Financial Development and the Volatility of Income*

Tiago Pinheiro         Francisco Rivadeneyra
Norwegian School of Economics         Bank of Canada
Marc Teignier         University of Alicante
March 20, 2012

WORK IN PROGRESS. PRELIMINARY AND INCOMPLETE

Abstract

This paper presents a general equilibrium model with collateral constraints which captures two empirically documented and opposing effects of the financial sector. First, the financial sector allows a better allocation of capital increasing the income level of the economy. Second, the financial sector amplifies the volatility of income by increasing the business cycle effects of technological shocks through the contraction or expansion of borrowing constraints. We parametrize the model to analyze this tradeoff at different levels of financial development. Our results show a non-monotonic relationship between the level and the volatility of income induced by the financial sector. More developed financial systems unambiguously increase the income level however the volatility can rise or fall depending on the degree of financial development.

JEL classification numbers: E32, E60.
Keywords: financial sector, income growth, income volatility, welfare costs.

*We thank the comments of James Chapman, Corey Garriott, Thor Koepl, Bob Lucas, Ben Moll, Brian Peterson, José-Víctor Ríos-Rull, Hajime Tomura, Randy Wright and seminar participants at the Bank of Canada, 2011 SED meetings, Econometric Society Asian Meetings and EFA meetings. The views presented here do not necessarily reflect the ones of the Bank of Canada.
“In the last 30 years, financial systems around the world have undergone revolutionary change. People can borrow greater amounts at cheaper rates than ever before, invest in a multitude of instruments catering to every possible profile of risk and return, and share risks with strangers from across the globe. Have these undoubted benefits come at a cost?”

Rajan (2005)

1 Introduction

The financial crisis of 2007-2009 renewed the attention to the linkages between the financial sector and the real economy. As a consequence of the crisis the world experienced a large and sharp contraction of output. The debate that followed regarding new regulation to prevent future financial sector crises has focused on capital adequacy of financial institutions and risk management practices. Yet little attention has been put into the possible effects of new regulation on financial development and economic growth. In this paper we aim at shedding light on this question by analyzing the relationship between financial sector development and the level and volatility of income in a a general equilibrium model with financial frictions in the form of collateral constraints.

We investigate the opposing effects of a financial sector expansion on the level and volatility of income through a relaxation of the borrowing constraint. In the model, the financial sector allows a better allocation of capital increasing the income level of the economy but, at the same time, it amplifies business cycle fluctuations.When aggregate productivity is hit by a negative shock, asset prices decrease and borrowing constraints get tighten, which further output because it reduces the capital allocation efficiency. We present a general equilibrium model which captures this tradeoff between the level of output and income volatility induced by the financial sector.

These two effects of financial deepening have been documented empirically in separate strands of literature. Regarding the relationship between the level of income and the development of the financial sector Beck et al. (2000) show a strong causal, significant and positive relationship between measures of financial development and the rate of per capita GDP growth in a large cross-section of countries. On the other hand, a period of financial liberalization in developing economies is commonly followed by financial crises that lead to a contraction of income (Aizenman (2004)). Also in developed economies credit expansions are one of the best predictors of financial instability and output volatility (Reinhart and Rogoff (2011)). Loayza and Ranciere (2005) argue that the apparent empirical contradiction between the positive and
negative effects of the financial sector can be resolved empirically if we distinguish between short and long-term effects of financial intermediation. Our paper makes a theoretical argument in this direction: the long-term benefits of the financial sector are in terms of higher level of income but short-term costs can show up in terms of added volatility of income.

We present a model with heterogenous agents and borrowing constraints based on Kiyotaki and Moore (1997) and Kiyotaki (1998). As in their setting, agents’ productivity is heterogeneous and financial markets intermediate loans from the least productive agents to the most productive ones. Capital markets are imperfect and credit constraints arise because agents’ borrowing is limited to a fraction of their wealth. Because the durable assets of the economy serve as collateral for loans, borrowers’ credit limits are affected by the prices of the collateralized assets. At the same time, the prices of these assets are affected by the size of the credit limits. The dynamic interaction between credit limits and asset prices amplifies and increases the persistence of productivity shocks thus raising the volatility of output. We generalize Kiyotaki (1998) by adding to the model an exogenous variation to the collateral constraint to capture, in a reduced form, possible cross-sectional differences in the institutional framework as well as financial innovation. We call this variation financial development as it can can ease or worsen the borrowing constraint problem.

We show that, depending on the value of the financial development parameter, the equilibrium is one of three types. First, the traditional credit constrained equilibrium type described by Kiyotaki and Moore (1997) arises with low values of financial development. The second equilibrium type occurs with larger values of the parameter, corresponding to a situation in which productive agents use all the available assets in the economy but remain credit constrained. The third equilibrium type occurs with yet larger values of the financial development parameter and corresponds to the case in which productive agents use all available assets and the financial sector allows them to become unconstrained, akin to Moll (2010).

The main point of the paper is to show that there is a non-monotonic relationship between financial development and the volatility of income. When a negative aggregate technology shock hits, the dynamic interaction between credit limits and asset prices can amplify the effect of the shock on output and wealth. We show that the degree of amplification depends on the level of financial development and that this relationship is non-monotonic.

This non-monotonicity is due to two opposing effects of the financial development parameter. In an environment with borrowing constraints, financial development increases the sensitivity of the credit limits to changes in the value of the collateral. Thus, when a technology shock reduces the value of the collateral, the credit limits fall more in economies with a more developed financial sector. This effect amplifies the impact of the shock. On the other hand, as the economy recovers from the shock and the collateral increases in value, the credit
limit increases more in economies with more financial development. This accelerates the recovery, thus increasing the value of the collateral and the credit limit at the time of the shock. This effect dampens the impact of the shock. We argue that this non-monotonic relationship is general to models with endogenous borrowing constraints.

We are not alone in exploring the relationship between income level and the volatility of income nor the connection between business cycle fluctuations and the financial sector. In relating financial frictions and economic development, our paper is closest to the work of Buera and Shin (2008). We are also related to a long list of theoretical work from the economic development literature like Banerjee and Duflo (2005), Buera (2008), Matsuyama et al. (2007). In relating financial frictions and output volatility our work is similar to Brunnermeier and Sannikov (2010), Gertler and Kiyotaki (2010) and Ranciere et al. (2008).

The rest of the paper is organized as follows. Section 2 reviews the literature that relates the financial sector and business cycles. Section 3 describes our model which builds upon the work of Kiyotaki (1998) and more generally on Kiyotaki and Moore (1997) while section 3.3 discusses the equilibria in detail. Section ?? describes the parametrization and discusses the main results of the cross-sectional analysis. Section 4.2 discusses the results of the dynamic exercises. Section 5 has some concluding remarks.

2 Related literature

Our paper mainly touches upon two separate strands of literature. First, the literature on the effects of financial intermediation on economic fluctuations. Second, the literature on the effects of financial intermediation on economic development.

2.1 Financial Intermediation and Business Cycles

The idea that the financial sector can be a source of economic fluctuations is not new but has been reexamined recently in light of the last global financial crisis and the recession that ensued. The work of Keynes and Minsky are early examples of this idea. A more formal treatment of this idea was developed by Bernanke and Gertler (1989) and Kiyotaki and Moore (1997). These two papers started a long literature on the “financial accelerator” by which the effects of a fundamental (technological) shock can be amplified by the financial sector. These papers develop theories of endogenously determined credit limits and are useful to understand how the financial system can amplify small technological shocks and make them more persistent. In these models, credit constraints arise because creditors cannot force debtors to repay debts unless they are secured by collateral. Bernanke and Gertler (1989)
focuses on the costly state verification problem between creditors and debtors which gives rise to the external financing premium, i.e. the difference between the costs of internal funding via retained dividends and the costs of external funding, while Kiyotaki and Moore (1997) focuses on the level of credit in the economy being constrained by the value of the collateral, which in turn depends on asset prices. Christiano et al. (2009) study the importance of the financial sector in explaining business cycle fluctuations by augmenting a standard monetary DSGE model to include a detailed banking sector. Their model suggest that financial intermediation turns diversifiable sources of idiosyncratic risk, into an aggregate systematic source or risk.

Diamond and Rajan (2001, 2000) argue that the fragility of the financial system, and in particular of banks, is a necessary implication of their maturity transformation activity because of the risk of bank runs (Diamond and Dybvig (1983)). Provided the banking system is necessarily fragile, from time to time shocks to the system will end up spilling over to the real economy increasing the volatility of income. The problem with this argument is that bank runs (and therefore spillovers) can be ruled out by simple government policies like deposit insurance.¹

The recent financial crisis also spurred interest in examining the possibility of new linkages between the financial sector and the real economy. One of the new channels being explored in the literature is the informational channel through dispersed, private or heterogeneous information. In a model with heterogeneous information, La’O (2010) shows how financial frictions in the form of collateral constraints can drive business cycle fluctuations even when fundamentals remain constant in a standard real business cycle model. Another novel channel of amplification is the one explored by Brunnermeier and Sannikov (2010). In their model, amplification of shocks occurs not only through the fall in the value of collateral, but also through the increased volatility in asset prices which in turn provides incentives for households to hold precautionary cash balances and fire sales. Gertler and Kiyotaki (2010) is another recent example of a model that attempts to connect financial frictions with business cycle fluctuations. They consider a mechanism of endogenous risk taking by which lower exogenous risk leads to greater leverage that may lead to higher total risk. In their model, securitization helps share idiosyncratic risk but amplifies endogenous risk. They calibrate their model and then generate a crisis experiment to show the effects on asset prices and output of an exogenous change to capital quality.

