Efficient Risk Sharing with Limited Commitment and Storage*

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Abstract

We extend the model of risk sharing with limited commitment by introducing both a public and a private (non-contractible and/or non-observable) storage technology. Public storage might be positive under general conditions, because public assets improve future risk sharing by relaxing participation constraints. This is in contrast to the case where hidden income or effort is the deep friction that limits risk sharing. If the return on storage is less than the discount rate, assets remain stochastic whenever only moderate risk sharing is implementable in the long run, but become constant if high but still imperfect risk sharing is the long-run outcome. If the return on storage is as high as the discount rate, perfect risk sharing is always self-enforcing in the long run. Further, higher consumption inequality implies higher public asset accumulation. Agents do not have an incentive to use their private storage technology, i.e., Euler inequalities are satisfied, at the constrained-efficient allocations of our model with public storage, while this is not true for the basic model under general conditions. Access to storage is welfare improving in the long run only if its return is sufficiently high that the benefits of public asset accumulation dominate the impact of the increase in the value of autarky.

Keywords: risk sharing, limited commitment, hidden storage, dynamic contracts

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1 Introduction

This paper extends the literature on risk sharing with limited commitment by introducing a storage technology. Storage potentially affects the constrained-efficient allocation through several channels. First, it allows the social planner to shift resources intertemporally. Second, it makes agents’ outside option more attractive as they have an instrument to smooth consumption in autarky. Third, if storage is not observable (and/or not contractible), it increases considerably the agents’ set of possible deviations. We provide a thorough analytical characterization of this environment. We also show that a constrained-efficient allocation can be decentralized as a competitive equilibrium with endogenous borrowing constraints similar to Alvarez and Jermann (2000).

The introduction of storage is important for several interrelated reasons. First, in several economic contexts where the model of risk sharing with limited commitment has been applied, agents are likely to have a way to transfer resources intertemporally using some aggregate technology, simple storage, savings, or a ‘backyard’ technology. In the context of village economies (Ligon, Thomas, and Worrall, 2002), households may keep grain or cash around the house for self-insure purposes, and there also exist community grain storage facilities. Households in the United States (Krueger and Perri, 2006) may keep savings in cash or ‘hide’ their assets abroad. Spouses within a household (Mazzocco, 2007) accumulate both joint assets and savings for personal use. Partners in a law firms also have both common and private assets. The public funds of the European Stability Mechanism, which has become operational in July 2012, can facilitate risk sharing across countries within the euro area. The insights we derive in this paper can be useful for all these applications.

Second, the availability of storage matters for the constrained-efficient allocation. We first show that, unless the return on storage is very low, public storage is used in equilibrium even though its return is less than the discount rate ($\beta(1 + r) < 1$). We characterize both the short- and long-run dynamics of assets, as well as the dynamics of consumption in this environment. It turns out that the level of risk sharing in the model with storage can be very different from the one in the basic model (Kocherlakota, 1996). In particular, if the return on storage is sufficiently high, the economy will converge to near perfect risk sharing even though risk sharing might be very limited in the case where public storage is not allowed.

Third, we show that even though agents would typically find it optimal to use a private

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1 In the existing models of risk sharing with limited commitment, only public and/or observable and contractible individual intertemporal technologies have been considered (Marcet and Marimon, 1992; Ligon, Thomas, and Worrall, 2000; Kehoe and Perri, 2002; Ábrahám and Cárceles-Poveda, 2006; Krueger and Perri, 2006).
storage technology in the basic model, this is not the case in the model when public storage is designed optimally. In other words, with optimal public asset accumulation the social planner can preempt the agents’ storage incentives.

Fourth, public and private storage was studied by Cole and Kocherlakota (2001) in a private information environment with full commitment. They show that public storage is never used and agents’ private savings incentives are binding in equilibrium, eliminating any risk sharing opportunity beyond self-insurance. We show that when the deep friction is limited commitment as opposed to private information, the results are very different. First, public storage is used in equilibrium. Second, private storage incentives do not bind. The main difference between the two environments is that in our environment more public storage helps to reduce the underlying limited commitment friction, while with private information public asset accumulation would make incentive provision for truthful revelation more costly.

Our starting point is the two-sided lack of commitment framework of Kocherlakota (1996), which we will often refer to as the basic model hereafter. Agents are infinitely lived, risk averse, and ex-ante identical. They receive a risky endowment each period. We assume that there is no aggregate uncertainty in the sense that the aggregate endowment is constant. Agents may make transfers to each other in order to smooth their consumption. These transfers are subject to limited commitment, i.e., each agent must be at least as well off as in autarky at each time and state of the world. The storage technology we introduce allows the planner and the agents to transfer resources from one period to the next and earn a net return $r$. We assume that agents are excluded from the returns of the publicly accumulated assets when they default (as in Krueger and Perri, 2006), although they may store privately before default to smooth consumption after default. This implies that the higher the level of aggregate assets is, the lower the incentives for default are in this economy.

To understand how public storage matters, note that limited commitment makes markets endogenously incomplete, i.e., consumption is volatile over time. This market incompleteness triggers precautionary saving/storage motives for the agents and the planner. At the same time, higher public assets reduce default incentives, thereby reduce consumption dispersion, and hence reduce the precautionary motive for saving. Further, as long as $\beta(1 + r) < 1$, i.e., agents are impatient relative to the return on storage, they would like to front load consumption. The long-run level of assets is determined by these conflicting forces. When the return on storage is high, a large stock of assets is accumulated and a considerable amount

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2See also Allen (1985) and Ábrahám, Koehne, and Pavoni (2011).
3The return $r$ can take any value such that $-1 \leq r \leq 1/\beta - 1$, where $\beta$ is the subjective discount factor. We say that the storage technology is efficient if $r = 1/\beta - 1$. 

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of additional risk sharing can be achieved. In fact, if \( \beta(1 + r) = 1 \), it is indeed optimal for the planner to fully complete the market by storage in the long run. That is, perfect risk sharing is self-enforcing in the long run in this case. This is because the trade-off between imperfect insurance and an inefficient intertemporal technology is no longer present.

To provide a full characterization of the dynamics of public assets and consumption, we establish two results: (i) There exists a threshold level of storage return above which the social planner will use public storage in the constrained-optimal allocation. (ii) Given inherited assets, storage will always be higher when cross-sectional consumption inequality is higher (due to higher current or past income inequality). This is because higher consumption inequality implies that there are higher benefits from further intertemporal consumption smoothing. One key implication of these results is that assets remain stochastic in the long run as long as the consumption distribution varies over time. This turns out to be the case if the return to storage is only moderately high. In this case the dissaving incentives are stronger than the precautionary motive for low levels of consumption dispersion, but the opposite is true when consumption is more dispersed. If the return on storage is high but \( \beta(1 + r) < 1 \) still, then assets converge to a constant level. This implies that consumption dispersion remains constant as well in the long run. Nevertheless, the economy is still characterized by partial insurance in this case, as individual consumptions change whenever an agent experiences the highest possible level of income, triggering his participation constraint to bind.

The introduction of public storage also has interesting implications for the dynamics of consumption predicted by the model when assets are stochastic in the long run. First, the amnesia property, i.e., the property that the consumption allocation only depends on the current income of the agent with a binding participation constraint and is independent of the past history of shocks whenever a participation constraint binds (Kocherlakota, 1996), does not hold. Second, the persistence property of the basic model, i.e., that the consumption allocation does not change for ‘small’ changes in the income distribution, does not hold either. There is a common intuition behind these results: the past history of shocks affects current consumptions through aggregate assets. Data on household income and consumption support neither the amnesia, nor the strong persistence property of the basic model (see Broer, 2012, for an extensive analysis). Hence, these differences are steps in the right direction for this framework to explain consumption dynamics.

We also show that constrained-efficient allocations can be decentralized as competitive equilibria with endogenous borrowing constraints (Alvarez and Jermann, 2000) and a com-
petitive financial intermediation sector which runs the storage technology (Ábrahám and Cárcceles-Poveda, 2006). In this environment, equilibrium asset prices will take into account the externality of aggregate storage on default incentives. In this sense, our paper provides a theory of endogenously growing (and shrinking) asset trees in general equilibrium.

We then consider hidden (non-contractible and/or non-observable) storage as well. Access to hidden storage not only changes the value of autarky, but it may also enlarge the set of possible deviations along the equilibrium path. That is, agents could default and store in every period either simultaneously or subsequently. This implies that, in principle, we need to consider a model where agents’ incentive to default on transfers and their incentive to store, as well as their incentive to store in autarky, are taken into account. Indeed, we show that, whenever the return on storage is high enough, and the basic limited commitment model exhibits relatively little risk sharing, the constrained-efficient allocation in the basic model without public storage is not incentive compatible if agents have access to hidden storage.\footnote{Note that this result does not hinge on how agents’ outside option is specified precisely: they may or may not be allowed to store in autarky, and they may or may not face an additional punishment for defaulting.}

This is the case because the constrained-efficient level of consumption dispersion triggers a precautionary saving motive whenever an agent has high consumption and the return on storage is high enough.

However, we show that at the constrained-efficient allocations with public storage agents no longer have an incentive to store in equilibrium. This is true because the planner has more incentive to store than the agents. First, the planner stores for the agents, because she inherits their consumption smoothing preferences. Second, storage by the planner makes it easier to satisfy agents’ participation constraints in the future. In other words, the planner internalizes the positive externality generated by aggregate assets on future risk sharing.

This result means that the characteristics of constrained-efficient allocations in a model with both public and private storage and a model with only public storage are the same. They correspond exactly as long as agents’ outside option is the same. This result also means in our model with limited commitment and public storage agents’ Euler inequalities are always satisfied. The Euler inequality cannot be rejected in micro data from developed economies, once labor supply decisions and demographics are appropriately accounted for (Attanasio, 1999). Therefore, we bring limited commitment models in line with this third observation about consumption dynamics as well.

A final question we ask is: what is the overall effect of access to storage on consumption dispersion and welfare? It is obvious that if storage is not available for the agents in autarky, access to storage strictly improves on welfare in the long run whenever it is used in equilib-
rium. It is also clear that whenever storage can be used in autarky, but public storage is not used in equilibrium, then access to storage reduces welfare, because storage reduces risk sharing by increasing the value of agents’ outside option. In all intermediate cases (public storage is used in equilibrium and private storage is available in autarky), the aggregate welfare consequences of access to storage depend on the sum of the above two forces. To see it another way, when the return on storage is relatively high we definitely have welfare gains in the long run, because the economy can get very close to perfect risk sharing. When the return on storage is lower, the negative effect of a better outside option dominates the positive affect of public assets on welfare. In the short run, public asset accumulation also has costs in terms of foregone consumption. Hence, it is a quantitative question whether access to storage improves welfare, taking into account the transition from the moment storage becomes available as well.

