(26) satisfies the*

[6] Whether a point in $\mathbb{R}^4_+$ belongs to the region $\overline{D}_1$ and the condition $\dot{n} \geq 0$ holds at equilibrium can be checked numerically: the first by looking at the characteristic equation; the second by computing the coefficients p_r as the positivity of $\dot{n}_t$ can be easily implied by suitable inequalities on them.

transversality condition (27). Such c_0 converges to the one defined in (41) as the distance $\sup_{t\in[-d,0]} |\bar{n}_t - \bar{n}_0 e^{g_e t}|$ *tends to* 0.

Moreover, if the parameters are on the boundary of D_1, then the nonlinear detrended system (31)-(32) have periodic solutions of period $\frac{2\pi}{\nu_2}$ arising through a Hopf bifurcation.

Finally the solution path (n_t, c_t) is the unique equilibrium path if the irreversibility constraint $\dot{n}_t \geq 0$ is respected for every $t \geq 0$.[7]

Proof. See Appendix. ■

The two fundamental properties of the model follow from this Proposition. Firstly, when parameters belong to the D-Subdivision D_1, as usual in endogenous growth models with one state variable, the transversality condition rules out the positive real root. Due to the existence of an infinite number of complex roots with negative real part, the economy converges by damping oscillations. Moreover, under log utility, consumption is expected to depend linearly on wealth. This is implicit in equation (41), where the right hand side of the equality, implicitly defines initial wealth as an equilibrium valuation of the flow of past innovation activities. Second, permanent cycles arise in endogenous growth models with implementation delays through a Hopf bifurcation. It is the case when parameters belong to the frontier between regions D_1 and D_3 –see Figure 1– where two complex roots cross the imaginary axes. In this case, the solution has two pure imaginary roots showing a permanent cycle. It is in this last sense that our results are in line with Schumpeter's statement that "wave-like fluctuations in business are the form economic development takes in the era of capitalism." As observable in Figure 1, for a given point (ρ, α), A close to $A^e_{\min}$, there exists a delay d that puts the economy in a permanent cycle equilibrium. In Section 5, we study the quantitative relevance of this finding.

4.6 Wave-Like Fluctuations

This section provides an intuitive explanation on why the model converges by damping oscillations and about the existence of permanent fluctuations. Let use (29) to rewrite the equilibrium system (32)-(31) as

$$\frac{\dot{\tilde{c}}_t}{\tilde{c}_t} = \left(\frac{\tilde{c}_t}{\tilde{n}_t} - \frac{Ae^{-g_e d} - g_e}{A}\right) \frac{(\rho + g_e)A}{Ae^{-g_e d} - g_e} \tag{43}$$

$$\frac{\dot{\tilde{n}}_t}{\tilde{n}_t} = -\left(\frac{\tilde{c}_t}{\tilde{n}_t} - \frac{Ae^{-g_e d}\,\tilde{n}_{t-d}/\tilde{n}_t - g_e}{A}\right) A. \tag{44}$$

From (29), $Ae^{-g_e d} - g_e > 0$, implying that the sign of $\dot{\tilde{c}}_t/\tilde{c}_t$ (resp. $\dot{\tilde{n}}_t/\tilde{n}_t$) depends positively (negatively) on the right hand side parentheses. Notice that both parentheses differ only on the $\tilde{n}_{t-d}/\tilde{n}_t$ term, reflecting the fact that current changes in technology take a delay d to be adopted.

[7] Uniqueness is guaranteed in a neighborhood of the balanced growth path. A deeper investigation of the global dynamics is left for future research.

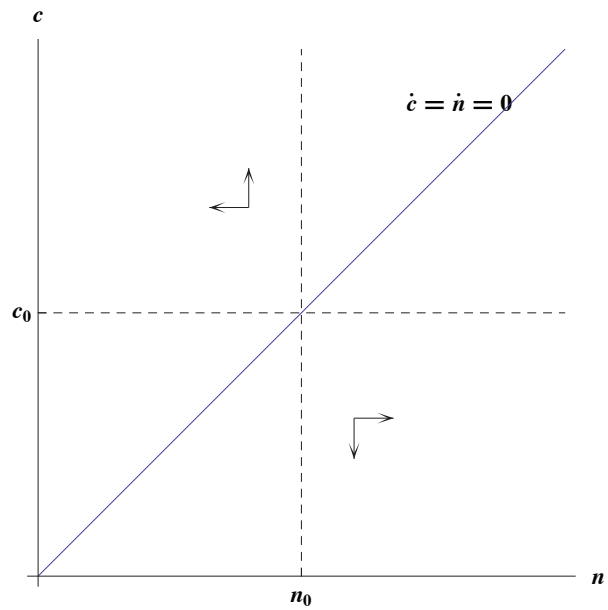

Figure 3: Phase diagram $d = 0$

Let first analyze the endogenous growth economy without implementation delays, i.e. the case $d = 0$. In this case, the right hand side parentheses on (43)-(44) become

$$\left(\frac{\tilde{c}_t}{\tilde{n}_t} - \frac{A - g_e}{A}\right),$$

the same in both equations. Figure 3 represents the behavior of the economy for any feasible (A, ρ, α), with g^e given by (29) under $d = 0$. The loci $\dot{\tilde{c}}_t = 0$ and $\dot{\tilde{n}}_t = 0$ are identical and the system diverges when (n_t, c_t) is not in these loci. As well-know in endogenous growth theory, for a given n_0, the initial consumption has to be c_0 making the economy jump to steady state at the initial time $t = 0$.

When d is strictly positive, the system (43)-(44) changes its nature and a phase diagram cannot be used to study global dynamics –see the Appendix for a formal analysis of the linearized system. However, the phase diagram in Figure 4 will help us understanding the oscillatory behavior of the economy. Notice that, $\tilde{n}_{t-d}/\tilde{n}_t$ in (44) is usually different from unity. It means that depending on the state of the cycle, $\dot{\tilde{n}}_t = 0$ may be above or below $\dot{\tilde{c}}_t = 0$. When it is below, a third region shows up in Figure 4, in which the system moves south-west. Notice that for a given n_{t-d}, the fact that n_t is reducing, tends to move the $\dot{\tilde{n}}_t = 0$ locus up. In the opposite case, the system moves north-east and the $\dot{\tilde{n}}_t = 0$ locus tends to move down. An equilibrium path, for given initial conditions, will tend to move then cyclically.