In sum, several strands of literature agree on the fact that the financial system can, at times and under certain conditions, generate and amplify business cycle fluctuations. However,

¹Bank runs may still be a relevant mechanism to explain potential spillovers since banks are not the only institutions performing maturity transformation in the financial system and traditionally these non-bank institutions do not participate in traditional deposit insurance mechanisms.
as argued by Matsuyama et al. (2007), models that study financial market imperfections still have not agreed upon the implications for growth and development. For example, do financial market frictions create recessions or boom-and-bust cycles? Our paper examines this quantitatively in a setting where the financial system is a source of amplification but not of risk itself.

2.2 Financial Intermediation and Development

Although our model has implications for the level of income, most of the literature on financial intermediation and development looks at the rate of growth of income. The theoretical linkages between financial intermediation and economic growth have been explored by Townsend (1983, 1978), Greenwood and Jovanovic (1990), Bencivenga and Smith (1991), among others. In general the positive effect of financial intermediation on growth has to do with at least one of the following: a) providing valuable aggregation of information; b) easing borrowing constraints; c) reducing moral hazard and other informational problems by scrutinizing investment projects; d) helping capital reallocation from low-return liquid assets to high-return ones; e) improving risk taking and risk sharing; or f) boosting technological innovation by identifying investment opportunities.

Evidently, both the growth rate of an economy and the development of the financial sector are endogenous, so economic growth may also affect financial development positively. One mechanism is that the financial services are costly to provide and develop only over time. Thus, a low income economy cannot afford to provide these services at the scale that a high income country can. This is the point of Townsend (1983). In this vain, Greenwood and Jovanovic (1990) develop a model where financial development promotes growth because it allows a higher rate of return on capital while growth allows the investment in financial development.

Bencivenga and Smith (1991) consider an endogenous growth model in which the introduction of a financial sector shifts the composition of savings towards capital accumulation which, under some general conditions, leads to a higher growth rate. This result is valid even in the special case of a model without externalities in which the steady state growth rate is larger when financial intermediation is present.

Empirically, the causal relationship between the growth rate of an economy and the level of financial development is hard to test because of the endogeneity problem. Still, Kuznets (1955) noticed that the size of the financial sector (defined as banking, insurance and real estate) as a share of GDP grows steeply with per capita income. Beyond the mere correlation, several studies have tried to identify the direction of causality. Rajan and Zingales (1998) explain this
correlation by arguing that a more developed financial system provides lower costs of capital to firms. Therefore, in their view, there is a causal effect from financial development to the growth rate. Using within country industry variations they identify that financial development affects growth more than the average in industries that technologically are more dependent on external financing.

More recent work has explored the role of financial frictions and growth, for example Buera and Shin (2008). Another angle to this idea is Moll (2010) which explores the role of entrepreneurship in easing borrowing constraints.

3 The Model

3.1 Overview of the Model

Kiyotaki and Moore (1997) and Kiyotaki (1998) present a dynamic model in which durable assets play a dual role, as factors of production and as collateral for loans. Our model is an extension to their work with the addition of an exogenous multiplicative parameter to the borrowing constraint. We call this parameter the degree of financial development.

In the model, there are two inputs in the production of final goods: physical capital, which is produced from the final good, and land, which is in fixed supply and normalized to 1. The economy has two types of agents, productive and unproductive, which are both risk averse and derive utility from consumption. Agents’ type evolves between the two states according to a Markov process and the realization of the productivity is known at the time they make their investment decisions.

Because of the heterogeneity in productivity, agents engage in borrowing and lending. We interpret the volume of lending and borrowing activity between agents in this economy as the size of the financial sector. This activity has a friction in the form of imperfect enforcement of debt contracts since borrowers can only recover a fraction of the non-collateralized credit if entrepreneurs can walk away from their debt. As a protection, lenders demand some collateral, which gives rise to the borrowing constraint faced by the productive agents.

We model this by assuming that agents cannot borrow more than a constant times the value of their collateral. Our variation is to introduce a parameter multiplying the value of the collateral. We interpret this scaling parameter as a measure of conditions that ease or aggravate the borrowing constraints, for example weakness of the legal system may further aggravate the constraint. On the other hand, we allow this parameter to represent the possibility of lending on margin in which case, the value of the collateral can be smaller than the amount borrowed. This variation gives rise to three types of equilibrium depending on
parameter values. We now turn to the details of the model.

### 3.2 Model Setup

Consider a discrete-time economy with a single homogenous good, a fixed asset, and a continuum of agents which live for ever and have preferences of the form:

$$
E_t \left( \sum_{\tau=0}^{\infty} \beta^\tau \ln c_{t+\tau} \right).
$$

In this economy, there are two production inputs, the fixed asset $k_t$, called land, which is in fixed supply and does not depreciate, and an intermediate good $x_t$, called investment, which fully depreciates every period.

Every agent shifts stochastically between two states, the productive and unproductive according to a Markov process. Productive agents become unproductive next period with probability $\delta$, and unproductive agents become productive with probability $n\delta$, where $n < 1$. We use the superscript $i$ for agent $i$, $p$ for the set of agents that are productive, and $u$ for the set of unproductive agents. The production function of both types exhibit constant returns to scale, but the total factor productivity of productive agents is higher than the one of unproductive agents:

$$
y_{i+1}^p \equiv \alpha \left( \frac{k_{i+1}^p}{\sigma} \right)^\sigma \left( \frac{x_{i+1}^p}{1 - \sigma} \right)^{1-\sigma} \quad \forall i \in p
$$

$$
y_{i+1}^u \equiv \gamma \left( \frac{k_{i+1}^u}{\sigma} \right)^\sigma \left( \frac{x_{i+1}^u}{1 - \sigma} \right)^{1-\sigma} \quad \forall i \in u
$$

where $\gamma < \alpha$ and $0 < \sigma < 1$.

To maximize their expected utility every agent chooses a path of consumption, land holdings, investment, output and debt subject to the technology constraints (1) and (2) and the corresponding flow-of-funds constraint:

$$
c_t + x_t + q_t (k_t - k_{t-1}) = y_t + \frac{b_{t+1}}{r_t} - b_t,
$$

for all $t > 0$, where $q_t$ denotes the price of land, $b_t$ is the debt repayment in period $t$, $b_{t+1}$ is the new debt in period $t$, and $r_t$ is the real interest rate at time $t$. More precisely, $q_t$ is the price of a unit of land at time $t$ that can be used in production during the period $t + 1$.

Every period, agents face the following borrowing constraint:

$$
b_{t+1} \leq \theta q_{t+1} k_t, \ \theta \in [0, \infty).
$$
This implies that the debt repayment cannot exceed the adjusted value of the collateral at period \( t+1 \), where the adjustment is given by \( \theta \). We allow this adjustment parameter to vary in the cross-section and regard \( \theta = 1 \) as the benchmark value, which corresponds to the model of Kiyotaki (1998). When \( \theta \) is smaller than 1, the collateral constraint is further reduced because lenders can only recover a fraction of the collateral. On the other hand, when \( \theta > 1 \), the borrowing constraint problem is reduced because it implies lenders can recover the whole collateral plus a fraction of the non-collateralized credit.

We interpret the parameter \( \theta \) as a reduced form summary of potential factors that allow expand or constrain the borrowing and lending between productive and unproductive agents. Factors that constrain the borrowing potential are for example weaknesses in the legal system that hinder lenders from collecting the collateral from borrowers. Factors that expand the borrowing potential are for example the ability of financial institutions to lend on margin. For the case when \( \theta > 1 \), note that some conditions have to be met for the existence of equilibrium which we clarify below.

### 3.3 Equilibrium types

Depending on the value of \( \theta \) and the productivity parameters \( \alpha \) and \( \gamma \), we can have one of three types of equilibrium which we describe in detail below. An equilibrium is defined in the standard way: a set of sequences of state-contingent allocations for each type of agent \( \{c_t, k_t, b_t\}_{t=0}^{\infty} \) and prices \( \{q_t, r_t\}_{t=0}^{\infty} \) such that i) unproductive agents maximize lifetime expected utility subject to their period budget constraints, ii) productive agents maximize lifetime expected utility subject to their budget and borrowing constraints, and iii) land and goods markets clear.\(^2\)

Given the logarithmic preferences, a simpler way to characterize the equilibrium is in terms of the aggregate wealth, the share of wealth that productive agents hold, the land price and the user cost of capital. We focus on describing the equilibrium using the real interest rate, whether the borrowing constraint is binding or not and the holdings of land. This characterization will allow us to provide intuition in the simulations we present in section ??.

We relegate proofs to the Appendix.

\(^2\)Note that all agents have a different history of shocks and, hence, a different wealth level. Still, even if the productive agents have different levels of fixed asset, in equilibrium they are all constrained or unconstrained. The reason is that if a productive agent is constrained, the other productive agents should keep increasing their borrowing until the point where their constraint is also reached, since the return they get is higher than the cost of borrowing.
3.3.1 Equilibrium Type I

Equilibrium type 1 occurs when the borrowing constraint is binding for all productive agents and investment is positive for all agents. In each period, both productive and unproductive agents optimally chose to consume a fraction $(1 - \beta)$ of their wealth $w_t$:

$$c_t = (1 - \beta) w_t.$$  \hspace{1cm} (5)

where agents’ wealth equals their output plus the value of the land minus the value of their debt,

$$w_i^t \equiv y_i^t + q_t k_{i-1}^t - b_i^t.$$ \hspace{1cm} (6)

**Unproductive agents.** The investment-to-land ratio of unproductive agents is given by

$$\frac{x_i^t}{k_i^t} = \frac{1 - \sigma}{\sigma} u_t \forall i \in u,$$ \hspace{1cm} (7)

where we define $u_t \equiv q_t - q_{t+1}/r_t$ as the user cost of land. In this case (verified in the Appendix), the real interest rate is given by:

$$r_t = \frac{1}{q_t} \left( q_{t+1} + \gamma \left( \frac{k_i^t}{\sigma} \right)^{\sigma-1} \left( \frac{x_i^t}{1 - \sigma} \right)^{1-\sigma} \right) = \gamma u_t^{-\sigma}.$$ \hspace{1cm} (8)

The real interest rate is given by the opportunity cost of unproductive agents’ land holdings. Intuitively, in this equilibrium since unproductive agents’ investment is positive, the rate of return of lending and investing has to be equal. Therefore, the real interest rate is given by the opportunity cost of unproductive agents. Moreover, the evolution of wealth of unproductive agents is given by:

$$w_i^{t+1} = \gamma u_t^\sigma \beta w_i^t \forall i \in u,$$ \hspace{1cm} (9)

This means that the wealth of unproductive agents next period is equal to their lending and investing this period, which equals a fraction $\beta$ of current wealth, times the rate of return of their lending and investing, given by the real interest rate.