For this reason, we also propose an algorithm to solve the model numerically, and present some computed examples to illustrate the effects of the availability of storage and its return on asset accumulation, risk sharing, and welfare.

The rest of the paper is structured as follows. Section 2 introduces and characterizes our model with public storage. Section 3 shows that hidden storage incentives are eliminated under optimal public storage. We also show that this is not the case in the basic model. Section 4 presents some computed examples. Section 5 concludes.

2 The model with public storage

We consider an endowment economy with two types of agents, \( i = \{1, 2\} \), each of unit measure, who are infinitely lived and risk averse. All agents are ex-ante identical in the sense that they have the same preferences and are endowed with the same exogenous random endowment process. Agents in the same group are ex-post identical as well, meaning that their endowment realizations are the same at each time \( t \).\(^5\) Let \( u() \) denote the utility function, that is strictly increasing, strictly concave, and continuously differentiable. Assume that the Inada conditions hold. The common discount factor is denoted by \( \beta \).

Let \( s_t \) denote the state of the world realized at time \( t \) and \( s^t \) denote the history of endowment realizations, that is, \( s^t = (s_1, s_2, ..., s_t) \). Given \( s_t \), agent 1 has income \( y(s_t) \), while agent 2 has income equal to \( (Y - y(s_t)) \), where \( Y \) is the aggregate endowment. Note that there is no aggregate uncertainty in the sense that the aggregate endowment is constant. We further

\(^5\)We will refer to agent 1 and agent 2 below. Equivalently, we could say type-1 and type-2 agents, or agent belonging to group 1 and group 2.
assume that income has a discrete support with \( N \) elements, that is, \( s_t \in \{s^1, \ldots, s^j, \ldots, s^N\} \) with \( y(s^j) < y(s^{j+1}) \), and is independently and identically distributed (i.i.d.) over time, that is, \( Pr(s_t = s^j) = \pi^j, \forall t \). The two-types-of-agents framework together with the no-aggregate-uncertainty assumption imposes some symmetry on both the income realizations and the probabilities. In particular, \( y(s^j) = Y - y(s^{N-j+1}) \) and \( \pi^j = \pi^{N-j+1} \). The i.i.d. assumption can be relaxed as long as weak positive dependence is maintained, i.e., the expected net present value of future lifetime income is weakly increasing in current income.

Suppose that risk sharing is limited by two-sided lack of commitment to risk sharing contracts, i.e., insurance transfers have to be voluntary, or, self-enforcing, as in Thomas and Worrall (1988), Kocherlakota (1996), and others. Each agent may decide at any time and state to default and revert to autarky. This means that only those risk sharing contracts are sustainable which provide a lifetime utility at least as great as autarky after any history of endowment realizations for each agent. We assume that the punishment for deviation is exclusion from risk sharing arrangements in the future. This is the most severe subgame-perfect punishment in this context. In other words, it is an optimal penal code in the sense of Abreu (1988). Note, however, that the qualitative results would remain the same under different punishments as long as the strict monotonicity of the autarky value is maintained. For example, agents could save in autarky (as in Krueger and Perri, 2006), or they might endure additional punishment from the community for defaulting (as in Ligon, Thomas, and Worrall, 2002).

We introduce a storage technology that makes it possible to transfer resources from today to tomorrow. Assets stored earn a net return \( r \), with \(-1 \leq r \leq 1/\beta - 1\). Note that with \( r = -1 \) we are back to the basic limited commitment model (Kocherlakota, 1996).

The constrained-efficient risk sharing contract is the solution to the following optimization problem:

\[
\max_{c_i(s^t)} \sum_{i=1}^{2} \lambda_i \sum_{t=1}^{\infty} \beta^t Pr(s^t) u(c_i(s^t)),
\]

where \( \lambda_i \) is the (initial) Pareto-weight of agent \( i \), \( Pr(s^t) \) is the probability of history \( s^t \) occurring, and \( c_i(s^t) \) is the consumption of agent \( i \) when history \( s^t \) has occurred; subject to the resource constraints,

\[
\sum_{i=1}^{2} c_i(s^t) \leq \sum_{i=1}^{2} y_i(s_t) + (1 + r)B(s^{t-1}) - B(s^t), \quad B(s^t) \geq 0, \forall s^t,
\]

where \( B(s^t) \) denotes public assets when history \( s^t \) has occurred; and the participation con-
constraints,
\[
\sum_{r=t}^{\infty} \sum_{s^r} \beta^{r-t} \Pr (s^r | s^t) u (c_i (s^r)) \geq U_{i}^{au} (s_t), \quad \forall s^t, \forall i, \tag{3}
\]
where \(\Pr (s^r | s^t)\) is the conditional probability of history \(s^r\) occurring given that history \(s^t\) occurred up to time \(t\), and \(U_{i}^{au} (s_t)\) is the expected lifetime utility of agent \(i\) when in autarky if state \(s_t\) has occurred today. In mathematical terms,
\[
U_{i}^{au} (s_t) = u (y(s_t)) + \frac{\beta}{1 - \beta} \sum_{j=1}^{N} \pi^j u (y(s^j)) \tag{4}
\]
and
\[
U_{2}^{au} (s_t) = u (Y - y(s_t)) + \frac{\beta}{1 - \beta} \sum_{j=1}^{N} \pi^j u (y(s^j)).
\]
Note that the above definition of autarky assumes that agents cannot use the storage technology in autarky. This is without a loss of generality in the sense that the same qualitative characterization would hold with any outside option which is strictly increasing in current income. We will return to the case of private storage in autarky (and possibly in equilibrium) in Section 3.

2.1 Characterization

We focus on the characteristics of constrained-efficient allocations. Our characterization is based on the recursive Lagrangian approach of Marcet and Marimon (2011). However, the same results can be obtained using the promised utility approach (Abreu, Pearce, and Stacchetti, 1990).

Let \(\beta^{t} \Pr (s^t) \mu_i (s^t)\) denote the Lagrange multiplier on the participation constraint, (3), and let \(\beta^{t} \Pr (s^t) \gamma (s^t)\) be the Lagrange multiplier on the resource constraint, (2), when history \(s^t\) has occurred. The Lagrangian is
\[
\mathcal{L} = \sum_{t=1}^{\infty} \sum_{s^t} \beta^{t} \Pr (s^t) \left\{ \sum_{i=1}^{2} \lambda_i u (c_i (s^t)) + \mu_i (s^t) \left( \sum_{r=t}^{\infty} \sum_{s^r} \beta^{r-t} \Pr (s^r | s^t) u (c_i (s^r)) - U_{i}^{au} (s_t) \right) + \gamma (s^t) \left( \sum_{i=1}^{2} (y_i (s_t) - c_i (s^t)) + (1 + r)B (s^{t-1}) - B (s^t) \right) \right\},
\]
with \( B(s^t) \geq 0 \). Using the ideas of Marcet and Marimon (2011), we can write the Lagrangian in the form

\[
\mathcal{L} = \sum_{t=1}^{\infty} \sum_{s^t} \beta^t \Pr(s^t) \left\{ \sum_{i=1}^{2} \left[ M_i(s^t) u(c_i(s^t)) - \mu_i(s^t) U^{au}_i(s_i) \right] \right. \\
+ \left. \gamma(s^t) \left( \sum_{i=1}^{2} (y_i(s_i) - c_i(s^t)) + (1 + r) B(s^{t-1}) - B(s^t) \right) \right\},
\]

where \( M_i(s^t) = M_i(s^{t-1}) + \mu_i(s^t) \) and \( M_i(s^0) = \lambda_i \).

The necessary first-order condition\(^6\) with respect to agent \( i \)'s consumption when history \( s^t \) has occurred is

\[
\frac{\partial \mathcal{L}}{\partial c_i(s^t)} = M_i(s^t) u'(c_i(s^t)) - \gamma(s^t) = 0.
\]

Combining such first-order conditions for agent 1 and agent 2, we have

\[
x(s^t) \equiv \frac{M_1(s^t)}{M_2(s^t)} = \frac{u'(c_2(s^t))}{u'(c_1(s^t))},
\]

where \( u_i(s^t) = \frac{\mu_i(s^t)}{M_i(s^t)} \)

and using the definitions of \( x(s^t) \) and \( M_i(s^t) \), we can obtain the law of motion of \( x \) as

\[
x(s^t) = x(s^{t-1}) \frac{1 - v_2(s^t)}{1 - v_1(s^t)}.
\]

The planner’s Euler inequality, i.e., the optimality condition for \( B(s^t) \), is

\[
\gamma(s^t) \geq \beta(1 + r) \sum_{s^{t+1}} \Pr(s^{t+1}|s^t) \gamma(s^{t+1}),
\]

which, using (5), can also be written as

\[
M_i(s^t) u'(c_i(s^t)) \geq \beta(1 + r) \sum_{s^{t+1}} \Pr(s^{t+1}|s^t) M_i(s^{t+1}) u'(c_i(s^{t+1})).
\]

Then, using (6) and (7), the planner’s Euler becomes

\[
u'(c_i(s^t)) \geq \beta(1 + r) \sum_{s^{t+1}} \Pr(s^{t+1}|s^t) \frac{u'(c_i(s^{t+1}))}{1 - v_i(s^{t+1})},
\]

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\(^6\)Under general conditions, these conditions are also sufficient together with the participation and resource constraints.

\(^7\)To reinforce this interpretation, notice that if no participation constraint binds in history \( s^t \) for either agent, i.e., \( \mu_1(s^t) = \mu_2(s^t) = 0 \) for all subhistories \( s^t \subseteq s^t \), then \( x(s^t) = \lambda_1/\lambda_2 \) is the initial relative Pareto weight of agent 1.
where $0 \leq v_i(s^{t+1}) \leq 1$. Given the definition of $v_i(s^{t+1})$ and equation (7), it is easy to see that (8) represents exactly the same mathematical relationship for both agents.

Equation (9) determines the choice of public storage, $B'$. It is clear that the higher the return on storage is, the more incentives the planner has to store. Second, whenever we do not have perfect risk sharing, that is, $c_i(s^{t+1})$ varies over $s^{t+1}$ for a given $s^t$, the planner will have a precautionary motive for saving, as it is typical in models with (endogenously) incomplete markets. Third, the new term compared to standard models is $1/(1 - v_i(s^{t+1})) \geq 1$. This term is strictly bigger than 1 for states when agent $i$’s participation constraint is binding. Hence, future binding participation constraints amplify the return on storage. This is the case, because higher storage will make the participation constraints looser by reducing the relative attractiveness of default. The planner internalizes this effect when choosing the level of public storage.