Existence of permanent cycles crucially depends on the implementation delay. For a relatively small d, the ratio $\tilde{n}_{t-d}/\tilde{n}_t$ tends to be close to unity, implying that oscillations dump and the

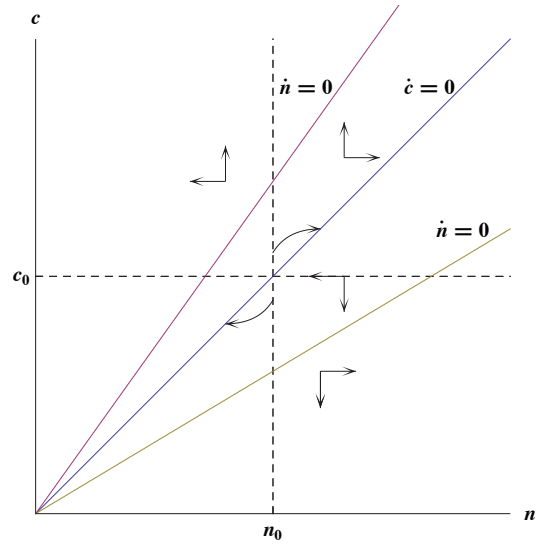

Figure 4: The behavior of the system under $d > 0$

economy converges to its steady state. As far as d increases, fluctuations persist for a longer time and they tend to be permanent when d converges to the Hopf bifurcation. In a permanent fluctuation equilibrium, the economy moves around its steady state.

Initial conditions determine the amplitude of the cycle. In the extreme case where the economy is on a Hopf bifurcation, but detrended initial conditions are constant, the term $\tilde{n}_{t-d}/\tilde{n}_t$ is one irrespective of the value of parameters and the economy will behave as in Figure 3, meaning that it will jump to steady state at the initial time. In the general case of varying initial conditions, the economy converges to a cycle which amplitude depends on the amplitude of the initial conditions.

5 Quantitative Analysis and Medium-Term Movements

In this section, we undertake a quantitative exercise to show that the conditions required for our economy to be on a permanent cycle equilibrium are quantitatively sensible. For this purpose, we set the model parameters to

$$d = 8.2, \quad \rho = 0.03, \quad \alpha = 0.9 \quad \text{and} \quad A = 0.786,$$

which allows us to replicate some key features of the US economy. The adopted value of d is consistent with Mansfield's estimations, and $\alpha = .9$ is in line with estimated markups in Basu and Fernald [9], implying a markup rate of 11%. By setting $\rho = .03$, A was chosen for the growth rate g_e to be equal to 2.4% as in Comin and Gertler [23]. Crucially, the model not only

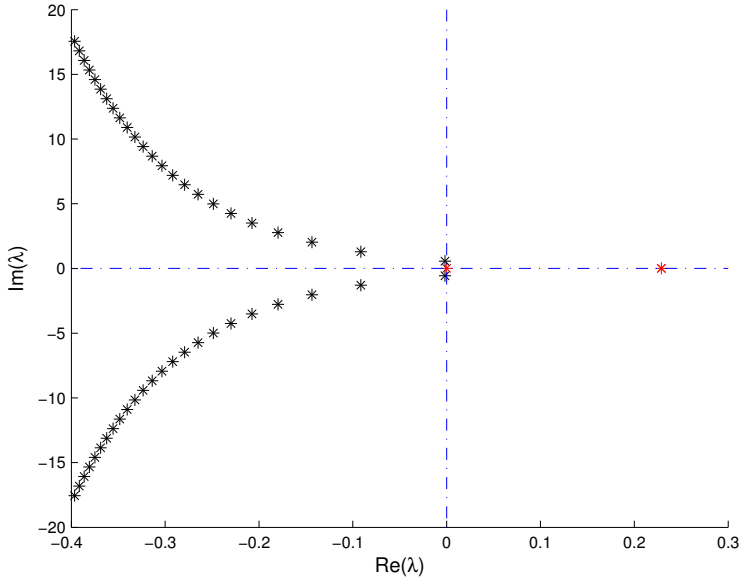

Figure 5: Spectrum of Roots.

matches the US long run growth but the economy is in D-Subdivision D_1, very close to the Hopf bifurcation.

We use the software DDE-BIFTOOL developed by Engelborghs and Roose [33] to compute the subset of the rightmost roots of the characteristic equations (34). The spectrum of roots is represented in Figure 5. As said above, the detrended system has a spurious zero root and a strictly positive real root, the latter being ruled out by the transversality condition. Under this parametrization, the spectrum shows two conjugate complex roots very close to the imaginary axes, all the other conjugate roots having strictly negative real part.

To set the initial conditions, we assume that during the years 1948 to 1959 the US economy faced a wave-like movement similar to the one estimated by Comin and Gertler for the same period. We interpret it as the US adjusting to the new economic environment emerging after World War II. Initial conditions are represented by the trigonometric function $\bar{n}_t = 1 + a\cos(bt/\pi)$, where parameter a is set to 0.0375 for the amplitude be close to 8%, and parameter b is set to 20/11 for the period be 11 years.[8]

To compute the numerical solution, we use the strategy proposed by Collard et al [22], which combines the method of steps suggested by Bellman and Cooke [11] with a shooting algorithm –see Judd [42]. We apply this strategy to the nonlinear system (32)-(31). The solution for detrended output, measured as $A\tilde{n}_{t-d}$ and normalized to turn around zero, is represented in

[8]The particular choice $n_0 = 1$ comes without any loss of generality, since the profile of the solution does not depend on the level of the state variable, as usual in endogenous growth models, but on the profile of the initial conditions.

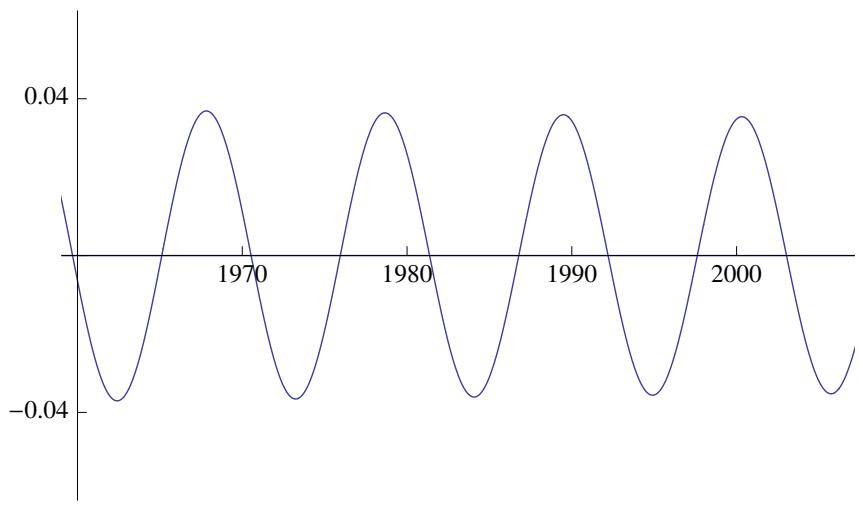

Figure 6: Simulated path for US output normalized around zero.

Figure 6. The decentralized equilibrium converges to a regular Juglar cycle with periodicity close to 11 years and an amplitude of around 8 percentage points. The periodicity of the cycle depends on model's parameters, in particular on the implementation delay d, but the amplitude of the cycle crucially depends on the amplitude of the initial conditions. Given that initial conditions are periodic with a periodicity close to the permanent cycle period, the economy converges to it very fast. The first recession and boom reflect the behavior of initial conditions, and maps on the following regular recessions around 1973, 1984, 1995 and 2006.