**Productive agents.** Now we turn to the choices of productive agents. In this benchmark equilibrium, the rate of return on investment of productive agents exceeds the real interest rate, therefore productive agents borrow up to the credit limit:

$$b_i^{t+1} = \theta q_{t+1} k_i^t \forall i \in p.$$ \hspace{1cm} (10)

In this equilibrium type, the optimal investment-to-land ratio of productive agents:

$$\frac{x_i^t}{k_i^t} = g(\omega_t) \forall i \in p.$$ \hspace{1cm} (11)
where \( g(\omega_t) \) is defined implicitly by:
\[
\frac{\sigma}{1 - \sigma} g(\omega_t) + q_{t+1} \frac{1 - \theta}{\alpha} \left( \frac{\sigma}{1 - \sigma} g(\omega_t) \right)^\sigma - \left( q_t - \theta \frac{q_{t+1}}{r_t} \right) = 0
\] (12)

where \( \omega_t \) stands for \((q_{t+1}, q_t, r_t; \alpha, \sigma, \theta)\). See the Appendix for the derivation.

Turning now to the optimal choices of productive agents, we can write their investment and land holdings as a proportion of their wealth by using the borrowing constraint (10), the flow of funds constraint (4), and the optimal consumption choice (5):
\[
x^i_t + (\theta u_t + (1 - \theta) q_{t+1}) k^i_t = \beta w^i_t.
\] (13)

Note that term in front of land is the user cost of capital plus an adjustment term which is positive for values of \( \theta > 1 \). Using the previous equation and equation (11) that defines the investment-to-land ratio of productive agents, we get the investment and land in terms of the their current wealth and the user cost of capital:
\[
k^i_t = \frac{1}{g(\omega_t) + (\theta u_t + (1 - \theta) q_t)} \beta w^i_t,
\] (14)
\[
x^i_t = \frac{g(\omega_t)}{g(\omega_t) + (\theta u_t + (1 - \theta) q_t)} \beta w^i_t.
\] (15)

The evolution of wealth of productive agents is given by:
\[
w^i_{t+1} = h(\omega_t) \beta w^i_t
\] (16)

where
\[
h(\omega_t) = \alpha \left( \frac{\sigma}{1 - \sigma} g(\omega_t) \right)^{-\sigma}.
\] (17)

and \( g(\omega_t) \) is defined in equation (12).

This means that the wealth of productive agents next period is equal to the non-consumed fraction of today’s wealth times the function \( h(\cdot) \), which is decreasing in the investment-to-land ratio of productive agents (see the Appendix).

Finally, in equilibrium the land market, the goods market, and credit markets must clear.

The perfect-foresight benchmark equilibrium at time \( t \) given state variables \((W_t, s_t, q_t)\) is described by a sequence of land prices, aggregate wealth, productive agents’ wealth ratio, and user cost of land, \( \{W_{t+1}, s_{t+1}, q_{t+1}, u_t \mid t = 0, 1, 2, \ldots\} \), satisfying the following dynamic equations:

1. Aggregate wealth \( W \) can be expressed as:
\[
\beta W_t = \frac{1 - \sigma}{\sigma} u_t + \frac{g(\omega_t) - \frac{1 - \sigma}{\sigma} u_t}{g(\omega_t) + \left( q_t - \theta \frac{q_{t+1}}{r_t} \right)} s_t \beta W_t + q_t.
\] (18)
2. Aggregate wealth evolution:

\[
W_{t+1} = \int_{i \in p} w_{t+1}^i di + \int_{i \in u} w_{t+1}^i di = (h(\omega_t) s_t + r_t (1 - s_t)) \beta W_t.
\]

(19)

where \( r_t \) is defined in equation (8) and \( h(\omega_t) \) is defined in equation (17).

3. Evolution of the productive agents’ share of aggregate wealth:

\[
s_{t+1} = \frac{\int_{i \in p} w_{t+1}^i di}{W_{t+1}} = \frac{(1 - \delta) h(\omega_t) s_t + n\delta r_t (1 - s_t)}{h(\omega_t) s_t + r_t (1 - s_t)}.
\]

(20)

4. Land price evolution:

\[
q_{t+1} = r_t (q_t - u_t).
\]

(21)

3.3.2 Equilibrium Type II: Constrained Productive Agents and Zero Investment of Unproductive Agents

The second type of equilibrium, which we call type II, is the one in which the borrowing constraint is binding for productive agents and investment is zero for unproductive agents. The benchmark equilibrium and the equilibrium type II only differ in the land holdings between both types of agents since all the land is owned by productive agents in each period.

As before, when productive agents are constrained, they borrow up to the credit limit and the credit market clearing condition \(-\int_{i \in u} b_i di = \int_{i \in p} b_i^i di\) together with the unproductive agents’ budget constraint determine the amount of land held by unproductive agents:

\[
\left( q_t + \frac{1 - \sigma}{\sigma} \right) \int_{i \in u} k_i^i = \beta (1 - s_t) W_t - \int_{i \in p} \frac{\theta q_{t+1} k_i^i}{r_t} di,
\]

(22)

which is a non-negative. Hence, for the equilibrium described in the previous subsection to be the prevailing one, the right hand side of equation (22) must be non-negative when the interest rate is the one defined in equation (8).

If this is not the case, then productive agents hold all the land and, using equation (22), the interest rate is equal to

\[
r_t = \frac{\theta q_{t+1}}{\beta (1 - s_t) W_t}.
\]

(23)

Note importantly that this does not imply that the borrowing constraint of productive agents is not binding and that productive agents can freely choose their optimal amount of borrowing. In particular, the ratio \( x_t/k_t \) is lower than in the unconstrained equilibrium case, since productive agents cannot buy as much intermediate good as it would be optimal for them.
As before, the perfect-foresight type II equilibrium is described by a sequence of land prices $q_t$, aggregate wealth $W_t$, and productive agents’ wealth ratio $s_t$, and user cost of capital $u_t$, \{\(q_t, W_t, s_t, u_t \mid t = 0, 1, 2, \ldots\)\}. It satisfies the same dynamic equations of equilibrium I (18)-(21) with the difference that the interest rate is given by (23) and the exception of the aggregate wealth definition:

1. Aggregate wealth $W$:
   \[
   \beta W_t = g_t(\omega_t) + q_t \tag{24}
   \]

Again, at any moment $t$, this is a system of five equations in five unknowns, $(W_{t+1}, s_{t+1}, q_{t+1}, g(\omega_t), u_t)$, since the variables $(W_t, s_t, q_t)$ are known at time $t$ and taken as exogenous.

### 3.3.3 Equilibrium Type III: Unconstrained Productive Agents

The third equilibrium type, which we call type III, occurs when the borrowing constraint of productive agents is not binding and the non-negativity constraints of unproductive agents are binding. In this type of equilibrium productive agents can achieve a higher rate of return and the real interest rate is equal to their opportunity cost of land. If the parameters of the model and the wealth distribution are such that the borrowing constraint does not bind, the land holdings of unproductive agents is zero, therefore do not undertake any investment. For productive agents, the first order conditions are the same as the ones for unproductive agents in equilibrium type II, which imply an investment-to-land ratio of productive agents:

\[
\frac{x_i^t}{k_i^t} = \frac{1 - \sigma}{\sigma} u_t \forall i \in p, \tag{25}
\]

where the real interest rate is

\[
r_t = \alpha u_t^{-\sigma}. \tag{26}
\]

Notice the slight difference in the equilibrium condition where the interest rate reflects the productivity level of productive agents, $\alpha$.

As before, the perfect-foresight type III equilibrium is described by a sequence of land prices $q_t$, aggregate wealth $W_t$, and productive agents’ wealth ratio $s_t$, and user cost of land $u_t$, \{\(q_t, W_t, s_t, u_t \mid t = 0, 1, 2, \ldots\)\} which satisfy the following dynamic system of equations:

1. Aggregate wealth is equal to:
   \[
   W_t = \frac{\alpha}{\sigma} u_{t-1}^{-\sigma} + q_t. \tag{27}
   \]

2. Aggregate wealth evolution:
   \[
   W_{t+1} = r_t/\beta W_t. \tag{28}
   \]

where $r_t$ is defined in (26).
3. Evolution of the productive agents’ share of aggregate wealth:

\[ s_{t+1} = (1 - \delta) s_t + n\delta (1 - s_t). \]  

(29)

3.3.4 Equilibrium type bounds

If we keep all the parameters constant except the borrowing constraint parameter \( \theta \), we can think of two equilibrium thresholds \( \theta_t^L \) and \( \theta_t^H \) such that

\[
\begin{aligned}
& \text{if } \theta < \theta_t^L & \text{the period-equilibrium type is } I \\
& \in (\theta_t^L, \theta_t^H) & \text{the period-equilibrium type is } II, \\
& > \theta_t^H & \text{the period-equilibrium type is } III,
\end{aligned}
\]  

(30)

where \( \theta_t^L \) denotes the upper bound of equilibrium type I and \( \theta_t^H \) denotes the lower bound of equilibrium type III.