Next, we introduce some useful notation and show more precisely the recursive formulation of our problem, which we will use to solve the model numerically. Let $y$ denote the current income of agent 1 and $V()$ denote the value function. The following system is recursive with $X = (y, B, x)$ as state variables:

$$x'(X) = \frac{u'(Y + (1 + r)B - B'(X) - c_1(X))}{u'(c_1(X))}$$

(10)

$$x'(X) = x \frac{1 - v_2(X)}{1 - v_1(X)}$$

(11)

$$u'(c_1(X)) \geq \beta(1 + r) \sum_{y'} \Pr(y') \frac{u'(c_1(X'))}{1 - v_1(X')}$$

(12)

$$u(c_1(X)) + \beta \sum_{y'} \Pr(y') V(X') \geq U^{au}(y)$$

(13)

$$u(Y + (1 + r)B - B'(X) - c_1(X)) + \beta \sum_{y'} \Pr(y') V(Y - y', B', 1/x') \geq U^{au}(Y - y)$$

(14)

$$B'(X) \geq 0.$$  

(15)

The first equation, (10), says that the ratio of marginal utilities between the two agents has to be equal to the current relative Pareto weight, and where we have used the resource constraint to substitute for $c_2(X)$. Equation (11) is the law of motion of the co-state variable, $x$. Equation (12) is the social planner’s Euler inequality, which we have derived above. Equations (13) and (14) are the participation constraints of agent 1 and agent 2, respectively, where $U^{au}(y) = U^{au}_1(s^j)$ and $U^{au}(Y - y) = U^{au}_2(s^j)$, with $y = y^j = y_1(s^j)$. Finally, equation (15) makes sure that storage is never negative.
Given the recursive formulation above, and noting that the outside option $U^{au}()$ is monotone in $y$ and takes a finite set of values, the solution can be characterized by a set of state-dependent intervals on the temporary Pareto weight. This is analogous to the basic model, where public storage is not considered (see Ljungqvist and Sargent, 2004, for a textbook treatment). The key difference is that these optimal intervals on the relative Pareto weight depend not only on current endowment realizations but also on $B$. To see this, note that we can express both $B'$ and the value function in terms of inherited assets and the end-of-period relative Pareto weight. The following lemma makes this statement more precise.

**Lemma 1.** $B'(\tilde{s}^j, B, \tilde{x}) = B'(\hat{s}^j, B, \hat{x})$ and $V(\tilde{s}^j, B, \tilde{x}) = V(\hat{s}^j, B, \hat{x})$ for all $(\tilde{s}^j, \tilde{x}), (\hat{s}^j, \hat{x})$ such that $x'(\tilde{s}^j, B, \tilde{x}) = x'(\hat{s}^j, B, \hat{x})$. That is, for determining public storage and agents’ expected lifetime utility, the current relative Pareto weight $x'$ is a sufficient statistic for the current income state, $s^j$, and last period’s relative Pareto weight, $x$.

**Proof.** Once we know $x'$, equations (10) and (12), which do not depend on $x$, give $c_1$ and $B'$. Then, the left hand side of (13) gives $V$. \qed

Lemma 1 implies that, with some abuse of notation, we can express agents’ life-time utility in terms of accumulated assets and the current Pareto weight, $V(B, x')$. It follows that the following conditions define the lower and upper bound of the optimal intervals as a function of $B$:

$$V(B, \underline{x}^j(B)) = U^{au}(y^j) \quad \text{and} \quad V\left(B, \frac{1}{\underline{x}^j(B)}\right) = U^{au}\left(Y - y^j\right).$$

Hence, given the inherited Pareto weight, $x_{t-1}$, and accumulated assets, $B$, the updating rule is

$$x_t = \begin{cases} 
\underline{x}^j(B) & \text{if } x_{t-1} > \underline{x}^j(B) \\
\overline{x}^j(B) & \text{if } x_{t-1} < \overline{x}^j(B) \\
x_{t-1} & \text{if } x_{t-1} \in [\underline{x}^j(B), \overline{x}^j(B)]
\end{cases}.$$ (17)

The ratio of marginal utilities is kept constant whenever this does not violate the participation constraint of either agent. When the participation constraint binds for agent 1, the relative Pareto weight moves to the lower limit of the optimal interval, just making sure that this agent is indifferent between staying and defaulting. Similarly, when agent 2’s participation constraint binds, the relative Pareto weight moves to the upper limit of the optimal interval. Thereby, ex ante it is guaranteed that as much risk sharing as possible is achieved while satisfying the participation constraints.

Note also that, as long as the value of autarky is strictly increasing in current income and the value function is strictly increasing in the current Pareto weight, we have that
\( \bar{\pi}^j(B) > \bar{\pi}^{j-1}(B) \) and \( \bar{x}^j > \bar{x}^{j-1} \) for all \( N \geq j > 1 \). It is easy to see that the two above conditions are satisfied in our model. It is also easy to see that, unless autarky is the only implementable allocation, we have that \( \bar{\pi}^j(B) > \bar{x}^j(B) \), for all \( j \).

Whenever the aggregate level of assets is constant over time \( (B^* \equiv B' = B) \), the above properties can be easily translated as properties of the consumption allocation. To see this, note that we can define the limits of the optimal consumption intervals as follows

\[
\bar{c}^j : \pi^j = \frac{u'(Y + rB^* - \pi^j)}{u'(\pi^j)} \quad \text{and} \quad \bar{c}^j : \bar{x}^j = \frac{u'(Y + rB^* - \bar{c}^j)}{u'(\bar{c}^j)}.
\]

Given the strict concavity of the value function, the above properties of \( \pi^j \) and \( \bar{x}^j \) will hold for \( \bar{c}^j \) and \( c^j \) as well.

However, when aggregate assets are changing, the determination of the consumption intervals is as follows:

\[
\bar{c}^j(B) : \pi^j(B) = \frac{u'(Y + (1 + r)B - B'(<\pi^j(B), B> - \pi^j(B)))}{u'(\pi^j(B))} \quad \text{and} \quad \bar{c}^j(B) : \bar{x}^j(B) = \frac{u'(Y + (1 + r)B - B'(<\bar{x}^j(B), B> - \bar{c}^j(B)))}{u'(\bar{c}^j(B))}.
\]

This implies that the monotonicity of consumption also depends on how aggregate asset accumulation responds to the temporary Pareto weight (or, in other words, to consumption inequality). We will establish this relationship below. Finally, note that symmetry implies that

\[
\bar{c}^j(B) = Y + (1 + r)B - B'(<\bar{\pi}^j(B), B> - \bar{\pi}^{N-j+1}(B)).
\]

Given the utility function, the income process, and \( B \), the intervals for different states may overlap or not depending on the discount factor, \( \beta \). If \( \beta \) is sufficiently large, then perfect risk sharing is self-enforcing by a standard folk theorem (Kimball, 1988). This is reflected in the fact that the larger \( \beta \) is, the wider these intervals are. Whenever all intervals overlap, that is, \( \bar{\pi}^1 \geq \bar{\pi}^N \), perfect risk sharing is implementable at the given asset level. However, as public assets are accumulated (or decumulated) these intervals change as well. Hence the joint dynamics of public storage and the intervals will determine the dynamics of consumption as well.

In order to better understand some key characteristics of this model, below we focus on the case where public storage is constant over time. Then, from the next section, we study in detail the joint dynamics of consumption dispersion and assets. The constant-assets case is particularly useful for three reasons. First, as we show later, under some conditions, the economy will converge (almost surely) to a constant level of public assets. Second, the
benchmark model without public storage is a special case of this economy with \( B' = B = 0 \).

Third, the characterization of this economy is very useful to understand the short- and long-run dynamics of assets.

We will focus on scenarios where the long-run equilibrium is characterized by imperfect risk sharing. As we have shown before, for the case of constant \( B \) we can express the intervals characterizing the constrained-efficient allocation equivalently in terms of consumption rather than in terms of the relative Pareto weight. That is, we assume from now on that \( \tau^1 < \zeta^N \). We do this both because there is overwhelming evidence from several applications (households in a village or in the United States, spouses in a household, countries) about less than perfect risk sharing, and because that case is theoretically not interesting, as it is equivalent to the well-known (unconstrained) efficient allocation of constant individual consumptions over time. It is not difficult to see that the law of motion described by (7) implies that, in the long run, risk sharing arrangements subject to limited commitment are characterized by a finite set of consumption values determined by the limits of the optimal consumption intervals. It turns out that considering two scenarios is enough to describe the general picture: (i) each agent’s participation constraint is binding only when his income is highest, and (ii) each agent’s participation constraint is binding in more than one state. Given this, to describe the constrained-efficient allocations in these two scenarios, it is sufficient to consider three income states, i.e., \( N = 3 \). Hence, for all our graphical and numerical examples, we will consider \( N = 3 \).

Consider an endowment process where each agent gets \( y^h \), \( y^m \), or \( y^l \) units of the consumption good, with \( y^h > y^m > y^l \), with probabilities \( \pi^h \), \( \pi^m \), and \( \pi^l \), respectively. Symmetry implies that \( y^m = (y^h + y^l)/2 \) and \( \pi^e \equiv \pi^h = \pi^l = (1 - \pi^m)/2 \), where the upper index \( e \) refers to the most extreme, i.e., most unequal, income distribution. We will refer to a state \( s^j \) when agent 1 has income \( y^j \), as before.

Given a constant \( B \) in the long run, denoted \( B^* \), the consumption intervals become wider either if we increase \( \beta \) for a given \( B^* \), as on the basic model, or because we increase \( B^* \) for a given \( \beta \). Both changes make autarky less attractive. The former because agents put higher weight on the lack of insurance in the future, the latter because the agent is excluded from the benefits of more public assets upon default. If partial insurance occurs there are two possible scenarios depending on the level of the discount factor and public assets. For higher levels of \( \beta \) and/or \( B^* \), \( \zeta^m \geq \zeta^h > \zeta^l \geq \zeta^m \). This means that the consumption interval for state \( s^m \) overlaps with the intervals associated with both the \( s^h \) and the \( s^l \) state. This is the case where each agent’s participation constraint binds for the highest income level only. Figure 1
Figure 1: The interval for state $s^m$ overlaps with the intervals for state $s^h$ and state $s^l$.

Figure presents an example satisfying these conditions.

Suppose current consumption of agent 1 is below $c^h$. Suppose that the type 1 agent’s initial consumption is below $c^h$. When agent 1 draws a high income realization (which occurs with probability 1 in the long run), his consumption jumps to $c^h$. Then it stays at that level until his income jumps to the lowest level. At that moment, agent 2’s participation constraint binds, because he has high income, and consumption of agent 1 will drop to $c^l$. Then we are back to where we started from. A very similar argument holds whenever type 1 agent’s initial consumption is above $c^h$. This implies that consumption takes only two values, $c^h$ and $c^l$ in the long run. When consumption changes, it always moves between these two levels, and the past history of income realizations does not matter. This is the amnesia property of the basic model (Kocherlakota, 1996). When state $s^m$ occurs after state $s^h$ or state $s^l$, the consumption allocation remains unchanged. That is, consumption does not react at all to this ‘small’ change in income. This is the persistence property of the basic model. Note that consumption also remains unchanged over time if the sequence $(h, m, h)$ or the sequence
(l, m, l) takes place.