As can be observed in Figure 6, the approximately 11-years period of the solution is larger than the 8.2-years implementation delay. In facts, it is easy to see that the implementation delay has to be close to three fourth of the cycle period. Figure 7 represents the stationary solution for a period just larger than a cycle. Let for example the economy be at the boom at time t, with c_t at its maximum level. Consequently, $\dot{\tilde{c}}/\tilde{c}$ has to be zero at time t. From the Euler-type equation, to $\dot{\tilde{c}}_t$ be zero, c_t/n_t has to be at its stationary value. As can be observed in Figure 7, equilibrium output crosses zero around $t+d$, meaning that c_{t+d}/n_t is around its stationary value. Since consumption fluctuates less than output, $c_t/n_t \approx c_{t+d}/n_t$ or equivalently c_t/n_t is crossing zero close to $t+d$. Consequently, d has to be close to 3/4 of the the 11-years cycle period. See the Appendix for a more formal argument.

6 Optimal Allocations and R&D Subsidies

An optimal allocation solves the following social planner problem[9]

[9]We implicitly assume that the solution is interior, meaning that $L_t \in (0,1)$. Bambi el al [7] in a similar framework explicitly states the needed parameter restriction.

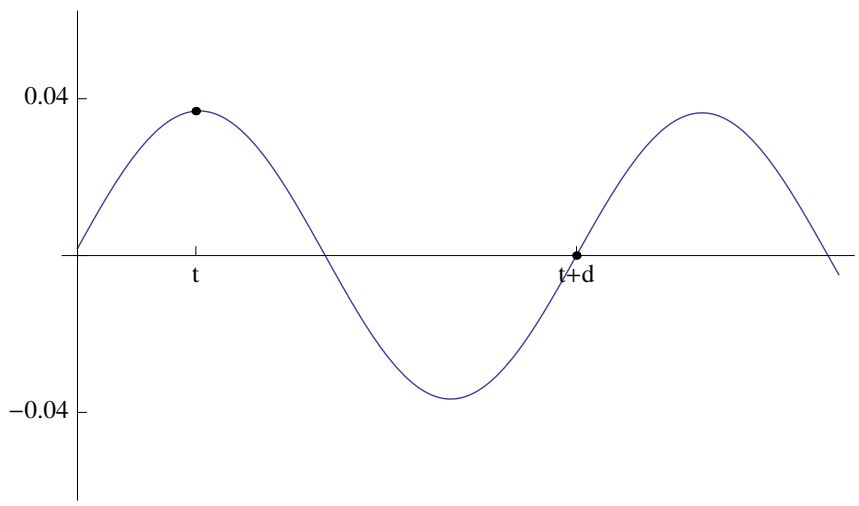

Figure 7: Periodicity of the cycle and the implementation delay.

$$\max \int_0^\infty \log(c_t)\mathrm{e}^{-\rho t}\mathrm{d}t$$

subject to the feasibility constraint

$$\dot{n}_t = A\left(n_{t-d} - c_t\right), \tag{7}$$

the irreversibility constraint $\dot{n} \geq 0$ and the initial condition $n_t = \bar{n}_t$, $\bar{n}_t \in C([-d, 0]; \mathbb{R}_+)$, with $\bar{n}_t$ the same as in the decentralized equilibrium. Notice that for $d = 0$ the variable change $\hat{c} = Ac$ renders this problem formally identical to the AK model in Rebelo [50].

Following Kolmanovskii and Myshkis [44] and operating as in the decentralized economy, optimality requires the Euler-type condition

$$\frac{\dot{c}_t}{c_t} = Ae^{-\rho d}\frac{c_t}{c_{t+d}} - \rho, \tag{45}$$

and the transversality condition

$$\lim_{t\to\infty} n_t c_t^{-1}\mathrm{e}^{-\rho t} = 0, \tag{46}$$

The social planner faces a trade-off between consuming at time t or saving and consuming at $t + d$. For this reason, in (45) the R&D productivity, A, is weighted by the ratio of marginal utilities of consuming at $t + d$ and t, which multiplied by $e^{-\rho d}$ represents the discount factor on a period of length d. It is useful to observe that, as in the AK model, the Euler-type mixed functional differential equation (45) does not depend on the state variable n. Consequently, since the social return to R&D is constant, the planner may allocate consumption over time without caring about the path of knowledge n. As shown in the appendix, optimal consumption is in its balanced growth path from time zero. However, since initial conditions affect production from

zero to time d, R&D has to adjust to fulfill the feasibility condition. This mechanism will repeat again and again making the optimal allocations to fluctuate, converging by damping oscillations.

An optimal allocation is then a path (c_t, n_t), for $t \geq 0$, verifying the mixed functional differential equations system (7) and (45), the transversality condition (46), the initial condition $n_t = \bar{n}_t$, $\bar{n}_t \in C([-d, 0]; \mathbb{R}_+)$, and the irreversibility constraint $\dot{n} \geq 0$. At a balanced growth path, from (45), consumption grows at the constant rate g solving

$$g + \rho = A\, \mathrm{e}^{-(g+\rho)d}. \tag{47}$$

The following parameter condition

$$A > \rho e^{\rho d} \equiv A^*_{\min} \tag{48}$$

is necessary and sufficient for g to be strictly positive. When $d = 0$, this condition collapses to the standard assumption in the AK model that $A > \rho$. Even if the transcendental equation (47) has an infinity of complex solutions, under assumption (48), existence and uniqueness of a real solution are trivial, since for $g > 0$ the right-hand-side of (47) is decreasing from $A\, \mathrm{e}^{-\rho d}$ to zero and the left-hand-side is increasing from ρ to infinity.

The main properties of the transitional dynamics are shown in the next proposition

Proposition 7 *Let's assume that $A > A^*_{\min}$, then the optimal equilibrium paths for n_t and c_t are*

$$n^*_t = a_L e^{gt} + \sum_{j=1}^{+\infty} a_j e^{z_j t} \tag{49}$$

$$c^*_t = c_0 e^{gt} \tag{50}$$

where g is the unique real solution of (47), a_L and $\{a_j\}_{j=1}^{+\infty}$ are the residues associated to the roots $\{z_j\}_{j=0}^{+\infty}$ of the characteristic equation $h(z) \equiv z - Ae^{-zd} = 0$,

$$a_L = A \sum_{j=0}^{+\infty} \frac{c^*_0}{(z_j - g)h'(z_j)} \qquad a_j = \frac{\bar{n}_0 + z_j \int_{-d}^{0} \bar{n}_s e^{-z_j s} ds}{h'(z_j)} - \frac{Ac^*_0}{(z_j - g)h'(z_j)} \tag{51}$$

with $z_0 = g + \rho$ and the initial value of consumption, c_0, equals to

$$c^*_0 = \frac{\rho}{A} \left(\bar{n}_{-d} + \int_{-d}^{0} \dot{\bar{n}}_s e^{(g+\rho)s} ds \right). \tag{52}$$

Proof. See Appendix. ■

Under log utility, consumption equals the return on wealth, the latter being represented by the term within brackets at the right hand side of (52) divided by the relative productivity A –see (7). Notice that initial wealth is the sum at time zero of the value of operative varieties n_{-d} plus the value of produced but still non operative varieties, i.e., those produced between $-d$ and zero. The factor $e^{(g+\rho)s}$, multiplying the mass of varieties $\dot{\bar{n}}_s$ created at time s, $s \in [-d, 0]$, discounts the varieties' value for the period still remaining until those varieties will become operative. The set of initial conditions which make the irreversibility constraint hold is characterized in Bambi et al [7].