**Equilibrium type I upper bound.** Equilibrium type I is feasible if the solution to equations (18) - (21) in section 3.3.1 is such that productive agents do not own all the productive inputs and, consequently, unproductive agents produce a positive amount of output. For equilibrium II to be feasible, on the other hand, it must be that unproductive agents optimally choose to lend their saving to productive agents instead of investing in land, which is the case when \( r_{t,II} \geq \gamma u_{t,II}^{-\sigma} \).

Thus, when \( \theta \) is such that \( r_{t,II} < \gamma u_{t,II}^{-\sigma} \) equilibrium type II is unfeasible. Let \( \theta_t^L \) denote the lower bound of equilibrium II, which must satisfy

\[ r_{t,II} (\theta_t^L) = \gamma u_{t,II} (\theta_t^L)^{-\sigma}. \]  

(31)

Using equations (12), (21), (23), and (24), we can rewrite (31) as

\[ \frac{\theta_t^L - \alpha [\frac{\sigma}{1 - \sigma} [\beta W_t - q_t]]^{-\sigma}}{1 - \theta_t^L} \frac{1}{(1 - s_t) \beta W_t} \left[ \frac{1}{1 - \sigma} q_t - \beta W_t \left[ (1 - s_t) + \frac{\sigma}{1 - \sigma} \right] \right] \left[ q_t - \beta \theta_t^L (1 - s_t) W_t \right]^{-\sigma} = \gamma, \]  

(32)

as we explain in detail in Appendix B.

**Equilibrium type III lower bound.** Equilibrium type III is feasible if the solution to equations (27) - (29) together with equation (21) is such that productive agents hold all the land and their borrowing constraints are not binding, so that \( \int_{i \in p} b_i^t di \leq \theta q_{t+1} \). Hence, the lower bound of equilibrium type III is the one determined by the following equation:

\[ \int_{i \in p} b_i^t (\theta_t^H) di = \theta_t^H q_{t+1} (\theta_t^H). \]  

(33)
Table 1: Description of the Steady State Equilibria

<table>
<thead>
<tr>
<th>Interest rate</th>
<th>I $r = \gamma u^{-\sigma}$</th>
<th>II $r = \frac{\theta q}{\beta(1-s)W}$</th>
<th>III $r = \alpha u^{-\sigma}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Land holdings</td>
<td>Interior solution</td>
<td>$K^U = 0, K^P = 1$</td>
<td>$K^U = 0, K^P = 1$</td>
</tr>
<tr>
<td>$s$</td>
<td>$s \in [n\delta, 1 - \delta]$</td>
<td>$s \in (1 - \delta, \bar{s})$</td>
<td>$s \in [\bar{s}, 1)$</td>
</tr>
<tr>
<td>$\theta$</td>
<td>$[0, \bar{\theta}]$ with $\bar{\theta} &gt; 1$</td>
<td>$\bar{\theta}, \infty$</td>
<td>$\bar{\theta}, \infty$</td>
</tr>
<tr>
<td>Description</td>
<td>Constrained/Interior borrowing</td>
<td>Constrained/All land</td>
<td>Unconstrained</td>
</tr>
</tbody>
</table>

If we then use equations (3), (25), (21), and (24), we get that the lower bound of equilibrium III is equal to

$$\theta^H_t = \frac{(1 - \sigma) \beta (1 - s_t) W_t}{q_t - \sigma \beta W_t},$$

as explained in the appendix. Note that if $\theta^L_t \leq \theta^H_t$, there is no coexistence of equilibrium types at a given point in time.

3.4 Steady State

From the equilibrium conditions of our setting we can see that, for each equilibrium type, there is a unique steady state $(\bar{q}, \bar{W}, \bar{s}, \bar{u})$ where the aggregate wealth, the productive agents’ wealth, the price of capital and $u_t$ are all constant. To distinguish the three types, let the superscripts I, II and III define the steady state variables types.

The equilibrium path has to satisfy a transversality condition ruling out exploding bubbles in the land price, which implies that, for a given initial conditions, the optimal path is the sequence $\{q_t, W_t, s_t, u_t \mid t = 0, 1, 2, ..\}$ that converges to the Steady State.

Table 1 summarizes the main equilibrium quantities for each steady state. The most important is the real interest rate which for steady state type I is given by the opportunity cost of land of unproductive agents. Let $\theta$ be the threshold between the equilibria I and II, and let $\bar{\theta}$ be the threshold between II and III. For $\theta < \bar{\theta}$, the real interest rate depends on the level of financial development and the share of wealth of productive agents. Finally for $\theta > \bar{\theta}$ the real interest rate is given by the opportunity cost of land of productive agents.
Table 2: Summary of the parameter values of the calibration

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>0.95</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.5</td>
</tr>
<tr>
<td>$\delta$</td>
<td>0.7</td>
</tr>
<tr>
<td>$n$</td>
<td>0.03</td>
</tr>
</tbody>
</table>

4 Numerical results

4.1 Parametrization

To understand the relation between financial development and the productivity difference, we plot the regions of each type of equilibria for the parameter values presented in table 2. For the share of land in the production function we set $\sigma = 0.5$. The discount factor is set to $\beta = 0.95$. For the Markov process we set the following values: $\delta = 0.7$ for the probability that a productive agent becomes unproductive, and $n = 0.03$ which is the ratio of productive to unproductive agents. Note that $n\delta$ is the probability that an unproductive agent becomes productive next period and that $n\delta + \delta < 1$. Figure 1 shows the different equilibria for a range of feasible values of $\theta$ and the productivity difference between productive and unproductive agents, $\alpha - \gamma$. Notice first that the upper bound between regions II and III does not depend on productivity, $\alpha - \gamma$. However the boundary between regions I and II does depend on productivity differences and occurs at a higher level of $\theta$ when the productivity difference between each type of agent is smaller.

4.2 Quantitative effects of a productivity shock

This section describes the exercise to evaluate the transition dynamics after an unexpected aggregate productivity shock. Figures 2 and 3 summarize the dynamics of different aggregate quantities after an unexpected shock of 1% to aggregate productivity for five different values of $\theta = [.75, .95, 1, 1.04, 1.12]$. These values were chosen so that given our parametrization of $\alpha$ and $\gamma$, the first three values of $\theta$ lie in region I, the fourth in region II and the last in region III. When the borrowing constraint parameter is below the benchmark level the computation of the dynamics are straightforward but when the parameter is above this level, the computation is more involved. For simplicity, we choose paths in which the equilibrium at every point in the transition is of the type achieved in steady state. We describe in detail the particular algorithm to compute the dynamics in the Appendix. We do not rule out other
Figure 1: Regions of each type of equilibria as a function of $\alpha - \gamma$ given $\theta$. This plot maps the corresponding combination to $\theta$ and $\alpha - \gamma$ to the type of equilibria with the rest of the parameters calibrated as in Table ??.

The upper bound between regions $II$ and $III$ is given by equation (??) which does not depend on $\alpha - \gamma$. The boundary between regions $I$ and $II$ occurs at a higher level of $\theta$ when the difference between the productivity levels of each type of agent is smaller.
potentially convergent paths with cycles or jumps between the three types of equilibria.

We are interested in showing the non-monotonic relationship between the transition dynamics and the particular value of $\theta$. Figure 2 shows the dynamics of the land price, $q_t$, and the dynamics of the real interest rate, $r_t$. On impact, the land price falls more when $\theta$ is larger if the equilibrium is in region $I$. In fact, the largest potential fall in the land price in our model occurs when the equilibria is around the threshold value $\theta_*$ between equilibria $I$ and $II$. For values of $\theta$ that lie in the region of equilibria $III$, with $\theta > \bar{\theta}$, the change in the land price on impact is smaller the larger the value of $\theta$. This reversion of the effect on land price is consistent with our idea that financial constraints have non-monotonic effects. Recall that the larger the value of $\theta$ the steady state level of wealth is higher. Intuitively when productive agents are unconstrained, shocks to productivity reduce the value of land through the reduction in output but do not interact with the borrowing constraint through the value of their collateral.

The mechanism is the following. Productivity shocks reduce the value of the collateral. As a result the level of borrowing is reduced. In the case of high level of $\theta$ borrowing is reduced more. With less borrowing, productive agents buy less land. Land is used more inefficiently. Its value is further depressed. On the other hand, a higher level of $\theta$ accelerates the recovery after a productivity shock. In other words, the higher the financial development, the more productive agents can borrow as their collateral gains value. This speeds recovery and a more efficient use of the collateral over time. The combination of these two effects provides the non-monotonicity of the dynamics with $\theta$.

Figure 3 shows the dynamics of the investment-to-land ratio of both types of agents, the share of wealth of productive agents and output. Output falls on impact 1% for the four paths as a response to the fall on productivity but clearly recovery occurs at different rates. For the two extreme values of $\theta$ recovery is almost at an identical pace, but recall that they converge to different absolute values of output and wealth. For the benchmark value $\theta = 1$ the recovery of output is the slowest of the four equilibria. The recovery path of output of equilibria II falls further after period 1 but recovers faster than the benchmark path. This fall is due to the reallocation of investment between productive and unproductive agents. When productive agents hold all the land but are still restricted a shock to the aggregate productivity has the effect of reducing the next period collateral value of land. This reallocation can be seen in the top two panels of the figure.