The key observation here is that, although individuals face consumption changes over time, the consumption distribution is time invariant. In every period, half of the agents consume $c^h$ and the other half consume $c^l$. Finally, note that exactly this case occurs for any $N$ if $c^2 \geq c^N > c^1 \geq c^{N-1}$.

For lower levels of $\beta$ and/or $B^*$, none of the three intervals overlap, i.e., $c^h > c^m > c^m > c^l$. Figure 2 shows an example of this second case. When all three intervals are disjunct, consumption takes four values in the long run. To see this, notice that the participation constraint of agent 1 binds both for medium and high level of income. That is, whenever his income changes his consumption will change as well, and similarly for agent 2.

In this second case, in state $s^m$ the past history determines which agent’s participation constraint binds, therefore consumption is Markovian. Current incomes and the identity of the agent with a binding participation constraint fully determine the consumption allocation. The dynamics of consumption exhibit amnesia in this sense here. Further, consumption
responds to every income change, hence the persistence property does not manifest itself.

The key observation for later reference is that the consumption distribution changes between \( \{c^m, c^m\} \) and \( \{c^l, c^h\} \). That is, the cross-sectional distribution of consumption is different whenever state \( s^m \) occurs from when an unequal income state, \( s^k \) or \( s^l \), occurs. If there are \( N > 3 \) income states, the cross-sectional consumption distribution changes over time whenever \( \bar{z}^2 < \bar{z}^N \) and \( \bar{z}^1 < \bar{z}^{N-1} \). The number of income states and the number of states where a participation constraint binds determine the possible number of long-run consumption levels, and consequently the persistence property may appear.

2.2 The dynamics of public assets

The next proposition provides a key property of the aggregate storage decision rule and characterizes the short-run dynamics of assets. It shows how public storage varies with the income and consumption distribution.

**Proposition 1.** \( B'(B, x') \) is strictly increasing in \( x' \) for \( x' \geq 1 \) and \( B'(B, x') > 0 \). That is, the higher cross-sectional consumption inequality is, the higher public asset accumulation is. \( B'(s^j, B, x) \geq B'(s^k, B, x), \forall (B, x), \) where \( j \geq N/2 + 1, k \geq N/2, \) and \( j > k \). The inequality is strict, i.e., \( B'(s^j, B, x) > B'(s^k, B, x), \) if the optimal intervals for states \( s^j \) and \( s^k \) do not overlap given \( B \). That is, aggregate asset accumulation is weakly increasing in cross-sectional income inequality.

**Proof.** A higher \( x' \geq 1 \) increases public storage for two complimentary reasons. First, higher consumption inequality today, which is uniquely determined by the current relative Pareto weight, \( x' > 1 \), increases the storage incentive of the agent with high current consumption. The planner inherits this incentive. Second, higher expected consumption inequality in the future provides further incentive to the planner to store, since public assets improve risk sharing (reduce consumption inequality) in the future. A higher \( x' > 1 \) implies higher consumption inequality tomorrow in expectation. To see this, note that three things can happen tomorrow with respect to the pattern of binding participation constraints: (i) no participation constraint binds tomorrow, thus consumption inequality remains the same as today, (ii) agent 1’s participation constraint is binding, thus consumption inequality either remains higher for a higher \( x' > 1 \) or no longer depends on \( x' \) when comparing a higher and a lower \( x' \), and (iii) agent 2’s participation constraint is binding, and either \( x'' > 1 \) and the same cases are possible as for (ii), or \( x'' < 1 \), in which case consumption inequality tomorrow does not depend on \( x' \). Therefore, to reduce consumption inequality tomorrow, the planner has more incentive to store when \( x' > 1 \) is higher.
From Lemma 1 we know that $B'(s^j, B, x) = B'(B, x')$. If $j > k$, and the optimal intervals for these two states do not overlap given $B$, then $x'$ must be higher in state $s^j$ than in state $s^k$, and we have already shown that assets depend positively on cross-sectional consumption inequality. If the optimal intervals overlap given $B$, then there exists $x$ for which $x' = x$ in both states $s^j$ and $s^k$. Aggregate savings are identical in the two states in this case.

We are now ready to characterize the long-run behavior of public assets.

**Proposition 2.** Assume that $\beta$ is such that agents obtain low risk sharing in the sense that the consumption distribution is time variant when $B' = B = 0$ is imposed.

(i) There exists $r_1$ such that for all $r \in [-1, r_1]$, $B' = 0$ for all income levels, that is, public storage is never used in the long run.

(ii) There exists a strictly positive $r_2 > r_1$ such that for all $r \in (r_1, r_2)$, $B$ remains stochastic but bounded in the long run.

(iii) For all $r \in [r_2, 1/\beta - 1)$, $B$ converges almost surely to a strictly positive constant where the consumption distribution is time invariant but perfect risk sharing is not achieved.

(iv) Whenever $r = 1/\beta - 1$, $B$ converges almost surely to a strictly positive constant and perfect risk sharing is self-enforcing.

If $\beta$ is such that the consumption distribution is time invariant when $B' = B = 0$ is imposed, then only (i), (iii), and (iv) occur.

**Proof.** Part (i). It is easy to see that $r_1$ is implicitly defined by the planner’s Euler inequality, (12), with equality when agent 1 has the highest possible income. That is, $r_1$ is implicitly given by

$$u'(c_1(y^N, 0, x_{t-1})) = \beta(1 + r_1) \sum_{s_{t+1}} \Pr(s_{t+1} | s^N) \frac{u'(c_1(y_{t+1}, 0, x^N))}{1 - v_1(y_{t+1}, 0, x^N)},$$

where $x_{t-1} \leq x^N$. If $r > r_1$, public assets will be positive at least when income inequality is highest, while public assets will be zero in the long run otherwise.

Next, we show that assets are bounded in the long run, which we need for parts (ii)-(iv). It is easy to see that there exists a high level of inherited assets, denoted $\hat{B}$, such that perfect risk sharing is at least temporarily enforceable, that is, $\pi^1(\hat{B}) \geq x^N(\hat{B})$. Therefore, if $r < 1/\beta - 1$, $B'(B, x') < B$ for all $B \geq \hat{B}$ and $\pi^1(B) \geq x' \geq x^N(B)$, i.e., assets have to
optimally decrease; and assets stay constant if \( r = 1/\beta - 1 \). This implies that assets are bounded above in the long run.

We now turn to part (iii). We first show that if the consumption distribution is time invariant, then there exists a unique constant level of assets, \( B^* \), such that all optimality conditions are satisfied. Afterwards, we show that assets converge almost surely to \( B^* \) starting from any initial level, \( B_0 \). Then, we establish that assets remain stochastic when the consumption distribution is time varying (case (ii)). Finally, we show that case (iii) occurs when the return on storage is high but less than the discount rate, while assets remain stochastic when the return is below a threshold, \( r_2 \).

Recall that if aggregate assets are constant, the optimal intervals for the relative Pareto weight are time invariant. Given that each agent’s participation constraint binds only for the highest income level in the long run, the optimality condition (10) and \( x^N(B^*) \) uniquely determine \( c^h(B^*) \), the time-invariant high consumption level. Then, using the planner’s Euler, we can determine the unique level of \( B^* \) such that all optimality conditions are satisfied. The planner’s Euler is

\[
u' \left( c^h \left( B^* \right) \right) = \beta (1+r) \left[ (1-\pi^c) \nu' \left( c^h \left( B^* \right) \right) + \pi^c \nu' \left( c^l \left( B^* \right) \right) \right].
\]

Dividing both sides by \( \nu' \left( c^h \left( B^* \right) \right) \), we obtain

\[
1 = \beta (1+r) \left[ (1-\pi^c) + \pi^c \frac{\nu' \left( c^l \left( B^* \right) \right)}{\nu' \left( c^h \left( B^* \right) \right)} \right] = \beta (1+r) \left[ (1-\pi^c) + \pi^e x^N \left( B^* \right) \right],
\]

(18)

where we have used (10). Note that \( x^N \left( B^* \right) \) is monotone and continuous in \( B^* \). Further, at \( B^* = 0 \) the right hand side of equation (18) is larger than 1 by assumption, and at \( B^* = \hat{B} \) the right hand side of (18) is smaller than 1, because \( x^N \left( \hat{B} \right) = 1 \) and \( B^* < \hat{B} \). Therefore, we know that there exists a unique \( B^* \) where the planner’s Euler is exactly satisfied by setting \( B' = B = B^* \).

Next, we show that assets converge almost surely to \( B^* \) starting from any initial level, \( B_0 \). We already know that \( B'(B_0, x') < B_0 \) for the ergodic range of \( x' \) when \( B_0 > \hat{B} \), i.e., when perfect risk sharing is (temporarily) self-enforcing, and \( B'(0, x') > 0 \) for some \( x' \) in the ergodic range of \( x' \), since \( r > r_1 \) by assumption. Consider \( B^* < B_0 < \hat{B} \) first, and assume that state \( N \) occurs, and agent 1’s participation constraint is binding. This is without loss of generality, because this occurs with probability 1 in the long run, and the problem is symmetric across the two agents. We know that the right hand side of (18) is smaller than 1, because \( x^N \left( B_0 \right) < x^N \left( B^* \right) \). Therefore, marginal utility tomorrow has to increase relative to marginal
utility today to satisfy the planner’s Euler, therefore $B'(B_0) < B_0$. What happens next period? The participation constraint will bind again even if the same state occurs. This is because $B'(B_0) < B_0$ implies $x^N(B'(B_0)) > x^N(B_0)$. Then assets will decrease again. What if some $s^j$ with $2 \leq j \leq N - 1$ occurs? We know that the participation constraints in these states are not binding for any $B \geq B^*$, because they are not binding for $B^*$. This means that now $x' = x = x^N(B_0) > x^N(B'(B_0))$. Then, by Proposition 1, storage will be lower than when the participation constraint is binding. Note that if states $s^2, ..., S^{N-1}$ occur repeatedly, assets will converge to a level below $B^*$. Then we are in the case where $B_0 < B^*$, which we study next.