6.1 Comparing Centralized and Decentralized Allocations

Optimal and equilibrium allocations differ in at least two dimensions. First, consumption is perfectly smoothed in the optimal allocation, but fluctuates at equilibrium. Second, the growth rates are different at the balanced growth path. We develop these two arguments below, before suggesting an optima R&D policy.

The fact that consumption does not fluctuate at the optimal allocation comes from the Euler-type equation (45), which due to linearity in the aggregate technology does not depend on the state of knowledge n. This is not the case at equilibrium, since the private return to R&D depends on future profits, which are a negative function of the future market share. At steady state

$$g_e\left(\mathrm{e}^{-\rho d}+\frac{\alpha}{A(1-\alpha)}\right)+\frac{\alpha\rho}{A(1-\alpha)} = A\,\mathrm{e}^{-(g_e+\rho)d}, \tag{29}$$

$$g+\rho = A\mathrm{e}^{-(g+\rho)d} \tag{47}$$

where g_e and g represent the equilibrium and optimal growth rates, respectively. As shown in Appendix, there exist cutoff level for α,

$$\underline{\alpha} \equiv \frac{g+\rho-ge^{-\rho d}}{2(g+\rho)-ge^{-\rho d}}, \tag{53}$$

$\underline{\alpha} \in (0, 1/2)$, such that the equilibrium growth rate, g_e, is equal to the optimal growth rate, g, iff $\alpha = \underline{\alpha}$. Equilibrium growth is smaller than optimal growth, i.e. $g_e < g$, iff $\underline{\alpha} < \alpha < 1$. Otherwise, it is larger. Remind that equilibrium profits are declining in α, which represents the inverse of the markup, and optimal growth does not depend on it.

This result is consistent with Benassy [12], who shows for $d = 0$ that the equilibrium growth rate is smaller than the optimal rate if and only if the knowledge externality, ν in equation (2), is small enough or, equivalently, the elasticity of substitution α is large enough. Since in our framework ν is assumed to be unity, let argue in terms of the elasticity of substitution for a given knowledge externality. For $d = 0$, $\underline{\alpha} = \left(1+\frac{A}{\rho}\right)^{-1}$, meaning that there is a range of parameters for which the optimal growth rate is smaller than the equilibrium growth rate at the balanced growth path. Increasing α makes goods more substitutable, reducing markups, the return to R&D and the growth rate. Consequently, there is a degree of substitutability beyond which the optimal growth rate is larger than the equilibrium rate.

Since returns to private R&D are different from public returns, optimality may be restored by the mean of a time dependent subsidy/tax scheme imposed on current R&D investments or, equivalently, on the return to R&D. By comparing the equilibrium (24) and the optimal (45) Euler-type conditions, it is easy to see that private and public returns equalize when the subsidy rate is

$$1+s_t = \frac{\alpha}{1-\alpha}\frac{n_t}{c_{t+d}} = \underbrace{\frac{\alpha}{1-\alpha}\frac{n}{c}}_{\text{constant}}\underbrace{\frac{n_t/c_{t+d}}{n/c}}_{\text{procyclical}},$$

where the stationary ratio n/c is defined in (28).

An optimal policy has two components. First, as in the expanding product variety model, it has to equalize the (average) private return to the (average) social return. The magnitude of it corresponds to the *constant* term in the equation above, which depends negatively on both the markup, $1/\alpha$, and the average market share of intermediary firms, c/n. Second, it has to compensate for the *countercyclical* fluctuations in the private return. The social return to R&D is constant and equal to A, but the private return fluctuates countercyclically, being small than the mean during expansions and large during contractions –due to consumption smoothing, market shares are small during booms. To render the equilibrium allocation optimal, the subsidy has to move procyclically to balance fluctuations in the private return.

6.2 Welfare Gains

This section suggests a R&D policy designed to partially remedy the distortions underlined in the previous section, with the purpose of undertaking some counterfactual exercise around the equilibrium computed in Section 5 and evaluate the corresponding welfare gains. The model is then extended to study a time varying R&D subsidy addressed to increase the average return to R&D and reduce the volatility of consumption. Let assume the R&D policy follows

$$1 + s_t = (1 + s) \left(\frac{c_t}{n_t} \right)^{\sigma - 1},$$

where s is a constant rate and $\sigma < 1$ represents the additional smoothing introduced by the R&D policy. The equilibrium Euler-type equation (24) becomes

$$\frac{\dot{c}_t}{c_t} = \frac{1 - \alpha}{\alpha} (1 + s) A \, \mathrm{e}^{-\rho d} \left(\frac{c_t}{n_t} \right)^{\sigma} - \rho.$$

Notice that an equilibrium without R&D policy requires $s = 0$ and $\sigma = 1$.

In order to make welfare comparisons, we compute a consumption equivalent measure defined as the constant rate at which consumption in the decentralized equilibrium should increase all over the equilibrium path to make equilibrium welfare equal to the corresponding welfare of the equilibrium path with subsidies. Since utility is logarithmic, our welfare measure collapses to

$$\omega = e^{\rho \, (W_{R\&D} - W_e)} - 1,$$

where $W_{R\&D}$ and W_e measure welfare, as defined by the utility function (1), evaluated at equilibrium with and without subsidies, respectively.

When the R&D policy pays a 10% average subsidy, $s = .10$, and the subsidy rate moves in order to smooth consumption, with a smoothing parameter $\sigma = 1/2$, the growth rate increases from 2.4% to 3.4%. In Figure 8, detrended consumption paths, relative to initial consumption, are represented for the economies with and without subsidies. The smother corresponds to the economy with procyclical subsidies. As can be observed, the subsidy halves consumption fluctuations. There are welfare gains of 9.6% as measured by ω. The order of magnitude

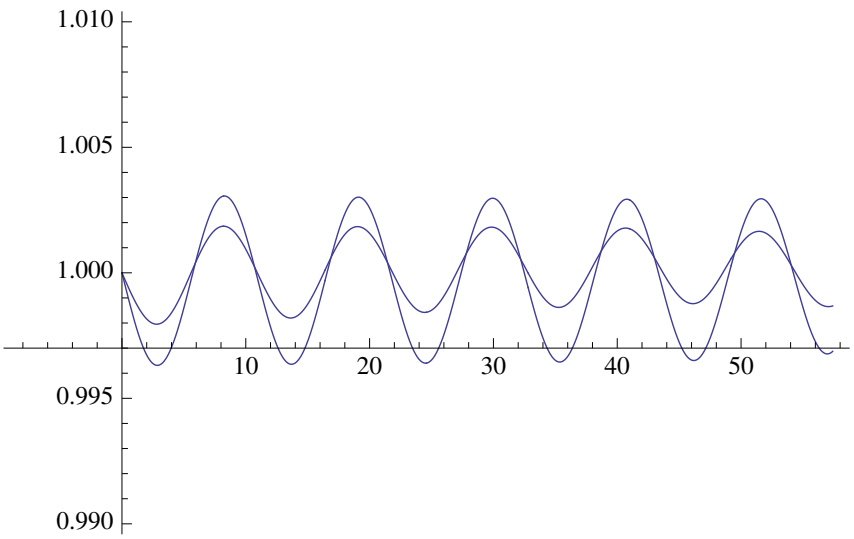

Figure 8: Consumption paths with and without subsidy.

is consistent with the findings in Barlevy [8]. If the 10% subsidy were constant, the growth rate would be 2.8% and the welfare gains 3.3%. Consequently, a 6.3% welfare gain may be attributed to consumption smoothing alone, however consumption smoothing affects welfare mainly through the raise in the growth rate.