Our model maintains the amplification mechanism observed in other models of endogenous collateral constraints. When the equilibrium is in $I$, a higher $\theta$ amplifies the effect of the shock in terms of current wealth. On the other hand, when the equilibrium is in region $III$, a higher value of the parameter $\text{dampens}$ the effect of the shock. One interpretation of this effect is the
Figure 2: Dynamics towards the steady state after a 1% unexpected shock to the aggregate productivity level. This plot shows that the fall in percentage terms from the steady state is larger at impact for larger $\theta$ but the recovery is faster. Notice that the steady state value of wealth is higher for a larger value of $\theta$. The right panel shows the level of real interest rate for the different paths. Frequency is quarterly.
Figure 3: Transition dynamics towards the steady state of the investment-to-land ratio, share of wealth of productive agents, and total output after a 1% unexpected shock to the aggregate productivity level. Frequency is quarterly.

negative effects on growth from financial liberalization. Empirically, Beck et al. (2000) have found that financial liberalization in economies with underdeveloped financial system usually is followed by financial crises. Developing the financial sector when it is underdeveloped amplifies the effects of real shocks in the economy. On the other hand, highly developed financial sector dampen the effect of real shocks.
Figure 4: Transition dynamics towards the steady state of the total amount of borrowing after a 1% unexpected shock to the aggregate productivity level for different values of financial development. This plot shows that the size of the financial sector, measured by the total borrowing and lending, falls the most relative to the steady state value when $\theta = 1$ after an aggregate productivity shock. Farther from this value, the response is smaller. Frequency is quarterly.
5 Concluding remarks

This paper develops a model of endogenous borrowing constraints to analyze the effects to income level and volatility from the variation of the borrowing constraints. One key difference in our model from previous models of borrowing constraints is that we allow for lending to occur in equilibrium when the value of the collateral is smaller than the value of the loan. We do this by introducing a parameter of financial development in the borrowing constraint of agents. Under certain conditions of the parameter values, in equilibrium lenders fully repay their debts contracts while taking advantage of the expanded lending capacity. We show that there are three types of equilibria in our model. The traditional equilibrium were the economy faces aggregate borrowing constraints, reduced potential output and lower level of wealth. The second type of equilibrium occurs when one set of agents hold all the capital available yet they are constrained and the economy achieves a lower steady state level of wealth. The third type of equilibrium occurs when the borrowing constraints are not binding.

With this model we analyze the relationship between the level and volatility of wealth in each type of equilibrium. In our model, the financial sector allows a better allocation of capital increasing the steady state level of income. At the same time it can also amplify business cycle fluctuations through the effects of the endogenous borrowing constraints as in the traditional models of endogenous collateral constraints. However, in our model the financial sector can also dampen business cycle fluctuations when the equilibria is of the third type as in economies close to the complete markets theoretical benchmark.

We parametrize the model to analyze empirically this relationship. Comparing a cross-section of equilibria with different values the borrowing constraint parameter, the quantitative effects to level and volatility of income suggest a non-monotonic tradeoff between the added volatility and the level of income induced by the financial sector. Finally we analyze the transition dynamics in our model after a unexpected and aggregate productivity shock. The well-know amplification channel of Bernanke and Gertler (1989) and Kiyotaki and Moore (1997) remains in our model for equilibria of the first type. On impact a large financial sector, measured by the size of the borrowing constraint parameter, amplifies the impact of the productivity shock up to the point when the financial sector actually dampens the shock.

Our simple framework allows us to embed the non-monotonic relationship between income level and volatility in a computation of the cost of business cycles like in Lucas (1990). In terms of present discounted value of wealth, agents prefer the unconstrained equilibrium over the rest. However for a particular level of relative risk aversion, agents are indifferent between the type one and type two equilibrium as the highly constrained equilibrium type one has lower response to shocks than type two.
In future work we will use an extension to our model allowing for capital accumulation to explore the tradeoff of the short-run costs of income volatility and long-run effects on income growth induced by the size of the financial sector. As argued by Barlevy (2004), the potential costs of business cycles under endogenous growth may be substantially higher than the ones computed by Lucas (1990) among others. We plan to conduct more detailed welfare calculations. Another related avenue we plan to explore is to analyze the relationship between the likelihood and severity of shocks with the level of the parameter $\theta$ to evaluate potential optimal choices. Finally, an important extension we will explore is to endogenize $\theta$ to recognize that changes in the level of financial development are a costly choice and to address the relationship between the the rate of growth of financial aggregates that is associated with sharp subsequent contractions in income.

References


23


A APPENDIX. Derivation of the equilibrium conditions

At each period, agents choose the amount of consumption good \( c_t \), debt to be repaid next period \( b_{t+1} \), intermediate good, \( x_t \), and land, \( k_t \), for next period's output to maximize their expected intertemporal utility. Their optimization is subject to the budget constraint in equation (3), the borrowing constraint in equation (4) and the non-negative production input constraints \( k^i_t \geq 0 \) and \( x^i_t \geq 0 \). The agents' optimization problem is

\[
\max_{\{c_t, x_t, k_t, b_{t+1}\}} \mathbb{E}_t \left( \sum_{t=0}^{\infty} \beta^t \ln c_t \right)
\]

subject to

\[
c_t + x_t + q_t (k_t - k_{t-1}) = y_t + \frac{b_{t+1}}{r_t} - b_t
\]

\[
b_{t+1} \leq \theta q_{t+1} k_t
\]

\[
c_t \geq 0, \; x_t \geq 0, \; k_t \geq 0.
\]

Let \( \mu^i_t \) and \( \lambda^i_t \) be the multipliers on the flow-of-funds and borrowing constraints of agent \( i \) at period \( t \), and let \( \eta^i_t \) and \( \psi^i_t \) be the multipliers of the non-negative input constraints of an arbitrary agent \( i \). The first order conditions from the utility maximization problem are the following four equations:

\[
c^i_t : \quad \frac{\beta^t}{c^i_t} - \mu^i_t = 0
\]

\[
k^i_t : \quad -\mu^i_t q_t + \mu^i_{t+1} \frac{\partial f^i(k^i_t, x^i_t)}{\partial k_t} + \mu^i_{t+1} q_{t+1} + \lambda^i_t \theta q_{t+1} + \eta^i_t = 0
\]

\[
x^i_t : \quad -\mu^i_t + \mu^i_{t+1} \frac{\partial f^i(k^i_t, x^i_t)}{\partial x_t} + \psi^i_t = 0
\]

\[
b^i_{t+1} : \quad \frac{\mu^i_t}{r_t} - \mu^i_{t+1} - \lambda^i_t = 0
\]

Depending on the parameters of the model and the wealth distribution, the constraints may or may not be binding, so the multipliers may or may not be equal to zero. This gives rise to different types of equilibrium, as we analyze next. We say the equilibrium is type I if the borrowing constraint for productive agents is binding but the non-negative production constraints for unproductive agents are not binding; type II if the constraints are all binding, and type III if the non-negative input constraints are binding for unproductive agents but the borrowing constraint is not.

\(^3\)To be rigorous, consumption must also be positive but this constraint is not included in the optimization problem.
A.1 Benchmark Equilibrium (Equilibrium type I)

A.1.1 Consumption policy function

For unproductive agents, using the first order conditions with respect to $c_i^t$ and $b_{i+1}^t$ when $\lambda_i^t = \psi_i^t = \eta_i^t = 0$, we get their the Euler equation when the equilibrium type is I:

$$c_{i+1}^t = \beta r c_i^t.$$  

And using then the first order conditions with respect to $k_i^t$, $x_i^t$ and $b_{i+1}^t$ when $\lambda_i^t = \psi_i^t = \eta_i^t = 0$ together with equations (13), (6) and (21), we get equations (8) and (7), as well as the flow of funds of unproductive agents when the equilibrium type is I:

$$w_{i+1}^t = r_t \left( w_i^t - c_i^t \right).$$  

It is then possible show that the policy function $c_i^t = (1 - \beta) w_i^t$ is consistent with these two conditions that summarize the optimal choices of unproductive agents when the equilibrium type is I. When the equilibrium type is II or III, the Euler equation of unproductive agents is exactly the same. The flow of funds also looks identical, given that they do not hold any land or investment good, so their wealth next period is equal to their current non-consumed wealth times the interest rate paid on savings.

For productive agents, using the first order conditions with respect to $c_i^t$ and $x_i^t$ when $\psi_i^t = \eta_i^t = 0$, together with equation (1), we get their Euler equation when the equilibrium type is I:

$$c_{i+1}^t = \beta \alpha \left( \frac{\sigma}{1 - \sigma} \frac{x_i^t}{k_i^t} \right)^{-\sigma} c_i^t.$$  

And using the first order conditions with respect to $k_i^t$, $x_i^t$ and $b_{i+1}^t$ when $\psi_i^t = \eta_i^t = 0$ together with equations (4), (4) and (21) we get the flow of funds of productive agents when the equilibrium type is I:

$$w_{i+1}^t = \alpha \left( \frac{\sigma}{1 - \sigma} \frac{x_i^t}{k_i^t} \right)^{-\sigma} (w_i^t - c_i^t).$$  

As before, it is straightforward to show that the policy function $c_i^t = (1 - \beta) w_i^t$ is consistent with these two conditions summarizing the optimal choices of productive agents in equilibrium type I. When the equilibrium is of type II, the Euler equation and flow of funds of productive agents are exactly the same as in type I, so their optimal policy function is also $c_i^t = (1 - \beta) w_i^t$. When the equilibrium is of type III, productive agents are no longer constrained, so their Euler equation and flow of funds are the same as the ones of unproductive agents as well as the optimal policy function of consumption.
A.1.2 Wealth evolution of unproductive agents

Using the first order conditions with respect to $k_t^i$, $x_t^i$ and $b_{t+1}^i$ when $\lambda_t^i = \psi_t^i = \eta_t^i = 0$, we obtain equation (7) shown in the text. If we then use equation (7) together with the first order conditions with respect to $x_t^i$ and $b_{t+1}^i$ we get the interest rate shown in equation (8).