Consider $0 \leq B_0 < B^*$ now, and suppose again that state $N$ occurs, and agent 1’s participation constraint is binding. We know that $x^N(B_0) > x^N(B^*)$ in this case. Using (18) again, it follows that $B'(B_0) > B_0$. Now, if the same state occurs tomorrow (in fact, any $s^j$ with $j \geq 2$), then the participation constraint will be slack. This means that now $x' = x = x^N(B_0) > x^N(B'(B_0))$. Then, by Proposition 1, storage will be higher than when the participation constraint is binding. This also implies that if state $s^1$ does not occur for many periods, assets converge to a level above $B^*$. Then once $s^1$ occurs, which happens with probability 1 in the long run, we are back to the case $B_0 > B^*$, and assets start decreasing.

Part (ii). Consider the case where in the long run there is a third state in which a participation constraint binds. In this case, each agent’s consumption takes at least four different values in the long run. These have to satisfy an additional participation constraint, an additional resource constraint, and an additional Euler, which is generically impossible for constant $B$.

Finally, we have to show that case (ii) occurs if $r_1 < r \leq r_2$, while case (iii) occurs if $r_2 < r < 1/\beta - 1$. It is easy to see that $B^*$ will be lower if $r$ is lower, where $B^*$ can be computed for any $r$ ignoring the participation constraints of states $s^j$ with $2 \leq j \leq N - 1$. However, as assets decrease, the optimal intervals become narrower, and eventually $\underline{c}^2 < \underline{c}^N$ and $\underline{c}^1 < \underline{c}^{N-1}$. Hence, $r_2$ is implicitly given by (18) such that $B^*$ is such that $\overline{c}^2 = \overline{c}^N$ and $\overline{c}^1 = \overline{c}^{N-1}$.

Part (iv). Note that when $\beta(1 + r) = 1$, the only way to satisfy agents’ Euler inequalities in all states is to provide them with a perfectly smooth consumption stream over time. Further, as long as a participation constraint binds given $B$, the planner has an incentive to

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8Note that this never happens in the basic model.
9Participation constraints in more states may be binding when $B$ is low, even if they only bind in states $s^j$ and $s^N$ for $B^*$. We know that assets will increase in the two most unequal states when $B < B^*$, therefore with probability 1 assets will reach a level where the participation constraints of the other states are no longer binding.
store more, because she does not face a trade-off between improving risk sharing and using an inefficient intertemporal technology.

The intuition behind Proposition 2 is that the social planner trades off two effects of increasing aggregate storage: it is costly as long as \( \beta(1 + r) < 1 \), but less so the higher \( r \) is, and it is beneficial because it reduces consumption dispersion in the future. The level of assets chosen just balances these two opposing forces. The relative strength of these two effects naturally depends on the return to storage, \( r \). Proposition 2 also says that aggregate assets will be constant in the long run if the cross-sectional consumption distribution does not change over time. This is what happens not just when consumption takes only one value, i.e., perfect risk sharing occurs (case (iv)), but also when it takes two values in the long run (cases (iii)). In the latter case half the agents consume \( c^h \) and the other half consume \( c^l \) in each period, only the identity of the agent with \( c^h \) changes over time. When participation constraints bind in more states and consumption has to take more than two values, the cross-sectional consumption distribution will change over time (case (ii)). Here, the relative strength of the above two forces that determine asset accumulation changes over time, implying that assets remain stochastic in the long run.

We now characterize the bounds of the stationary distribution of assets when they are stochastic in the long run. Let \( \underline{B} (\overline{B}) \) denote the lower (upper) limit of the stationary distribution of assets.

**Proposition 3.** The lower limit of the stationary distribution of aggregate assets, \( \underline{B} \), is either strictly positive and is implicitly given by

\[
u^' (\tau_m (\underline{B})) = \beta(1 + r) \sum_{j=1}^{N} \pi^j u^' \left( c^j \left( \frac{\underline{B}}{x^m (\underline{B})} \right) \right), \tag{19}\]

where the upper index \( m \) refers to the least unequal income state, or is zero, and the Euler equation above holds as strict inequality. The upper limit of the stationary distribution of aggregate assets, \( \overline{B} \), is implicitly given by

\[
u^' \left( c^N \left( \frac{\overline{B}}{x^N (\overline{B})} \right) \right) = \beta(1 + r) \sum_{j=1}^{N} \pi^j u^' \left( \frac{c^j \left( \frac{\overline{B}}{x^N (\overline{B})} \right)}{1 - \nu^j} \right). \tag{20}\]

Proof. From Proposition 1 it is clear that \( \underline{B} \) will be approached if the least unequal income state, denoted \( s^m \),\(^{10}\) happens repeatedly, while \( \overline{B} \) can only be approached with state \( s^N \) (or \( s^1 \)) happening many times in a row.

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\(^{10}\)Note that \( s^m \) refers to two states when \( N \) is even, \( s^{N/2} \) and \( s^{N/2+1} \).
If $B$ is part of the stationary distribution, then it must be that $B \geq \bar{B}$. This means that there are less and less resources available over time while assets approach $\bar{B}$, thus the relevant participation constraint will always bind along this path. Therefore, $x' = \bar{x}^m(B)$ along this path, and the planner’s Euler is (19) if $\bar{B} > 0$, or $\bar{B} = 0$.\footnote{We will give an example for each of these cases in the next section, where we present some computed examples.}

The upper limit of the stationary distribution, $\bar{B}$, is approached from below, thus along that path, the relevant participation constraint is slack.\footnote{It will become clear that the upper limit is reached if state $s^N$ or state $s^1$ occurs many times in a row, but not if the economy alternates between these two states.} As a result, when $B$ converges to its upper limit, $\bar{x} \equiv x' = \bar{x}^h(B_1)$ where $B_1$ is the level of inherited assets when we switch to state $s^N$ (or $s^1$). Denote by $\bar{B}$ the level of assets where $B$ might converge to from below when state $s^N$ occurs many times in a row. $\bar{B}$ is the solution to the following system:

$$
\frac{u'(c^1(\bar{B}, \bar{x}))}{u'(c^N(\bar{B}, \bar{x}))} = \bar{x}
$$

$$
c^N(\bar{B}, \bar{x}) + c^1(\bar{B}, \bar{x}) = Y + r\bar{B}
$$

$$
u'\left(c^N(\bar{B}, \bar{x})\right) = \beta(1 + r) \sum_{j=1}^{N} \pi^j u'\left(c^j(\bar{B}, \bar{x})\right).
$$

(21)

When is $\bar{B}$ equal to $\bar{B}$, the upper limit of the stationary distribution? Using Proposition 1, we know that $B'(B, \bar{x})$ is highest when $\bar{x}$ is highest. At which asset level $B_1$ within the stationary distribution of assets should we switch to state $s^N$ in order to have $\bar{x} = \bar{x}^N(B_1)$ the highest possible? This happens when $B_1$ is at the lower limit of the stationary distribution, i.e., when $B_1 = \bar{B}$. In that case, $\bar{x} = \bar{x}^N(\bar{B})$. Then, replacing for $\bar{x}$ in (21) gives (20).

Finally, we illustrate asset dynamics on two figures. First, Figure 3 illustrates the short-run dynamics of assets in the case when they are constant in the long run. The solid (blue) line represents $B'(B, \bar{x}^N(B))$, i.e., we compute $B'$ assuming that the relevant participation constraint is binding. Suppose that state $N$ occurs when inherited assets are at the level $B_0 < B^*$. Then public storage will be $B'(B_0, \bar{x}^N(B_0))$. Next period, if any state $s^j$ with $j \geq 2$ occurs, no participation constraint is binding, thus assets will be $B'(B, \bar{x}^N(B_0)) > B'(B, \bar{x}^N(B))$. This is represented by the dot-dashed (red) line. As long as state $s^1$ does not occur, assets stay on this line and eventually converge to the level $\bar{B} > B^*$. Now, assume that state $s^1$ occurs when inherited assets are $\bar{B}$. By symmetry, storage will be $B'(\bar{B}, \bar{x}^N(\bar{B}))$. Next period, if either state $s^1$ or $s^N$ occurs, the participation constraint binds again, and assets
Figure 3: Short-run asset dynamics when assets are constant in the long run

follow the solid (blue) line. If any other state occurs, assets will decrease more, and if this continues happening, assets will approach a level lower than $B^*$ (not represented).

Second, Figure 4 illustrates both the short- and long-run dynamics of public assets in the case where they are stochastic in the long run. For simplicity, we consider three income states. This means that there are two types of states: two with high income and consumption inequality (states $s^h$ and $s^l$) and one with low income and consumption inequality (state $s^m$). The solid (red) line represents $B'(B, x^h(B))$, i.e., storage in state $s^h$ (or $s^l$) when the relevant participation constraint is binding. Similarly, the dot-dashed (blue) line represents $B'(B, x^m(B))$, i.e., storage in state $s^m$ when the relevant participation constraint is binding. Starting from $B_0$, if state $s^m$ occurs repeatedly, assets converge to the lower limit of their stationary distribution, $\bar{B}$. The relevant participation constraint is always binding along this path, because inherited assets keep decreasing. The dashed (green) line represents the scenario where state $s^h$ (or state $s^l$) occurs when inherited assets are at the lower limit of the
stationary distribution, $\bar{B}$, and then the same state occurs repeatedly. This is when assets will approach the upper limit of their stationary distribution, $\bar{B}$. The relevant participation constraint is not binding from the period after the switch to $s^h$, therefore storage given inherited assets is described by the function $B' \left( B, x^h(\bar{B}) \right)$. Finally, without loss of generality assume that state $s^l$ occurred many times while approaching $\bar{B}$, and suppose that state $s^h$ occurs when inherited assets are (close to) $\bar{B}$. In this case, $x' = x^h(\bar{B}) < x^h(\bar{B})$, and assets will decrease. They will then converge to a level $\tilde{\bar{B}}$ from above with the relevant participation constraint binding along this path. The same will happen whenever $B > \tilde{\bar{B}}$ when we switch to state $s^h$ (or $s^l$). $\tilde{\bar{B}}$ is implicitly given by

$$ u'(x^h(\tilde{\bar{B}})) = \beta (1 + r) \sum_{j \in \{l, m, h\}} \pi^j u'(c^j(\tilde{\bar{B}}, x^h(\tilde{\bar{B}}))) . $$

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2.3 The dynamics of individual consumptions

Having characterized assets, we now turn to the dynamics of consumption. One key property of the basic model is that whenever either agent’s participation constraint binds ($v_1(X) > 0$ or $v_2(X) > 0$), the resulting allocation is independent of the preceding history. In our formulation, this implies that $x'$ is only a function of $s'$ and the identity of the agent with a binding participation constraint. This is often called the amnesia property (Kocherlakota, 1996), and typically data do not support this pattern, see Broer (2012) for the United States and Kinnan (2012) for Thai villages. Allowing for storage helps to bring the model closer to the data in this respect.

**Proposition 4.** The amnesia property does not hold when public assets are stochastic in the long run.

*Proof.* $x'$ and hence current consumption depend on both current income and inherited assets, $B$, when a participation constraint binds. This implies that the past history of income realizations affects current consumptions through $B$.