7 Conclusions

This paper studies the relation between Schumpeterian wave-like business fluctuations and economic development in an endogenous growth framework with implementation delays. The paper shows that the equilibrium path admits a Hopf bifurcation where consumption, research and output permanently fluctuate around a positive trend. The main mechanism relating growth to wave-like fluctuations is based on the assumption that innovations being fundamental for economic growth require long implementation and adoption lags. A simple quantitative exercise shows that such an endogenous mechanism relating the sources of growth and business fluctuations is not only theoretically possible but quantitatively relevant.

Additionally, the paper makes some welfare considerations. Firstly, it shows that detrended consumption is constant from the initial time in an optimal allocation, and both R&D and output converge by oscillations. Second, it proves that a procyclical subsidy/tax scheme would restore optimality. Finally, it quantitatively find that a procyclical 10% subsidy rate halving consumption fluctuations will increase the growth rate from 2.4% to 3.4% with a 9.6% increase in welfare.

Appendix

More on Equilibrium

For completeness, we now present the theorem giving the optimality conditions for the *unconstrained* households problem who maximize (13) subject to the instantaneous budget constraint (14), the control constraint $c_t \geq 0$ ($t \geq 0$) and the initial condition $n_t = \bar{n}_t$, for $t \in [-d, 0]$, where $\bar{n}_t$ is a known continuous positive function defined on the t domain. For brevity we will call this problem (UHP).

Theorem 1 *Assume that the function r_t is bounded and that the function w_t is such that*

$$w_t \leq k_1 e^{k_2 t}, \qquad \forall t \geq 0,$$

for suitable constants $k_1, k_2 > 0$. Then an admissible state control path (n_t, c_t) for the problem (UHP) is optimal if there exists an absolutely continuous costate function μ_t such that

$$\frac{e^{-\rho t}}{c_t} = \mu_t \tag{54}$$

$$\dot{\mu}_t = -r_{t+d}\mu_{t+d}, \tag{55}$$

and, for every other admissible path $(\hat{a}_t, \hat{c}_t)$, the transversality condition

$$\overline{\lim}_{t\to\infty}(\hat{a}_t - a_t)\mu_t \geq 0, \tag{56}$$

holds. The conditions (54) and (55) above are also necessary.

Proof. It is a special case of Theorem 3.1 of [4]. ■

A straightforward consequence of such result (that, we recall, holds only for the unconstrained problem) is that, if we find a solution (a_t, c_t, μ_t) to the system of equations (14)-(54)-(55) satisfying the new transversality condition

$$\lim_{t\to\infty} a_t\mu_t = 0, \quad \text{or, equivalently,} \quad \lim_{t\to\infty} a_t {c_t}^{-1}\, \mathrm{e}^{-\rho t} = 0, \tag{57}$$

then such solution is optimal. If, moreover, the state constraint $a_t \geq 0$ is satisfied for every $t \geq 0$, then the such solution is optimal for the constrained problem, too.

Main Propositions

Proof of Proposition 2. The fact that the system (31)-(33) admits a solution with a series expansion follows e.g. from Corollary 6.4 (p.168) of [29]. Here we explicitly compute the coefficients of such solutions by using the Laplace transform. We first differentiate equation (31) and then substitute there $\dot{\tilde{c}}_t$ from (33) and $\tilde{c}_t$ from (31). We the find the following second order delay differential equation for $\tilde{n}_t$

$$\ddot{\tilde{n}}_t - \rho\dot{\tilde{n}}_t - Ae^{-g_e d}\dot{\tilde{n}}_{t-d} - \left(g_e(g_e+\rho) + \frac{\alpha(g_e+\rho)^2 e^{\rho d}}{1-\alpha}\right)\tilde{n}_t + A(g_e+\rho)e^{-g_e d}\tilde{n}_{t-d} = 0, \tag{58}$$

where the initial data, in terms of the initial data of the system (31)-(33), are $\tilde{n}_t = \bar{n}_t e^{-g_e t}$ for $t \in [-d, 0]$ and $\dot{\tilde{n}}_0 = A(\bar{n}_{-d} - c_0) - g_e \bar{n}_0$. Recalling that the Laplace transformation of a function f (with subexponential growth at infinity) is defined as $\mathtt{L}(f)(\lambda) = \int_0^\infty f(t)e^{-\lambda t}dt$, observing that the solution of (58) satisfies such subexponential growth (see e.g. Theorem 5.4, p.34 of [29]), we have, for λ sufficiently big,

$$\begin{aligned}
\mathtt{L}(\dot{\tilde{n}}_t)(\lambda) &= -\tilde{n}_0 + \lambda \mathtt{L}(\tilde{n}_t)(\lambda) \\
\mathtt{L}(\ddot{\tilde{n}}_t)(\lambda) &= -\dot{\tilde{n}}_0 + \lambda \mathtt{L}(\dot{\tilde{n}}_t)(\lambda) = -\dot{\tilde{n}}_0 - \lambda \tilde{n}_0 + \lambda^2 \mathtt{L}(\tilde{n}_t)(\lambda) \\
\mathtt{L}(\tilde{n}_{t-d})(\lambda) &= e^{-\lambda d}\left[\int_{-d}^0 \tilde{n}_t e^{-\lambda t}dt + \mathtt{L}(\tilde{n}_t)(\lambda)\right] \\
\mathtt{L}(\dot{\tilde{n}}_{t-d})(\lambda) &= -\tilde{n}_{-d} + \lambda \mathtt{L}(\tilde{n}_{t-d})(\lambda) = -\tilde{n}_{-d} + \lambda e^{-\lambda d}\left[\int_{-d}^0 \tilde{n}_t e^{-\lambda t}dt + \mathtt{L}(\tilde{n}_t)(\lambda)\right]
\end{aligned}$$

So applying the Laplace transform to the equation (58) we have

$$\mathtt{L}(\tilde{n}_t)(\lambda) \cdot h(\lambda) = \phi(\lambda)$$

where $h(\lambda)$ is the left hand side of the characteristic equation (34) and

$$\phi(\lambda) = \dot{\tilde{n}}_0 + \tilde{n}_0(\lambda - \rho) - Ae^{-g_e d}\tilde{n}_{-d} + Ae^{-(g_e+\lambda)d}(\lambda - g_e - \rho)\int_{-d}^0 \tilde{n}_t e^{-\lambda t}dt$$

or, in terms of $\bar{n}_t$ and c_0,

$$\phi(\lambda) = -Ac_0 + \bar{n}_0(\lambda - \rho - g_e) + Ae^{-(g_e+\lambda)d}(\lambda - g_e - \rho)\int_{-d}^0 \bar{n}_t e^{-(g_e+\lambda)t}dt.$$