Using equations (2) and (7), we can rewrite next period production as:

$$y_{t+1}^i = \gamma u_t^{1-\sigma} \left( \frac{k_t^i}{\sigma} \right).$$

Moreover, from the budget constraint in equation (3), and using equations (5) and (6), $b_{t+1}^i$ can be written in the following form:

$$b_{t+1}^i = r_t c_t^i + r_t x_t^i + r_t q_t (k_t^i - k_{t-1}^i) - r_t y_t^i + r_t b_t^i$$

$$= r_t w_t^i + r_t (x_t^i + q_t k_t^i - \beta w_t^i) - r_t (y_t^i + q_t k_{t-1}^i - b_t^i)$$

$$= r_t (x_t^i + q_t k_t^i - \beta w_t^i).$$

Then, using this last result and the evolution of $q_{t+1}$ in equation (21), we can rewrite next period wealth as

$$w_{t+1}^i = y_{t+1}^i + r_t (q_t - u_t) k_t^i - r_t (x_t^i + q_t k_t^i - \beta w_t^i)$$

$$= y_{t+1}^i - r_t u_t k_t^i - r_t x_t^i + r_t \beta w_t^i$$

Finally, using this last equation together with equation (35), the real interest rate in equation (8), the optimal investment-to-land ratio of unproductive agents in equation (7) in the next wealth equation, we get:

$$w_{t+1}^i = \gamma u_t^{1-\sigma} \left( \frac{k_t^i}{\sigma} \right) - \gamma u_t^{1-\sigma} k_t^i - \frac{1-\sigma}{\sigma} u_t^{1-\sigma} k_t^i + \gamma u_t^{\sigma} \beta w_t^i,$$

which is equation (9) in the text.

A.1.3 Derivation of $g(\omega_t)$

Using the first order condition with respect to $k_t$ and $b_{t+1}$ when $\psi_t^i = \eta_t^i = 0$, dividing by $\mu_t$ and replacing the Lagrange multiplier $\lambda_t$, we obtain:

$$\frac{\mu_{t+1}}{\mu_t} \alpha \left( \frac{k_t^i}{\sigma} \right)^{\sigma-1} \left( \frac{x_t^i}{1-\sigma} \right)^{1-\sigma} = \frac{\mu_{t+1}}{\mu_t} q_{t+1} \theta + q_t - \theta \frac{q_{t+1}}{r_t}.$$

28
Now, replace $\mu_{t+1}/\mu_t$ from the first order condition of $x_t$

$$\frac{x_t}{k_t} \frac{\sigma}{1 - \sigma} = (1 - \theta) \left[ \alpha \left( \frac{k_t}{\sigma} \right)^{\sigma} \left( \frac{x_t}{1 - \sigma} \right)^{-\sigma} \right]^{-1} + q_t - \theta \frac{q_{t+1}}{r_t},$$

which can be rewritten as

$$\frac{x_t}{k_t} \frac{\sigma}{1 - \sigma} + \frac{1 - \theta}{\alpha} q_{t+1} \left( \frac{x_t}{k_t} \frac{\sigma}{1 - \sigma} \right)^{-\sigma} = q_t - \theta \frac{q_{t+1}}{r_t} = 0.$$

Using our definition of the investment-to-land ratio, it yields the equation that determines the optimal investment-land ratio of productive agents:

$$\frac{\sigma}{1 - \sigma} g(\omega_t) + q_{t+1} \frac{1 - \theta}{\alpha} \left( \frac{\sigma}{1 - \sigma} g(\omega_t) \right)^{-\sigma} = q_t + \theta \frac{q_{t+1}}{r_t} = 0.$$

### A.1.4 Wealth evolution of productive agents

Following a similar strategy as for unproductive agents, we can get the wealth dynamics for productive agents deriving the factor demand function and using the assumption of the binding borrowing constraint.

Start by replacing the borrowing constraint in equation (4) in the budget constraint in equation (3) and the definition of wealth in equation (6) to obtain:

$$c_i^t + x_i^t + q_i k_i^t = w_i^t + \frac{\theta q_{t+1} k_i^t}{r_t}.$$  

If we then replace the optimal consumption rule from equation (5) and rearrange, we obtain:

$$x_i^t + k_i^t \left( q_t + \frac{\theta q_{t+1}}{r_t} \right) = \beta w_i^t.$$  

Equations ((15)) and ((14)) follow easily using the optimal investment-to-land ratio of productive agents in ((11)):

$$k_i^t = \frac{1}{g(\omega_t) + (\theta u_t + (1 - \theta) q_t) \beta w_i^t},$$  

$$x_i^t = \frac{g(\omega_t)}{g(\omega_t) + (\theta u_t + (1 - \theta) q_t) \beta w_i^t}.$$

Finally using the production function of productive agents in equation (1), we can rewrite next period output as:

---

4Note that a solution to this equation exists and is unique when $\theta \leq 1$ because the LHS of the previous equation strictly increases with $g(\omega_t)$.
\[ y_{i+1}^j = \frac{\alpha}{\sigma(1-\sigma)^{1-\sigma}g(\omega)^{1-\sigma}} g(\omega) + (\theta u_t + (1 - \theta) q_t) \beta w_i^j. \]

Using the last result, together with real interest rate of the benchmark equilibrium in equation \((8)\) and the investment-to-land ratio of productive agents in equations \((11)\) and \((12)\):

\[
w_{i+1}^j = y_{i+1}^j + q_{t+1}k_{i+1}^j - b_{t+1}
= y_{i+1}^j + (1 - \theta) q_{t+1}k_{i+1}^j
= \frac{\alpha}{\sigma(1-\sigma)^{1-\sigma}g(\omega)^{1-\sigma}} g(\omega) + (\theta u_t + (1 - \theta) q_t) \beta w_i^j + \frac{(1 - \theta) q_{t+1}}{g(\omega) + (\theta u_t + (1 - \theta) q_t) \beta w_i^j} \beta w_i^j
= \frac{\alpha}{\sigma(1-\sigma)^{1-\sigma}g(\omega)^{1-\sigma}} g(\omega) + (\theta u_t + (1 - \theta) q_t) \beta w_i^j + \alpha \left( \frac{\sigma}{1-\sigma} g(\omega) - \frac{\sigma}{1-\sigma} g(\omega) + (\theta u_t + (1 - \theta) q_t) \right) \beta w_i^j
= \alpha \left( \frac{\sigma}{1-\sigma} g(\omega) \right)^{-\sigma} \beta w_i^j.
\]

Hence,

\[ h(\omega_t) = \alpha \left( \frac{\sigma}{1-\sigma} g(\omega_t) \right)^{-\sigma}. \]

### A.1.5 Aggregate wealth evolution

Let \( s_t \) be the wealth share of productive agents, i.e. \( s_t = \frac{\int_{i\in p} w_{i+1}^j di}{W_t} \) and \( 1 - s_t = \frac{\int_{i\in u} w_{i+1}^j di}{W_t} \). Using equations \((16)\) and \((9)\), aggregate wealth evolution can be written as

\[
W_{t+1} = \int_{i\in p} w_{i+1}^j di + \int_{i\in u} w_{i+1}^j di
= \int_{i\in p} h(\omega_t) \beta w_{i+1}^j di + \int_{i\in u} \gamma u_{i+1}^{-\sigma} \beta w_{i+1}^j di
= (h(\omega_t) s_t + \gamma u_{i+1}^{-\sigma} (1 - s_t)) \beta W_t,
\]

which is equation \((19)\).

### A.1.6 Evolution of productive agents’ wealth share

Using equations \((16)\) and \((9)\) and the fact that a fraction \( \delta \) of productive agents at \( t \) become unproductive at \( t + 1 \) and a fraction \( n\delta \) of unproductive agents at \( t \) become productive at
\[ s_{t+1} = \frac{\int_{i \in P} w^i_{t+1} di}{W_{t+1}} \]
\[ = \frac{(1 - \delta) h(\omega_t) s_t \beta W_t + n \delta \gamma u_t^{-\sigma} (1 - s_t) \beta W_t}{(h(\omega_t) s_t + \gamma u_t^{-\sigma} (1 - s_t)) \beta W_t} \]
\[ = \frac{(1 - \delta) h(\omega_t) s_t + n \delta \gamma u_t^{-\sigma} (1 - s_t)}{h(\omega_t) s_t + \gamma u_t^{-\sigma} (1 - s_t)} \],

which is equation (20).

### A.1.7 Evolution of land price

Using the definition of \( u_t \), we get that

\[ q_{t+1} = r_t (q_t - u_t) , \]

which gives us equation (21) once we replace \( r_t \) using equation (8):

\[ q_{t+1} = \gamma u_t^{-\sigma} (q_t - u_t) . \]

### A.1.8 Aggregate Wealth

Equation (18) is derived using the definition of aggregate wealth and the land market clearing condition \( \int_{i \in u} k^i_t di + \int_{i \in p} k^i_t di = 1 \), as well as equations (5), (7), (11) and (14):

\[ W_t = \int_i y^i_t di + q_t \int_i k^i_{t-1} di - \int_i b^i_t di \]
\[ = \int_i c^i_t di + \int_i x^i_t di + q_t \int_i k^i_t di \]
\[ = (1 - \beta) \int_i w^i_t di + \int_{i \in u} x^i_t di + \int_{i \in p} x^i_t di + q_t \]
\[ = (1 - \beta) W_t + \frac{1 - \sigma}{\sigma} u_t \int_{i \in u} k^i_t di + g(\omega_t) \int_{i \in p} k^i_t di + q_t \]
\[ = (1 - \beta) W_t + \frac{1 - \sigma}{\sigma} u_t + \left( g(\omega_t) - \frac{1 - \sigma}{\sigma} u_t \right) \int_{i \in p} k^i_t di + q_t \]
\[ = (1 - \beta) W_t + \frac{1 - \sigma}{\sigma} u_t + \frac{g(\omega_t) - \frac{1 - \sigma}{\sigma} u_t}{g(\omega_t) + (\theta u_t + (1 - \theta) q_t) \beta s_t W_t + q_t} \beta s_t W_t + q_t \]

\[ \Rightarrow \]
\[ \beta W_t = \frac{1 - \sigma}{\sigma} u_t + \frac{g(\omega_t) - \frac{1 - \sigma}{\sigma} u_t}{g(\omega_t) + (q_t - \theta^{q_{t+1}})} s_t \beta W_t + q_t , \]

which is equation (18).
A.2 Equilibrium type II

As it was the case in equilibrium type I, the credit constraint is binding for productive agents, so their optimization conditions are the same and the evolution of their wealth is also given by equation (16):

\[ w_{i+1}^i = h(\omega_t)^\beta w_i^i \forall i \in p. \]

Unproductive agents, on the other hand, do not produce anything so their wealth evolution is determined by the equilibrium interest rate

\[ w_{i+1}^i = r_t \beta w_i^i \forall i \in u, \]

where \( r_t \) is specified in equation (23).