Another property of the basic model is that whenever neither participation constraint binds ($v_1(X) = v_2(X) = 0$), the consumption allocation is constant and hence exhibits an extreme form of persistence. This can be seen easily: (11) gives $x' = x$, and the consumption allocation is only a function of $x'$ with constant aggregate income. This implies that for ‘small’ income changes which do not trigger a participation constraint to bind, we do not see any change in individual consumptions. It is again not easy to find evidence for this pattern in the data, see Broer (2012). In our model, even if the relative Pareto weight does not change, (10) does not imply that individual consumptions will be the same tomorrow as today. This is because $(1 + r)B - B'(X)$ is generically not equal to $(1 + r)B' - B''(X')$ when assets are stochastic in the long run.

**Proposition 5.** The persistence property does not hold when public assets are stochastic in the long run.

*Proof.* Even though $x' = x$, when neither participation constraint binds, consumption is only constant if net savings are identical in the past and the current period. This is not the case when $B$ is stochastic.

The last two propositions imply that the dynamics of consumption in the our model are richer and closer to the data than in the basic model.
2.4 Welfare

It is clear that access to public storage reduces consumption dispersion and improves welfare, because zero assets can always be chosen. If public storage is positive for at least the most unequal income state, then welfare strictly improves.

2.5 Computation

We use the recursive system given by equations (10)-(15) to solve the model numerically. We discretize $x$ and $B$ ($y$ is assumed to take a finite number of values). We have to determine $x'$ and $B'$ on a 3-dimensional grid on $X = (y, B, x)$. The initial values for $V'(X')$, $c_1(X')$, and $v_1(X')$ are from the solution of a model where the participation constraints are ignored. We iterate until the value and policy functions converge.

As we proceed, we use the characteristics of the solution. In particular, we know that if agent 1’s participation constraint binds at $\tilde{x}$, it will bind at all $x < \tilde{x}$. Similarly, if agent 2’s participation constraint binds at $\hat{x}$, it will bind at all $x > \hat{x}$. At each iteration, at each income state and for each $B$, we solve directly for the limits $\tilde{x}$ and $\hat{x}$ using (13) and (14) with equality, respectively, first assuming that $B' = 0$. Afterwards, we check whether the planner’s Euler is satisfied at the limits. If not, we solve a 2-equation system of (12) and (13) (or (14)), with unknowns $(x', B')$. Finally, we solve for a new $B'$ at points on the $x$ grid where neither participation constraint binds, i.e., at the interior of the optimal interval for $(y, B)$ of the current iteration.

2.6 Decentralization

Ábrahám and Cárceles-Poveda (2006) show how to decentralize a limited commitment economy with capital accumulation which is similar to the one studied in this paper. In particular, they introduce competitive intermediaries and show that a decentralization with endogenous debt constraints that are ‘not too tight’ (that make the agents just indifferent between defaulting and participating), as in Álvarez and Jermann (2000), is possible. However, Ábrahám and Cárceles-Poveda (2006) use a neoclassical production function where wages depend on aggregate capital. This implies that there the value of autarky depends on aggregate capital as well. They show that if the intermediaries are subject to endogenously determined capital accumulation constraints, then this externality can be taken into account, and the constrained-efficient allocation can be decentralized as a competitive equilibrium.\footnote{This is also the case in the two-country production economy of Kehoe and Perri (2004).\footnote{Chien and Lee (2010) achieve the same objective by taxing capital instead of using a capital accumulation...}}
Note that public storage can be thought of as a form of capital, $B$ units of which produce $Y + (1 + r)B$ units of output tomorrow and which fully depreciates. Hence, the results above directly imply that a competitive equilibrium corresponding to the constrained-efficient allocation exists. In particular, households trade Arrow securities subject to endogenous borrowing constraints that prevent default, and the intermediaries also sell these Arrow securities to build up public storage. The key intuition is that equilibrium Arrow security prices take into account binding future participation constraints, as these prices are given by the usual pricing kernel. Moreover, agents will not hold any ‘shares’ in public storage, hence their autarky value is not affected. Finally, no arbitrage or perfect competition will make sure that the intermediaries make zero profits in equilibrium. Capital accumulation constraints are not necessary, because in our model public storage does not affect the outside option of the agents.

3 The model with both public and private storage

In this section, we provide the formulation and analytical characterization of our model with limited commitment and private and public storage. We add agents’ Euler inequalities as constraints to the problem given by the objective function (1) and the constraints (2) and (3). We maintain the assumption of a constant aggregate endowment to exclude the precautionary motive for storage at the aggregate level, and to thus isolate the asset accumulation incentives due to endogenously incomplete markets.

The social planner’s problem is

$$\max_{\{c_i(s^t), B(s^t)\}} \sum_{i=1}^2 \lambda_i \sum_{t=1}^{\infty} \sum_{s^t} \beta^t \Pr(s^t) u(c_i(s^t))$$

s.t. $$\sum_{i=1}^2 c_i(s^t) \leq \sum_{i=1}^2 y_i(s_t) + (1 + r)B(s^{t-1}) - B(s^t), \ \forall s^t,$$

$$(PI) \sum_{r=t}^{\infty} \sum_{s^r} \beta^{r-t} \Pr(s^r | s^t) u(c_i(s^r)) \geq U_i^{au}(s_t), \ \forall s^t, \forall i,$$

$$u'(c_i(s^t)) \geq \beta(1 + r) \sum_{s^{t+1}} \Pr(s^{t+1} | s^t) u'(c_i(s^{t+1})), \ \forall s^t, \forall i,$$

$$B(s^t) \geq 0, \ \forall s^t.$$

The social planner chooses the consumption of each agent given any possible history of income states, $c_i(s^t), \forall i$, and current gross storage, $B(s^t)$, given any possible history of income states, $s^t$. The consumption of each agent is subject to the budget constraint.
and maximizes a weighted sum of agents’ lifetime utilities. The first constraint, (23), is the resource constraint, where \( B(s^{t-1}) \) denotes assets inherited from the previous period. The next constraint, (24), is the participation constraint, where \( \tilde{U}_{i}^{au}(s_{t}) \) is the value function of autarky when storage is allowed. Equation (25) is agents’ Euler inequality. Finally, (26) is the planner’s borrowing constraint.

A few remarks are in order about this structure before we turn to the characterization of constrained-efficient allocations. First, agents can store in autarky, but they lose access to the benefits of the public asset.\(^{15}\) This implies that \( \tilde{U}_{i}^{au}(s^{j}) = V_{i}^{au}(s^{j}, 0) \), where \( V_{i}^{au}(s^{j}, b) \) is defined as

\[
V_{i}^{au}(s^{j}, b) = \max_{b'} \left\{ u(y_{i}(s^{j}) + (1 + r)b - b') + \beta \sum_{k=1}^{N} \pi^{k}V_{i}^{au}(s^{k}, b') \right\},
\]

where \( b \) denotes private savings. Since \( V_{i}^{au}(s^{j}, 0) \) is increasing (decreasing) in \( j \) for agent 1 (2), it is obvious that if we replace for the autarky value in the basic model, provided that a solution exists, or in the model of Section 2 with the one defined here, the same characterization holds.

Second, if neither the social planner nor the agents can store in the optimal contract, but agents can and would store in autarky, then no feasible solution exists for sufficiently low \( \beta \)s, i.e., when the solution of the basic model is ‘close’ to autarky. Note, however, that since the return on public and private storage is the same, public storage can replicate any allocation which agents can achieve using private storage. Further, equation (27) implicitly assumes that public storage does not affect the value of autarky. As mentioned above, when agents default they are not only excluded from future risk sharing but also from the benefits of the public asset.

Third, we use a version of the first-order condition approach (FOCA) here. That is, these constraints only cover a subset of possible deviations. In particular, we check whether the agent is better off staying in the risk arrangement rather than defaulting with any possible level of storage (condition (24)), and we check whether he is happy with not storing given that he does not default (condition (25)). It is not obvious whether these conditions are sufficient,\(^{16}\) because multiple and multi-period deviations are not considered by these constraints. In particular, the agent can store in the current period (to increase his value of autarky in future periods) and default in a later period. For now, we assume that these deviations are

\(^{15}\)This is the same assumption as in Krueger and Perri (2006), where agents lose access to the benefits of a tree after defaulting. In this paper the ‘tree’ is endogenous.

\(^{16}\)In fact, Kocherlakota (2004) shows that in an economy with private information and hidden storage the first-order condition approach can be invalid.
not profitable given the contract which solves Problem P1. Given this assumption, we will characterize the solution. In Section 3.4 we will show that agents indeed have no incentive to use these more complex deviations.

Fourth, both the participation constraints (24) and the Euler constraints (25) involve future decision variables. Given these two types of forward-looking constraints, a recursive formulation using either the promised utilities approach (Abreu, Pearce, and Stacchetti, 1990) or the Lagrange multipliers approach (Marcet and Marimon, 2011) is difficult. Euler constraints have been dealt with using the agent’s marginal utility as a co-state variable in models with moral hazard and hidden storage, see Werning (2001) and Ábrahám and Pavoni (2008). In our environment, this could raise serious tractability issues, since we would need two more continuous co-state variables, in addition to the state variable to make the participation constraints recursive.

In this paper, we follow a different approach that avoids these complications. In particular, we ignore agents’ Euler inequalities first. Then we verify that the solution of the simplified problem satisfies those Euler constraints. That is, instead of Problem P1, we consider the following simpler problem:

\[
\max_{\{c_i(s^t), B(s^t)\}} \sum_{i=1}^{2} \lambda_i \sum_{t=1}^{\infty} \sum_{s^t} \beta^t \Pr(s^t) u_i(c_i(s^t))
\]

\[(P2) \quad \text{s.t.} \quad \sum_{i=1}^{2} c_i(s^t) \leq \sum_{i=1}^{2} y_i(s_i) + (1 + r)B(s^t-1) - B(s^t), \ \forall s^t,
\]

\[
\sum_{r=t}^{\infty} \sum_{s^r} \beta^{r-t} \Pr(s^r | s^t) u_i(c_i(s^r)) \geq \tilde{U}_i^{au}(s_i), \ \forall s^t, \forall i,
\]

\[B(s^t) \geq 0, \ \forall s^t.\]

This is the problem we studied in Section 2.

Comparing the planner’s Euler, (9), to the standard Euler inequalities for the agents, (25), gives the following result:

**Proposition 6.** When the planner’s Euler is satisfied, the agents’ Eulers are satisfied as well. Therefore, the solution of the model with hidden storage, P1, corresponds to the solution of the simplified problem, P2.