Since $\tilde{n}_t$ is a continuously differentiable function in $[0, +\infty)$,[10] and therefore certainly continuous and of bounded variation on any finite interval, then we can use the inversion formula for the Laplace transformation to obtain that, for $t > 0$, (see e.g. [11], Theorem 6.3, p. 175-176)

$$\tilde{n}_t = \int_{a-i\infty}^{a+i\infty} \frac{\phi(\lambda)}{h(\lambda)} e^{\lambda t} d\lambda. \tag{59}$$

Then one can compute such complex integral by means of the Residue Theorem as in [11], Section 6.7 (in particular Theorem 6.5) obtaining

$$\tilde{n}_t = \sum_{r=0}^{\infty} p_r(t) e^{\lambda_r t} \tag{60}$$

where $\{\lambda_r\}_{r\in\mathbb{N}}$ is the sequence of the roots of the characteristic equation (34) and the $p_r(t)$ are polynomials of degree less or equal to $k(r)-1$ where $k(r)$ is the multiplicity of λ_r. More precisely they are given by (setting for simplicity $k(r) = k$ in the formula below)

$$e^{-\lambda_r t} \cdot \lim_{\lambda\to\lambda_r} \frac{1}{(k-1)!}\frac{d^{k-1}}{d\lambda^{k-1}}\left((\lambda - \lambda_r)\frac{\phi(\lambda)e^{\lambda t}}{h(\lambda)}\right),$$

[10] See the previously mentioned theorem of existence and uniqueness of solution in Bellman and Cooke [11].

so, when $k(r) = 1$, p_r is independent of t and is given by $p_r = \frac{\phi(\lambda_r)}{h'(\lambda_r)}$. Finally the solution of $\tilde{c}_t$ can be derived from (35) and (31). ■

Proof of Proposition 3.
First, to prove this statement we study the function $h(\lambda)$ for $\lambda \in \mathbb{R}$. It is easy to check that

$$h(0) = 0, \qquad \lim_{\lambda \to +\infty} h(\lambda) = +\infty, \qquad \lim_{\lambda \to -\infty} h(\lambda) = +\infty.$$

Moreover

$$h'(\lambda) = 2\lambda - \rho + Ae^{-(g_e+\lambda)d}\left[-1 + d(\lambda - g_e - \rho)\right]$$

with

$$h'(0) = -\rho + Ae^{-g_e d}\left[-1 - d(g_e + \rho)\right] < 0, \qquad \lim_{\lambda \to +\infty} h'(\lambda) = +\infty, \qquad \lim_{\lambda \to -\infty} h'(\lambda) = -\infty.$$

and

$$h''(\lambda) = 2 + Ade^{-(g_e+\lambda)d}\left[2 - d(\lambda - g_e - \rho)\right].$$

with

$$h''(0) = 2 + Ade^{-g_e d}\left[2 + d(g_e + \rho)\right] > 0, \qquad \lim_{\lambda \to +\infty} h''(\lambda) = 2, \qquad \lim_{\lambda \to -\infty} h''(\lambda) = +\infty.$$

By simple computations it is easy to prove that the function $h''(\lambda)$ has a minimum point at $\bar{\lambda} = g_e + \rho + \frac{3}{d}$ and that the value of the minimum is $2 - Ade^{-(2g_e+\rho)d-3}$. We have now two cases.

- If $2 - Ade^{-(2g_e+\rho)d-3} \geq 0$ then the minimum value of $h''(\lambda)$ is positive so $h''(\lambda) > 0$ for every $\lambda \in \mathbb{R}$. This implies that h' is strictly increasing and there exists a unique point $\hat{\lambda} > 0$ such that $h'(\hat{\lambda}) = 0$. The claim follows from the fact that $h(0) = 0$.

- If $2 - Ade^{-(2g_e+\rho)d-3} < 0$ then the minimum value of $h''(\lambda)$ is negative so there exists an interval $(\bar{\lambda}_1, \bar{\lambda}_2)$ such that $h''(\lambda) > 0$ iff $\lambda \in (\bar{\lambda}_1, \bar{\lambda}_2)$. Since $h''\left(g_e + \rho + \frac{2}{d}\right) = 2 > 0$ then $\bar{\lambda}_1 > g_e + \rho + \frac{2}{d}$. Now this means that $h'(\lambda)$ is strictly increasing on $(-\infty, \bar{\lambda}_1)$ and on $(\bar{\lambda}_2, +\infty)$ and strictly decreasing on $(\bar{\lambda}_1, \bar{\lambda}_2)$. Since $h'(0) < 0$ and $h'\left(g_e + \rho + \frac{2}{d}\right) = 2g_e + \rho + Ae^{-(2g_e+\rho)d-2} > 0$ then h' has a unique zero in the interval $\left(0, g_e + \rho + \frac{2}{d}\right)$. Moreover from the expression of h' it easily follows that, for every $\lambda > g_e + \rho + \frac{2}{d}$ it must be $h'(\lambda > 0)$. So, as before, there exists a unique point $\hat{\lambda} > 0$ such that $h'(\hat{\lambda}) = 0$. The claim follows again from the fact that $h(0) = 0$.

Finally, the fact that $\lambda_0 > g_e + \rho$ follows from computing $h(g_e + \rho)$ and easily finding that it is negative.
Second, it is clear that all the other roots of (34) are not real and that for any root $\lambda_r = \mu_r + i\nu_r$ also its conjugate $\mu_r - i\nu_r$ is a root. Assume that for some r, $\lambda_r = i\nu_r$ with $\nu > 0$. If such root is not simple then the number ν must solve the equations $h(i\nu) = h'(i\nu) = 0$. With standard computations, that we omit for brevity, it is not difficult to see that this is impossible. ■

Proof of Proposition 4. We start focusing our analysis on the quasi-polynomial $h(\lambda)$ when $A = A^e_{min}$, and then $g_e = 0$; in this case, if we fix any $d > 0$, the characteristic equation becomes a continuous function of only two parameters, (ρ, α), and the computation are easier (moreover also a visual representation of the results can be provided). Then we we will use this case to infer the result in the general case using the fact that the stability results obtained for this restriction still hold for any $d > 0$ and for a sufficiently small variation of A.
So fix $d > 0$ and set $A = A^e_{min}$. We have

$$h(\lambda) = \lambda^2 - \rho\lambda - \lambda e^{-\lambda d}\hat{\alpha}\rho e^{\rho d} + e^{-\lambda d}\hat{\alpha}\rho^2 e^{\rho d} - \hat{\alpha}\rho^2 e^{\rho d} = 0 \tag{61}$$

where $\hat{\alpha} = \frac{\alpha}{1-\alpha} \in [0, +\infty)$ since $\alpha \in [0, 1)$. We also extend the domain of ρ to the interval $(-\varepsilon, 1 + \varepsilon)$, with ε positive and infinitely small, in order to pin down more easily the different D_j.