As a result, the aggregate wealth evolution is given by

\[
W_{t+1} \equiv \int_{i \in p} w_{i+1}^i di + \int_{i \in u} w_{i+1}^i di \\
= \int_{i \in p} h(\omega_t)^\beta w_i^i di + \int_{i \in u} r_t \beta w_i^i di \\
= (h(\omega_t)s_t + r_t (1 - s_t))^\beta W_t,
\]

which is equation (??). And the evolution of productive agents’ wealth share is given by

\[
s_{t+1} \equiv \frac{\int_{i \in p} w_{i+1}^i di}{W_{t+1}} \\
= \frac{(1 - \delta) h(\omega_t)s_t + n\delta r_t (1 - s_t)}{h(\omega_t)s_t + \gamma r_t (1 - s_t)},
\]

which is equation (??).

When the equilibrium is type II, \( \int_{i \in u} k_i^i di = \int_{i \in p} c_i^i di = 0 \), \( \int_{i \in u} k_i^i di = 1 \) and \( \int_{i \in p} x_i^i di = g(\omega_t) \). Thus, aggregate wealth can be written as

\[
W_t \equiv \int_i y_i^i di + q_t \int_i k_{i-1}^i di - \int_i b_i^i di \\
= \int_i c_i^i di + \int_i x_i^i di + q_t \int_i k_i^i di \\
= (1 - \beta) \int_i w_i^i di + \int_{i \in u} x_i^i di + \int_{i \in p} x_i^i di + q_t \\
= (1 - \beta) W_t + g(\omega_t) + q_t \\
\Rightarrow \beta W_t = g(\omega_t) + q_t,
\]

which is equation (24).
A.3 Equilibrium type III

When the equilibrium type is III, unproductive agents do not produce anything, as it was the case in equilibrium type II. Thus, their wealth evolution is determined by the equilibrium interest rate

\[ w_{t+1}^i = r_t \beta w_t^i \forall i \in u, \]

where \( r_t \) is specified in equation (26).

For productive agents, on the other hand, the borrowing constraint is no longer binding, so that

\[ \frac{x_t^i}{k_t^i} = \frac{1 - \sigma}{\sigma} u_t \forall i \in p, \]

as it was the case with unproductive agents in equilibrium type I. As a result, using equations (3), (5), (6), and (21)

\[
\begin{align*}
    w_{t+1}^i &\equiv y_{t+1}^i + q_{t+1}k_t^i - r_t b_{t+1}^i \\
    &= y_{t+1}^i + r_t (q_t - u_t) k_t^i - r_t (x_t^i + q_t k_t^i - \beta w_t^i) \\
    &= \alpha u_t^{1-\sigma} \left( \frac{k_t^i}{\sigma} \right) - \alpha u_t^{1-\sigma} k_t^i - \alpha \frac{1 - \sigma}{\sigma} u_t^{1-\sigma} k_t^i + \alpha u_t^{-\sigma} \beta w_t^i \\
    &= \alpha u_t^{-\sigma} \beta w_t^i \forall i \in p.
\end{align*}
\]

Hence, the evolution of aggregate wealth is given by

\[
\begin{align*}
    W_{t+1} &\equiv \int_{i \in p} w_{t+1}^i di + \int_{i \in u} w_{t+1}^i di \\
    &= \int_{i \in p} \alpha u_t^{-\sigma} \beta w_t^i di + \int_{i \in u} \alpha u_t^{-\sigma} \beta w_t^i di \\
    &= \alpha u_t^{-\sigma} \beta W_t,
\end{align*}
\]

and the evolution of productive agents’ wealth share is given by

\[
\begin{align*}
    s_{t+1} &\equiv \frac{\int_{i \in p} w_{t+1}^i di}{W_{t+1}} \\
    &= \frac{(1 - \delta) \alpha u_t^{-\sigma} s_t \beta W_t + n \delta \alpha u_t^{-\sigma} (1 - s_t) \beta W_t}{\alpha u_t^{-\sigma} \beta W_t} \\
    &= (1 - \delta) s_t + n \delta (1 - s_t).
\end{align*}
\]

When the equilibrium type is III, \( \int_{i \in u} k_t^i di = \int_{i \in p} x_t^i di = 0 \), \( \int_{i \in u} k_t^i di = 1 \) and \( \int_{i \in p} x_t^i di = \frac{1 - \sigma}{\sigma} u_t \).
Thus, aggregate wealth can be written as

\[ W_t \equiv \int y^i_t di + q_t \int k_{i-1}^i di - \int b^i_t di \]
\[ = \int c^i_t di + \int x^i_t di + q_t \int k^i_t di \]
\[ = (1 - \beta) \int w^i_t di + \int_{i \in u} x^i_t di + \int_{i \in p} x^i_t di + q_t \]
\[ = (1 - \beta) W_t + \frac{1 - \sigma}{\sigma} u_t + q_t \]

$$\Rightarrow \beta W_t = \frac{1 - \sigma}{\sigma} u_t + q_t,$$

which is equation (24).

**A.4 Equilibrium type bounds**

**A.4.1 Equilibrium type II lower bound**

As explained in section 3.3.4, the equilibrium II lower bound \( \theta_{t,I}^L \) is defined as \( r_{t,II} \left( \theta_{t,I}^L \right) u_{t,II} \left( \theta_{t,I}^L \right) \beta = \gamma \). Using equations (12), (21), and (24), we get that

\[ u_{t,II} = \left( q_t - \frac{\beta}{\beta} (1 - s_t) W_t \right), \]

and

\[ q_t+1 = \frac{\alpha \left( q_t - \beta (1 - s_t) W_t - \frac{\sigma}{1 - \sigma} g (\omega_t) \right)}{\left( \frac{\sigma}{1 - \sigma} (1 - \sigma) \right) (W_t - q_t)} \]
\[ = \frac{\alpha \left( q_t - \beta (1 - s_t) W_t - \frac{\sigma}{1 - \sigma} (\beta W_t - q_t) \right)}{\left( \frac{\sigma}{1 - \sigma} (1 - s_t) \right) (W_t - q_t)}. \]

Hence, using equation (23), we can rewrite the condition that defines \( \theta_{t,I}^L \) as:

\[ \theta_{t,I}^L = \frac{\alpha \left( q_t - \frac{\beta}{\beta} (1 - s_t) W_t \right)^{\sigma} (q_t - \beta (1 - s_t) W_t - \frac{\sigma}{1 - \sigma} (\beta W_t - q_t))}{1 - \theta_{t,I}^L} \]
\[ \left( \frac{\sigma}{1 - \sigma} (1 - s_t) \right)^{\sigma} (\beta W_t - q_t) = \gamma \]

**A.4.2 Equilibrium type III lower bound**

When equilibrium is of type III, productive agents are no longer credit constrained, so that their optimal borrowing is lower than their credit limit \( \theta q_{t+1} k_{t}^i \), so that

\[ \int_{i \in p} b_{t+1}^i di < \theta q_{t+1}. \]
Using equations (4) and (25), we can rewrite $b_{i+1}'$ as $r_t (\beta w_i' + k_i' (q_t + \frac{1-\sigma}{\sigma} u_t))$, so that
\[
\int_{s \in p} b_{i+1}' di = r_t \left(-\beta s_t W_t + q_t + \frac{1-\sigma}{\sigma} u_t\right).
\]
This implies that productive agents are not credit constrained if
\[
-\beta s_t W_t + q_t + \frac{1-\sigma}{\sigma} u_t < \theta q_{t+1}^{\theta} r_t
\]
and, using also equation (21) we get
\[
-\beta s_t W_t + q_t + \frac{1-\sigma}{\sigma} u_t < \theta (q_t - u_t),
\]
Then, from equation (33), the equilibrium III lower bound must satisfy
\[
-\beta s_t W_t + q_t + \frac{1-\sigma}{\sigma} u_t (\theta^H_t) = \theta^H_t (q_t - u_t (\theta^H_t))
\]
Using also equation (27), we get that
\[
\beta (1 - s_t) W_t = \theta^H_t (q_t - u_t (\theta^H_t))
\]
which, using equation (27) again can be rewritten as
This implies that the equilibrium III lower bound is equal to
\[
\theta^H_t = \frac{(1-\sigma) \beta (1 - s_t) W_t}{q_t - \sigma \beta W_t}.
\]

A.5 Steady State

A.5.1 Type I Steady State

From equations (18) - (21) together with equation (8), we can derive the conditions that define type I steady-state variables $(W', s', q', u')$:

- Aggregate wealth:
\[
W = \left[\beta - \beta \frac{g(\omega) - \frac{1-\sigma}{\sigma} u}{g(\omega) + \theta u - (1 - \theta) q}\right]^{-1} \left(\frac{1-\sigma}{\sigma} u + q\right),
\]
\[
1 = \beta \left[h(\omega) s + \gamma u^{-\sigma} (1 - s)\right]
\]
where
\[
h(\omega) = \alpha \left(\frac{\sigma}{1-\sigma} g(\omega)\right)^{-\sigma}
\]
and
\[
\frac{\sigma}{1-\sigma} g(\omega) + \frac{(1-\theta)}{\alpha} q \left(\frac{\sigma}{1-\sigma} g(\omega)\right)^{\sigma} - \theta u - (1 - \theta) q = 0.
\]
• Productive agents’ wealth share:

\[ s = \frac{(1 - \delta) h(\omega) s + n \delta \gamma u^{-\sigma} (1 - s)}{h(\omega) s + \gamma u^{-\sigma} (1 - s)}. \]  

(38)

• Land price:

\[ q = \frac{u}{1 - \frac{u^\sigma}{\gamma}}. \]  

(39)

To simplify the computational analysis, we can rewrite the steady-state equations as a function of the share of productive agents’ wealth, \( s \). From equation (37), we get

\[ \gamma (1 - s) u^{-\sigma} = \frac{1}{\beta} - s h(\omega). \]

Plugging this in equation (38), we get

\[ h(\omega) = \frac{n \delta - s}{\beta s [n \delta - (1 - \delta)]}. \]

Using equation (17) we can express \( g(\omega) \) as a function of \( h(\omega) \)

\[ g(\omega) = \frac{1 - \sigma}{\sigma} \left( \frac{\alpha}{h(\omega)} \right)^{1/\sigma}, \]

using (37) we get

\[ u = \left( \frac{\gamma (1 - s)}{1/\beta - s h(\omega)} \right)^{1/\sigma}, \]

and using (39) we get

\[ q = \frac{u}{1 - \frac{u^\sigma}{\gamma}}. \]

We could now use all these results in equation (12) to solve for \( s \).