**Proof.** The planner’s Euler is (9), while the agents’ Euler is (25). The right hand side of (9) is bigger than the right hand side of (25), for \(i = \{1, 2\}\), since \(0 \leq v_i(s^{t+1}) \leq 1, \ \forall s^{t+1}\). Therefore, (9) implies (25). \(\square\)
Given this result, the characteristics of the constrained-efficient allocations of Problem $P1$ are the same as those of Problem $P2$, which is the same problem that we studied in Section 2. Proposition 6 also means that private storage does not matter as long as public asset accumulation is optimal.

The intuition behind this result is that the planner has more incentive to store than the agents. She stores for the agents, because she inherits the agents consumption smoothing preferences. Thereby she can also eliminate the agents’ incentive to store in a hidden way. Further, comparing (9) and (25) again, it is obvious that the planner has more incentive to store than the agents in all but the most unequal states. In particular, the presence of $1/(1 - v_i(s^{t+1})) > 1$ in the planner’s Euler indicates how increasing assets helps the planner to relax future participation constraints, and thereby improve future risk sharing, or, make markets more complete. In other words, the planner internalizes the positive externality of asset accumulation on future risk sharing.

Next, we relate the case with both private and public storage to the case with private but without public storage. The following result follows from Proposition 6.

**Corollary 1.** The planner stores in equilibrium whenever an agent’s Euler inequality is violated at the constrained-efficient allocation of the basic model with storage in autarky.

Corollary 1 is interesting if private storage matters, i.e., agents’ Euler inequalities are violated, in the basic model under general conditions. This is what we establish next.

### 3.1 Does hidden storage matter in the basic model?

In this section, we identify conditions under which agents would store at the constrained-efficient solution of the basic model without public storage. We assume that partial insurance occurs at the solution. If Euler inequalities are violated, the solution is not robust to deviations when private storage is available.

We first consider the benchmark case where agents have access to an efficient intertemporal technology, i.e., storage earns a return $r$ such that $\beta(1 + r) = 1$. Afterwards, we study the general case. We only examine whether agents would use the available hidden intertemporal technology at the constrained-efficient allocation of the basic model. We do not make any assumption about the number of income states, except that income may take a finite number of values and the support of the income distribution is bounded.

**Lemma 2.** Suppose that partial insurance occurs and the hidden storage technology yields a return $r$ such that $\beta(1 + r) = 1$. Then agents’ Euler inequalities are violated at the constrained-efficient allocation of the basic model.
Proof. We show that the Euler inequality is violated at the constrained-efficient allocation at least when an agent receives the highest possible income, $y^N$. If partial insurance occurs, as opposed to full insurance, then it must be that there exists some state $s^j$ where the agent consumes $c^j < c^N$. Then,

$$u'(c^N) < \sum_j \Pr(s^j) u'(c^j),$$

that is, the Euler inequality is violated.

Proposition 7. There exists $\tilde{r} < 1/\beta - 1$ such that for all $r > \tilde{r}$ agents’ Euler inequalities are violated at the constrained-efficient allocation of the basic model.

Proof. $\tilde{r}$ is defined as the solution to

$$u'(c^N) = (1 + \tilde{r}) \sum_j \Pr(s^j) u'(c^j).$$ \hfill (28)

For $\tilde{r}$ close to $-1$, the right hand side is close to zero. By Lemma 2, the right hand side is greater than the left hand side if $\tilde{r} = 1/\beta - 1$. It is obvious that the right hand side is continuous and increasing in $\tilde{r}$. Therefore, there is a unique $\tilde{r}$ solving equation (28), and agents’ Euler inequalities are violated for higher values of $r$.

The intuition behind this result is that whenever partial insurance occurs, the agent enjoying high consumption today faces a weakly decreasing consumption path. Therefore, if a storage technology with sufficiently high return is available, the agent will use it for self-insurance purposes. We can also show that the threshold $\tilde{r}$ in Proposition 7 can be negative. In particular, we show that agents would use a storage technology with $r = 0$ under general conditions. A necessary condition is that the consumption distribution is time varying in the long run. The proofs are available upon request.

3.2 The dynamics of individual consumptions revisited

We have shown in Section 2.3 that, introducing public storage, we overturn two counterfactual properties of consumption dynamics in the basic model, the amnesia and persistence properties. We can improve on the basic model with respect to a third aspect of the dynamics of consumption. In particular, the Euler inequality cannot be rejected in household survey data from developed economies, once household demographics and labor supply are appropriately accounted for (see Attanasio (1999) for a comprehensive review of the literature). Since in our model with public storage agents’ Euler inequalities are satisfied, while they are violated in the basic model, we bring limited commitment models in line with this third observation as well.
3.3 Welfare revisited

In Section 2.4 we have argued that access to public storage unambiguously reduces consumption dispersion and improves welfare. It is clear that adding hidden storage counteracts these benefits of storage, because it increases the value of agents’ outside option, which in itself increases consumption dispersion and reduces welfare. The overall effects of access to both public and private storage are thus ambiguous in general, and will depend on the return to storage, \( r \). We first compare welfare at the long-run stationary distribution of our model with both public and private storage and the basic model without storage. Afterwards, we discuss the effects of the transition from the moment when storage becomes available.

Proposition 8.

(i) There exists \( r_1 \) such that for all \( r \in [-1, r_1] \) storage is not used even in autarky, therefore access to storage is welfare neutral.

(ii) There exists a strictly positive \( r_2 > r_1 \) such that for all \( r \in (r_1, r_2] \) storage is used in autarky but not in equilibrium, therefore consumption dispersion increases and welfare deteriorates as a result of access to storage.

(iii) There exists a strictly positive \( r_3 > r_2 \) such that for all \( r \in (r_2, r_3) \) public storage is (at least sometimes) positive, but access to storage is still welfare reducing. Access to storage is welfare neutral in the long run at the threshold \( r = r_3 \).

(iv) For all \( r \in (r_3, 1/\beta - 1] \) access to storage is welfare improving in the long run. Consumption dispersion may or may not be lower than in the basic model without storage.

Proof. It is easy to see that storage is never used when its return is close to -1. It is similarly easy to see that storage in equilibrium implies storage in autarky. This follows from the fact that the planner’s and the agents’ Euler is the same when income inequality is highest, i.e. when the incentive to store is highest, and agents’ Euler inequality is more stringent in autarky than in equilibrium with some risk sharing. Then, if storage only takes place in autarky, the only effect of storage is that the value of agents’ outside option increases, which reduces risk sharing and welfare. As \( r \) further increases to above the threshold \( r_2 \), the planner’s Euler becomes binding and public storage is sometimes positive. However, at this point the negative effect of the increase in the value of autarky dominates the positive effect of the (small) stock of public assets on risk sharing, therefore welfare still goes down as a result of access to storage. Finally, if \( r = 1/\beta - 1 \), perfect risk sharing occurs and aggregate
consumption is $Y + rB^*$ rather than $Y$, therefore welfare improves in the long run. Then, for any $r$ in a small neighborhood of $1/\beta - 1$ the positive effect of the increase in aggregate consumption dominates the negative effect of the increase in the value of autarky, thus welfare improves. For such $r$, consumption dispersion is small. However, it might be that perfect risk sharing occurs without access to storage. In this case access to storage increases consumption dispersion at the same time as improving welfare for $r < 1/\beta - 1$ sufficiently high.

Even when welfare improves in the long run, accumulating public assets has short-run costs, since it reduces aggregate consumption in the short run. It is clear that whether access both public and private storage improves welfare taking into account the transition from the moment when storage becomes available will depend on $r$. However, it is difficult to establish analytically the short-run costs of asset accumulation, since they depend on the path of shocks. We will compute the average total change in welfare from the introduction of storage for some numerical examples in Section 4.

3.4 Validity of the first-order condition approach

Until now we have assumed that by introducing agents’ participation constraints and Euler inequalities (equations (24) and (25), respectively) in Problem $P_1$, we guarantee incentive compatibility. In other words, we have assumed that the constrained-optimal allocation can be obtained by checking that agents have no incentive to default given that they do not store, and that they have no incentive to store given that they do not default. In principle, they may still find it optimal to use more complicated ‘double’ deviations involving both storage and default, potentially in different time periods, given some history of income shocks.

First, we need to consider contemporaneous joint deviations: the agent defaults and saves at the same time.\footnote{In the literature with private information, a similar joint deviation, shirking (or reporting a lower income) and saving, is the relevant deviation. Detailed discussion of these joint deviations can be found for the hidden income case in Cole and Kocherlakota (2001), and for the hidden action (dynamic moral hazard) case in Kocherlakota (2004) and Ábrahám, Koehne, and Pavoni (2011).} Since in the participation constraint (24) we use $\bar{U}_{it}^a(s_t)$, the value of autarky when the agent can store (see equation (27)), this contemporaneous double deviation is already taken into account. Further, note that in autarky the agent is allowed to store whenever this makes him better off. Therefore, the ‘default today and store later’-type of double deviations are already taken care of as well. This implies that the only potentially profitable double deviations we still need to consider are those which involve private asset accumulation first and default in a later period.
We demonstrate that ‘store today and default later’-type of double deviations cannot be profitable in the simplest possible case: only two consumption levels occur in the long run, $c^h$ and $c^l$ with switching probability $\pi^e$. It is not difficult to generalize the argument to more consumption levels. Let $V^h$ denote the expected lifetime utility of an agent who consumes $c^h$ today. Since $c^h$ is pinned down by the binding participation constraint of agents when their income reaches its highest level, we know that $V^h = V_1^{au}(s^N, 0)$, where $V_1^{au}()$ is defined in equation (27).

Now, we formally define the problem of an agent who is facing this consumption process and has the option of storing today and defaulting later. We denote by $W^h(b)$ (or $W^l(b)$) the value function for an agent who is entitled to receive $c^h$ ($c^l$) in the current period, has $b$ units of assets accumulated, and decides not to default today. These value functions are defined recursively as

$$W^h(b) = \max_{b' \geq 0} \left\{ u(c^h + (1 + r)b - b') + \beta \left[ \sum_{j=2}^{N} \pi^j \max\{W^h(b'), V_1^{au}(s^j, b')\} \right] \right\}$$

and

$$W^l(b) = \max_{b' \geq 0} \left\{ u(c^l + (1 + r)b - b') + \beta \left[ \sum_{j=1}^{N-1} \pi^j \max\{W^l(b'), V_1^{au}(s^j, b')\} \right] \right\} + \pi^N \max\{W^h(b'), V_1^{au}(s^N, b')\}.$$

We define the solution of the above optimization problems as $g^h(b)$ and $g^l(b)$, respectively.

**Lemma 3.** $g^h(0) = 0$. That is, the agent assigned to consume $c^h$ today will not store, even if defaulting later is an option.