Let us start with the analysis of the two *extreme cases*: $\hat{\alpha} = 0$ and $\rho = 0$. When $\hat{\alpha} = 0$ then the parameters space is partitioned in two regions, D_1 if $\rho \in (0, 1 + \varepsilon)$ and D_0 if $\rho \in (-\varepsilon, 0]$; in fact when $\hat{\alpha} = 0$ then $h(\lambda) = \lambda(\lambda - \rho) = 0$ and there are only two real roots $\lambda_0 = \rho$, and $\lambda_1 = 0$. On the other hand $\rho = 0$ implies $h(\lambda) = \lambda^2 = 0$.

Let us now focus on the purely imaginary roots $\lambda = iv$ when $\hat{\alpha}$ and ρ can take any values in their respective domains in order to identify the D-curves, $\hat{\alpha} = \hat{\alpha}(v)$ and $\rho = \rho(v)$, separating different D_j regions. The characteristic equation writes

$$h(iv) = -v^2 - \rho iv - \hat{\alpha}\rho e^{\rho d} v\,[i\cos(vd) + \sin(vd)] + \hat{\alpha}\rho^2 e^{\rho d}\,[\cos(vd) - i\sin(vd)] - \hat{\alpha}\rho^2 e^{\rho d} = 0 \tag{62}$$

Observe also that $h(iv) = U(v) + iW(v) = 0$ with:

$$U(v) = 0 \quad \Leftrightarrow \quad v^2 + \hat{\alpha}\rho^2 e^{\rho d} + \hat{\alpha}\rho e^{\rho d}\,[v\sin(vd) - \rho\cos(vd)] = 0 \tag{63}$$

$$W(v) = 0 \quad \Leftrightarrow \quad v + \hat{\alpha} e^{\rho d}\,[v\cos(vd) + \rho\sin(vd)] = 0 \tag{64}$$

From $W(v) = 0$ follows immediately that

$$\hat{\alpha} = \frac{-v}{e^{\rho d}\,[v\cos(vd) + \rho\sin(vd)]} \tag{65}$$

when $v\cos(vd) + \rho\sin(vd) \neq 0$; then substituting (65) into (63) leads to

$$\rho = \rho(\omega) = \pm\frac{1}{d}\sqrt{\frac{\omega^2\cos(\omega)}{1 - \cos(\omega)}} \qquad with \;\; \omega = vd \tag{66}$$

Then substituting back (66) into (65) leads to

$$\hat{\alpha}(\omega) = \frac{-\omega}{e^{\sqrt{\frac{\omega^2\cos(\omega)}{1-\cos(\omega)}}}\left[\omega\cos(\omega) + \sqrt{\frac{\omega^2\cos(\omega)}{1-\cos(\omega)}}\cdot\sin(\omega)\right]} \tag{67}$$

Relations (66) and (67) determine the point $(\rho_1, \hat{\alpha}_1) = (\rho(\omega_1), \hat{\alpha}(\omega_1))$ of a D curve for $\omega = \omega_1$. If ω varies in its domain $\omega \in \left(\frac{(2k-1)\pi}{2}, \frac{(2k+1)\pi}{2}\right) \setminus 2k\pi$ with $k = \ldots, -2, -1, 0, 1, 2, \ldots$, we obtain all the D-curves.[11] Besides these curves, the D-subdivision may contain some straight lines for the values of ω which imply an indeterminate form of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ to $\rho(\omega)$ or $\hat{\alpha}(\omega)$. However in our specific case, the only indeterminate form emerges at $\omega = 0$ which implies $h(0) = 0$ confirming the presence of a zero root in all the parameters space. Then the properties of the parametric D-curves can be analytical derived; among them, we show in the following why the region $[-\varepsilon, 0] \times [0, +\infty]$ in the parameters space $(\rho, \hat{\alpha})$ is a subset of D_0.

To show this fact we will prove that if $\rho \to 0^-$ then $\hat{\alpha} \to \pm\infty$ and then no D-curve can be in the region under analysis. From (66) it is clear that $\rho \to 0^-$ if and only if $\omega \to \frac{(2k+1)\pi}{2}$, and then we have to study the following limit:

$$\begin{aligned}
\lim_{\omega \to \frac{(2k+1)\pi}{2}} \hat{\alpha}(\omega) &= \lim_{\omega \to \frac{(2k+1)\pi}{2}} \frac{-\omega}{e^{\rho(\omega)d}\left[\omega\cos(\omega) \pm |\omega|\sqrt{\frac{\cos(\omega)}{1-\cos(\omega)}} \cdot \sin(\omega)\right]} \\
&= \lim_{\omega \to \frac{(2k+1)\pi}{2}} \frac{-\omega}{e^{\rho(\omega)d}\left[\omega\cos(\omega) \pm \frac{|\omega|}{|\sin(\omega)|}\sqrt{\cos(\omega)\cdot(1+\cos(\omega))} \cdot \sin(\omega)\right]}
\end{aligned}$$

if k is even then $\pm$ otherwise $\mp$; let's assume k even, then

$$\begin{aligned}
\lim_{\omega \to \frac{(2k+1)\pi}{2}} \hat{\alpha}(\omega) &= \lim_{\omega \to \frac{(2k+1)\pi}{2}} \frac{-1}{\cos(\omega) \pm \sqrt{\cos(\omega)(1+\cos(\omega))}} \\
&= \lim_{\omega \to \frac{(2k+1)\pi}{2}} \frac{1}{\sqrt{\cos(\omega)}} \cdot \lim_{\omega \to \frac{(2k+1)\pi}{2}} \frac{1}{\sqrt{\cos(\omega)} \pm \sqrt{1+\cos(\omega)}} = \mp\infty
\end{aligned}$$

On the other hand if k is odd then $\pm\infty$.
Each curve separating two regions is obtained by studying the values that the two parameters can have in each of the intervals of v. It is also clear that the D_1 region changes as shown in Figure 2 because $\frac{\partial\rho(\omega)}{\partial d} < 0$, while $\frac{\partial\hat{\alpha}(\omega)}{\partial d} = 0$.

Now we pass to the general case. Here we can still use the D-subdivision method to locate the characteristic roots but the regions D_1 and D_3 are four-dimensional since the parameters are ρ, α, A and d. Taking into account that the characteristic roots are continuous functions of such parameters[12] then the D_1 and D_3 subdivisions are nonempty open regions in the space R^3_+ since their projection on the plane $A = A^e_{min}$ is nonempty due to the first part of the proof. ■

Proof of Proposition 5. Given our assumptions, the only positive root to be ruled out in order to have convergence to the balanced growth path is λ_0. To do that we have to specify c_0 as in (41) so that $p_0 = 0$. Uniqueness of the equilibrium path is a direct consequence of the fact that (41) is the only choice of the initial condition of consumption which rules out λ_0.

[11] The domain of ω excludes the points $2k\pi$ and $\frac{(2k+1)\pi}{2}$ which are discontinuity for $1 - \cos(\omega)$ and $\frac{\omega}{d}\cos(\omega) + \rho(\omega)\sin(\omega) = 0$ respectively.