A.5.2 Type II Steady State

From equations (18)-(21) together with equations (24) and (23), we can derive the equations that define the type II Steady-State variables \((W^{II}, s^{II}, q^{II}, u^{II})\):

• Aggregate wealth:

\[ \beta W = g(\omega) + q, \]  

(40)

\[ 1 = (h(\omega) s + r (1 - s)) \beta \]  

(41)

where

\[ r = \frac{\theta q}{\beta (1 - s) W}. \]
• Productive agents’ wealth share:

\[
    s = \frac{(1 - \delta) h(\omega) s + n\delta r (1 - s)}{h(\omega) s + r (1 - s)}.
\]  

(42)

• Land price:

\[
    q = u + \frac{\beta (1 - s) W}{\theta}.
\]  

(43)

To simplify the computational analysis we can express all the variables as a function of \(s\) and use then equation these results to get an expression with \(s\) as the only unknown variable. First, from equations (40) and (42),

\[
    h(\omega) = \frac{n\delta - s}{\beta s [n\delta - (1 - \delta)]}.
\]

Then, from equations (40), (41), and (43), together with (23), we get

\[
    W = \frac{\theta g(\omega)}{\beta (\theta - 1/\beta + sh(\omega))},
\]

\[
    q = \left(\frac{\theta}{\theta - 1/\beta - sh(\omega)} - 1\right) g(\omega)
\]

and

\[
    (1 - \theta)q + \theta u = -\left(\frac{\theta s}{\theta - 1/\beta - sh(\omega)} - 1\right) g(\omega).
\]

Plugging these results in equation (12), we get

\[
    \beta W = \left(\frac{1}{\beta (1 - \sigma) (1 - \delta - n\delta)} - 1\right) s^2
\]

\[
    + \left(\frac{\theta - 1/\beta}{1 - \sigma} - \frac{2n\delta}{\beta (1 - \sigma) (1 - \delta - n\delta)} + n\delta + (1 - \theta) (1 - \delta - n\delta)\right) s
\]

\[
    + \frac{n\delta}{1 - \sigma} \left(\frac{n\delta}{\beta (1 - \delta - n\delta)} + \frac{1}{\beta - \theta}\right) = 0
\]

A.5.3 Type III Steady State

From equations (27)-(29) together with equations (??) and (26), we can derive the equations that define the type III Steady-State variables \((W^{III}, s^{III}, q^{III}, u^{III})\):

• Aggregate wealth:

\[
    \beta W = \left(\frac{1 - \sigma}{\sigma} + \frac{1}{1 - \beta}\right)^{1/\sigma} (\alpha\beta)^{1/\sigma}.
\]  

(44)
• Productive agents’ wealth share:

\[ s = \frac{n}{1+n}. \]  

(45)

• Land price:

\[ q = \frac{(\alpha \beta)^{\frac{s}{1-\beta}}}{1-\beta}. \]  

(46)

A.5.4 Steady-State types thresholds

From equation (31), the threshold \( \bar{\theta} \) between type I and type II steady states is determined by:

\[ r^{II} (\bar{\theta}) = \gamma u^{II} (\bar{\theta})^{-\sigma} \]

\[ \Leftrightarrow \]

\[ \frac{\bar{\theta} q^{II} (\bar{\theta})}{\beta (1 - s^{II} (\bar{\theta})) W^{II} (\bar{\theta})} = \gamma u^{II} (\bar{\theta})^{-\sigma}. \]  

(47)

Note that \( \bar{\theta} \) has no closed form solution.

From equation (34), we get that the threshold \( \bar{\theta} \) between type II and type III steady states is equal to

\[ r^{III} (\bar{\theta}) (1 - s^{III} (\bar{\theta})) \beta W^{III} (\bar{\theta}) = \bar{\theta} q^{III} (\bar{\theta}). \]

Using equations (44) - (46), we find that

\[ \bar{\theta} = \frac{r^{III} (1 - s^{III}) \beta W^{III}}{q^{III}} \]

\[ \Leftrightarrow \]

\[ \bar{\theta} = \frac{1}{1+n} \left( 1 + \frac{1-\beta}{\beta \sigma} \right). \]  

(48)

A.5.5 Steady State uniqueness

[TBA]

B APPENDIX. Computational details

Since we only consider equilibria where there are no exploding bubbles in the land price,

\[ \lim_{t \rightarrow \infty} \frac{q_t}{r_0 r_{t-1} \ldots r_{t-1}} = 0, \]

the equilibrium path is the one that converges to the steady state, where all variables are constant. There are three types of steady states, as described in section 3, and the parameter values determine to which one the economy converges.
To examine the dynamics, Kiyotaki (1998) makes a linear approximation of the dynamical system around the steady state and takes the two stable eigenvalues. It then expresses the system in terms of the deviations from the steady state, and analyzes the impact of a small, unanticipated, temporary productivity shock. It finds that the land price and the aggregate wealth fall proportionately more than the temporary productivity shock, while they would only decrease in the same proportion without the credit constraint.

Another possible way to do the analysis, is to parametrize the model and solve it numerically.

B.1 Model simulation

The initial conditions of the economy are given exogenously, and at each period during the transition path there are also three possible types of equilibria, as described in section 2. The state variables at each period together with the parameter values, determine which one is the prevailing equilibrium at each period.

- Simulations steps:
  1. Specify value of model parameters: \((\sigma, \gamma, \beta, \delta, n; \theta)\)
  2. Specify value of initial conditions \((Y_0^j, B_{-1}^j, K_{-1}^j)_{j=u,p}\)
     (a) initial wealth \(W_0\), which depends on the initial output of agents, given by the initial stocks of land, \(k_{-1}^i\), and intermediate good, \(x_{-1}^i\):
        
        \[
        W_0 = \int_{i\in p} \alpha \left( \frac{k_{-1}^i}{\sigma} \right)^{\sigma} \left( \frac{x_{-1}^i}{1-\sigma} \right)^{1-\sigma} \, di + \int_{i\in u} \gamma \left( \frac{k_{-1}^i}{\sigma} \right)^{\sigma} \left( \frac{x_{-1}^i}{1-\sigma} \right)^{1-\sigma} \, di + q_0
        \]

        (b) initial \(s_0\), which depends on the initial outputs and initial debts:
        
        \[
        s_0 = (1-\delta) \left( Y_0^p + q_0 K_{-1}^p - B_0^p \right) + n\delta \left( Y_0^u + q_0 K_{-1}^u - B_0^u \right) \frac{W_0}{W_0}
        \]

    3. Solve for the value of the variables of interest \((q, W, s, u)\) in the steady state:
       (a) Solve for the steady state thresholds \(\bar{\theta}\) and \(\bar{\theta}\) using equations (??) and (??) or (??). Compare them with the actual \(\theta\) and determine which one is the prevailing steady state type using the rule in (??).
       (b) Solve then for the steady state value of variables \((q^*, W^*, s^*, u^*)\) using the right system of equations:
i. If steady state type is I, use system in subsection 3.1.

ii. If steady state type is II, use system in subsection 3.2.

iii. If steady state type is III, use system in subsection 3.3.

4. Use a forward shooting algorithm to solve for the time path of the variables of interest, \( \{q_t, W_t, s_t, u_t \mid t = 0, 1, 2, \ldots T \} \), where \( T \) is a large enough number:

(a) Guess an initial value for the variable \( q_0 \), and use it to obtain values for

\[
W_0 = Y_0^u + Y_0^p + q_0, \\
S_0 = \frac{(1-\delta)(Y_0^p + q_0K_{-1}^p - B_0^p) + n\delta(Y_0^u + q_0K_{-1}^u - B_0^u)}{W_0}.
\]

(b) Solve for \( \theta^L_t \) and \( \theta^H_t \) using equations (??) and (??).

(c) For \( t = 1, 2, \ldots T \), use the appropriate equation system to solve for \( (W_t, s_t, q_t, u_t) \) given \( (W_{t-1}, s_{t-1}, q_{t-1}) \):

i. If \( \theta \leq \theta^L_t \), use the system of equations in (19) - (18).

ii. If \( \theta \in (\theta^L_t, \theta^H_t) \), use the system of equations in (??) - (??).

iii. If \( \theta \geq \theta^H_t \), use the system of equations in (28) - (29).

5. Check whether \( (q_t, W_t, s_t, u_t) \) is close enough to the steady state \( (q^*, W^*, s^*, u^*) \) for some \( t \in [1, T] \).

(a) If it is, stop the code because we found a good enough approximation to the solution.

(b) If it is not, go back to step 3 trying a different initial guess for \( q_0 \).