[12] This is a simple consequence of the Implicit Function Theorem, see e.g. [45], p.55. or [32].

Oscillatory convergence follows from the properties of the spectrum of roots as discussed in the previous proposition. Finally the general equilibrium path converges to the balanced growth path and for this reason it respects the transversality conditions. In fact, convergence implies that $\lim_{t\to\infty} \frac{n_t}{c_t} = \frac{(1-\alpha)A}{\alpha(g_e+\rho)e^{\rho d}}$ and then $\lim_{t\to\infty} \frac{n_t}{c_t} e^{-\rho t} = 0$. ■

Proof of Proposition 6. First of all we observe that, thanks to Diekmann et al. [29], Theorem 6.8 p.240, since the characteristic equation admits a strictly positive real root λ_0, then all equilibrium points of the detrended system (31)-(32) are unstable. Moreover thanks to Theorem 6.1 p.257 and to Theorem 5.3 p.266 the detrended system (31)-(32) admits, in a neighborhood of any equilibrium point, a stable manifold $\mathcal{W}_S$ and a center manifold $\mathcal{W}_C$ which contains the set of initial conditions $(\bar{n}_t, c_0)$ which gives rise to a BGP (i.e. that satisfy Proposition 1).

Assume first that parameters are in the D_1-subdivision. If the initial conditions $(\bar{n}_t, c_0)$ belong to the linear stable manifold then there exists a small real number δ (which goes to zero when the distance between $\bar{n}_t$ and the BGP goes to zero) such that $(\bar{n}_t, c_0+\delta)$ belong to $\mathcal{W}_S$ (see Theorem 6.1 (ii) in Diekmann et al. [29]). Now, given an initial datum $\bar{n}_t$, choose c_0 as in (41). If we start the linearized system (31)-(33) from such $(\bar{n}_t, c_0)$ then we know that the solution converges to a BGP so $(\bar{n}_t, c_0)$ belongs to the linear stable manifold. Thanks to the Theorem 6.1 (iv), p.257 in Diekmann et al. [29], we get that, if $\bar{n}_t$ is sufficiently close to a BGP, then for suitable small δ as above, the solution of the nonlinear detrended system (31)-(32) converges to a BGP, too. This in particular implies that the transversality condition

$$\lim_{t\to\infty} \tilde{n}_t \tilde{c}_t^{-1} \, \mathrm{e}^{-\rho t} = 0, \tag{68}$$

holds. As a conclusion, if we prove that the solution of the nonlinear detrended system (31)-(32) satisfies the irreversibility constraint $\dot{\tilde{n}}_t + g_e \tilde{n}_t \geq 0$ $(t \geq 0)$, then this is an equilibrium associated to $\bar{n}_t$.

If the parameters are in the boundary of the D_1 subdivision then all the above considerations remain true except for the fact that we have two purely imaginary elements $(\pm i\nu_2)$ of the spectrum coming out. In this case, given any $\bar{n}_t$, the linearized detrended system (31)-(33) starting from $(\bar{n}_t, c_0)$ where c_0 is given by (41) has a solution whose principal components (the one coming from the eigenvalues with 0 real part) oscillates around a BGP. Arguing as above we then get that, for a suitable small δ, the solution of the nonlinear detrended system with datum $(\bar{n}_t, c_0 + \delta)$ remain bounded and satisfies the transversality condition (68). Moreover we can apply Theorem 2.7 p.291 in Diekmann et al. [29] to get that, on the center manifold, periodic solutions arise (the so-called Hopf bifurcation). As before, if we prove that the solution of the nonlinear detrended system (31)-(32) satisfies the irreversibility constraint $\dot{\tilde{n}}_t + g_e \tilde{n}_t \geq 0$ $(t \geq 0)$, then this is an equilibrium associated to $\bar{n}_t$. ■

Proof of Proposition 7. The proof follows from the maximum principle approach developed by Bambi [6] and the dynamic programming approach in Bambi et al. [7], Theorem 4. ■

Proof of the results based on equation 53. Let's assume $g = g_e$. Combining (29) and

(47) to solve for α gives $\underline{\alpha}$ as defined above. Notice that from (47), g does not depend on α, meaning that $\underline{\alpha}$ in (53) only depends on the other three parameters A, d, ρ. It is straightforward to observe that $\underline{\alpha}$ is always smaller than 1/2. Finally $g_e < g$ iff $\alpha > \underline{\alpha}$, since from (29)

$$\frac{dg_e}{d\alpha} = -\frac{\frac{(g_e+\rho)e^{\rho d}}{(1-\alpha)^2}}{1 + dAe^{-g_e d} + \frac{\alpha}{1-\alpha}e^{\rho d}} < 0,$$

and g in (47) does not depend on α. ■

Proof: The periodicity of the cycle is larger than the implementation delay. Remember that the behavior of n is governed by the feasibility condition (7), whose detrended version is in (31). Let first show that the solution cannot be periodic of period d. We can prove it by contradiction. Suppose the solution is periodic of period d, then $\tilde{n}_t = \tilde{n}_{t-d}$, implying that (31) becomes

$$\frac{\dot{\tilde{n}}_t}{\tilde{n}_t} = Ae^{-g_e d} - g_e - \frac{\tilde{c}_t}{\tilde{n}_t}.$$

Firstly, when detrended $\tilde{n}_t$ is at its maximum value, because of consumption smoothing the ratio $\tilde{c}_t/\tilde{n}_t$ is at its minimum value, implying that the growth rate is maximal at this point. Second, since the solution is periodic, it has to be that the growth rate $\dot{\tilde{n}}_t/\tilde{n}_t = 0$ at a maximum, but positive before. This contradicts the result that the growth rate is maximal at the maximum.
Let us now show that if a periodic solution exists, it has to be that the period is larger than d. Since the solution is periodic, $\tilde{n}_t$ has to be bounded, meaning that $\tilde{n}_t \in [n^{\min}, n^{\max}]$. Since the period of the solution is different from d, $\tilde{n}_{t-d} \neq n^{\max}$. Let us call t_m at a time t at which $\tilde{n}_t = n^{\max}$. From (31), at any t larger than but close to t_m

$$Ae^{-g_e d}\Delta\tilde{n}_{t-d} - \Delta\tilde{c}_t = \Delta\dot{\tilde{n}}_t + g_e\Delta\tilde{n}_t,$$

where Δx_t refers to the discrete change in variable x with respect to t_m. The right-hand-side is strictly negative, since $\tilde{n}_t$ is decreasing and concave at the right of the maximum, meaning that $\Delta\tilde{n}_t < 0$ and $\Delta\dot{\tilde{n}}_t < 0$. From consumption smoothing, we know that detrended output $Ae^{-g_e d}\tilde{n}_{t-d}$ reacts more than consumption, meaning that the left-hand-side has the same sign as $\Delta\tilde{n}_{t-d}$, which has to be negative then. Consequently, when n_t is at $n^{\max}$, n_{t-d} has to be close, but at the right of the previous spike, which proves that the period of the solution is larger than d. ■

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