**Proof.** Assume indirectly that $g^h(0) > 0$, that is, the agent saves today but does not default. Two cases are possible: either (i) the agent defaults in some state(s) tomorrow, or (ii) the agent does not default in any state tomorrow but he does later.

In case (i), the agent must default when his income is the highest possible tomorrow, i.e., when he earns $y^N$. Let $c^{au}(y^j, b)$ denote the consumption level chosen by the agent in autarky given that his income is $y^j$ and he has accumulated $b$ units of assets. Remember that $\pi^e = \pi^N = \pi^1$. Storing $g^h(0)$ today and defaulting tomorrow if his income is $y^N$, the agent's
Euler is
\[ u'(c^h - g^h(0)) = \beta(1 + r) \left[ \pi^c u'(c^{au}(y^N, g^h(0))) \right] \\
+ (1 - 2\pi^c) u'(c^h + (1 + r)g^h(0) - g^h(g^h(0))) \\
+ \pi^c u'(c^l + (1 + r)g^l(0) - g^l(g^l(0))) \right], \tag{31} \]

We will show that the agent would want to ‘borrow’ given this consumption path, which he can do by reducing \( g^h(0) \). Given that
\[ u'(c^h) = \beta(1 + r) \left[ (1 - \pi^c) u'(c^h) + \pi^c u'(c^l) \right], \]
a sufficient condition for this is that
\[ c^{au}(y^N, g^h(0)) > c^h, \]
\[ c^h + (1 + r)g^h(0) - g^h(g^h(0)) > c^h, \]
and
\[ c^l + (1 + r)g^l(0) - g^l(g^l(0)) > c^l. \]

Consumption cannot decrease in the agent’s ‘income,’ i.e., it cannot be that he chooses a consumption lower than \( c^l \) when he has access to \( c^l + (1 + r)g^h(0) \) units of the consumption good rather than only \( c^l \) units. To see the first condition, we first show that \( c^{au}(y^N, 0) > c^h \). Assume indirectly that this is not true. Given that the participation constraint holds with equality when the agent’s income is \( y^N \), this implies that the benefits of being in the risk sharing arrangement occur today while its costs occur in the future relative to autarky. This in turn implies that risk sharing must increase when the discount factor decreases. This contradicts the folk theorem. Recall that in Proposition 2 we have shown that as \( \beta \) increases to a level such that \( \beta(1 + r) = 1 \), perfect risk sharing is the long-run outcome. Intuitively, a higher \( \beta \) means a better enforcement technology in models of risk sharing with limited commitment. Now, clearly, \( c^{au}() \) is increasing in its second argument, therefore we also know that \( c^{au}(y^N, g^h(0)) > c^h \) holds for any \( g^h(0) \geq 0 \). This means that (31) is a strict inequality, thus the agent wishes to increase current consumption, which he can do by reducing \( g^h(0) \). A similar argument can be used if the agent would want to default in more states tomorrow.

In case (ii), substituting in the future Euler equations, we can use an almost identical argument as above. For example, take the case when the agent would save in periods 0 and 1 and default in the high state in period 2 only if the income delivered by the optimal allocation
\( (c^h) \) remains high in both periods. The Euler equation in period 0 is

\[
u' (c^h - g^h(0)) = \beta(1 + r) \left[ (1 - \pi^e) u' (c^h + (1 + r)g^h(0) - g^h (g^h(0))) + \pi^e u' (c^l + (1 + r)g^h(0) - g^l (g^h(0))) \right].
\] (32)

When the current state is \( h \), the Euler equation in period 1 is

\[
u' (c^h + (1 + r)g^h(0) - g^h (g^h(0))) = \beta(1 + r) \left[ \pi^e u' (c^h (g^h(0))) \right]
+ (1 - 2\pi^e) u' (c^h + (1 + r)g^h (g^h(0)) - g^h (g^h(0)))
+ \pi^e u' (c^l + (1 + r)g^h (g^h(0)) - g^l (g^h(0)))\]
\] (33)

and when the current state is \( l \), it is

\[
u' (c^l + (1 + r)g^h(0) - g^l (g^h(0))) = \beta(1 + r) \left[ \pi^e u' (c^l + (1 + r)g^l (g^h(0)) - g^l (g^h(0))) \right]
+ (1 - \pi^e) u' (c^l + (1 + r)g^l (g^h(0)) - g^l (g^h(0)))\]
\] (34)

Using equations (33) and (34) to substitute for the marginal utilities on the right hand side of (32) gives the two-period-ahead Euler equation in this case. Note that when the agent neither saves nor defaults for two periods, the two-period-ahead Euler equation is given by

\[
u' (c^h) = \beta(1 + r) \left[ (1 - \pi^e) \beta(1 + r) \left[ (1 - \pi^e) u' (c^h) + \pi^e u' (c^l) \right] + \pi^e \beta(1 + r) \left[ (1 - \pi^e) u' (c^l) + \pi^e u' (c^h) \right] \right].
\] (35)

Now, comparing the right hand sides of (32) after substitution and (35) term by term we can use practically the same argument as above to show that \( g^h(0) = 0 \).

\[\Box\]

**Proposition 9.** The first-order condition approach is valid.

**Proof.** The first-order condition approach is valid if \( g^h(0) = g^l(0) = 0 \), \( V^h = W^h(0) \), and \( V^l = W^l(0) \). It is easy to see that \( g^l(0) = 0 \). Lemma 3 shows that \( g^h(0) = 0 \). Replacing these solutions into (29) and (30), the first two conditions follow. \[\Box\]

### 4 Computed examples

[INCOMPLETE]

In this section we use the algorithm described in Section 2.5 to solve for the constrained-efficient allocations in economies with limited commitment and access to public and hidden storage. We determine agents’ value of autarky allowing them to store. We show that
aggregate savings due to the planner’s desire to complete the market can be significant in magnitude. We also illustrate how risk sharing, welfare, and the dynamics of consumption, are affected by the availability of storage with different returns $-1 \leq r \leq 1/\beta - 1$. Note that with $r = -1$ we are back to the basic model without storage. We consider two discount factors, low ($\beta = 0.7$) and high ($\beta = 0.8$). In the former case risk sharing is partial without storage, while in the latter case perfect risk sharing occurs without access to storage. Access to storage may still improve welfare, as we show below.

We assume that agents’ per-period utility function is of the CRRA form with a coefficient of relative risk aversion equal to 1, i.e., $u() = \ln()$. Income of both agents is i.i.d. over time, and may take three values, $\{0.2, 0.5, 0.8\}$, with equal probabilities. Remember that income is perfectly negatively correlated across the two agents, thus aggregate income is 1 in all three states. Remember also that $s^l$, $s^m$, and $s^h$ denote the state where agent 1 earns low, medium, and high income, respectively.

We present the simulation results on a few figures. First, let us look at the behavior of assets in the long run. Figure 5 shows the limits of the stationary distribution of assets, the first panel for $\beta = 0.7$ and the second for $\beta = 0.8$. Assets naturally increase with their return, $r$. When the storage technology is efficient $(r = 1/\beta - 1)$, assets reach at least 36.97 percent of aggregate income in the long run when $\beta = 0.8$ ($\beta = 0.7$). Depending on the history of shocks, assets may reach a higher level even if their initial level is zero. When the discount factor is high ($\beta = 0.8$), the participation constraints in state $s^m$ do not bind in the long run for any return on storage. With $r = 0.16$, for example, the planner’s savings amount to 16.56 percent of aggregate income, while with $r = 0.11$ they are 2.90 percent of aggregate income. When $\beta = 0.7$, for intermediate values of $r$ the participation constraints bind in all three states, and assets remain stochastic in the long run. For example, with $r = 0.16$ savings by the planner vary between 6.43 and 7.41 percent of aggregate income. When the interest rate is $r = 0.11$, savings vary between 0 and 1.38 percent. This last example shows that 0 can be part of the stationary distribution of aggregate assets.

In terms of how much risk sharing is achieved in the long run, the availability of a storage technology with low return does not change anything, since it is not used even in autarky. Note that when $\beta = 0.8$ consumption is also perfectly smooth without any storage. For returns above a threshold, an increase in $r$ then reduces risk sharing, since it raises the value of autarky, while storage is not used in equilibrium. As $r$ increases further, consumption dispersion starts to decrease in the long run as a result of a (sometimes) positive stock of public assets. Consumption becomes perfectly smooth (again) as $r$ reaches $1/\beta - 1$. Figure 6
Figure 5: Assets in the long run

(a) $\beta = 0.7$

(b) $\beta = 0.8$

shows the possible long-run consumption values. Note that the role of the discount factor is the same in our model with storage as in the basic model. In particular, a higher $\beta$ means a better enforcement technology and more risk sharing. When assets are stochastic in the long run (which happens for intermediate values of $r$ when $\beta = 0.7$), consumption depends on inherited assets. The variation of consumption is small in these computed examples. For example, with $r = 0.11$, the coefficient of variation is 2.8 percent in state $s^i$.

Figure 6: Consumption in the long run

(a) $\beta = 0.7$

(b) $\beta = 0.8$

Welfare in the long is determined by the variability of consumption and aggregate con-
sumption available. Long-run welfare is lower if a storage technology with intermediate return is available than if it is not possible to transfer resources from today to the future. This happens when storage is only used in autarky, and when the negative effect of storage in autarky outweighs the positive effect of higher aggregate consumption. Welfare is strictly higher with storage than without it for \( r \) sufficiently high, see Figure 7. This happens when the benefits from returns on aggregate storage outweigh the negative effect of the increase in the value of agents’ outside option. When \( \beta = 0.7 \) the threshold return above which long-run welfare improves is . When \( \beta = 0.8 \) it is .

Figure 7: Welfare in the long run

Finally, we compute average welfare from the moment the storage technology becomes available. We do this to take into account the costs of asset accumulation. Figure 8 shows the results. Comparing Figure 7 and Figure 8, the threshold return above which the availability of storage improves welfare is higher when we take the transition into account as well. The thresholds above which overall welfare increases as a result of the availability of the storage technology are and when \( \beta = 0.7 \) and 0.8, respectively.

5 Concluding remarks

This paper has shown that some implications of the basic limited commitment model with no private or public storage are not robust to hidden storage. When public storage is allowed though, the incentive for private storage is eliminated in the constrained-optimal allocation.
The intertemporal technology is used in equilibrium even though the aggregate endowment is constant and the return is lower than the discount rate, i.e., $\beta(1 + r) < 1$. Further, when income inequality is not the highest, the planner has more incentive to store than the agents. The reason for additional storage by the planner is that public assets relax future participation constraints, and thus improve risk sharing.

We have also shown that aggregate assets might be stochastic in the long run. This happens when each agent’s participation constraint binds at more than one income level, i.e., when the cross-sectional consumption distribution varies over time. Given inherited assets, public storage is higher when consumption inequality is higher. Finally, the dynamics of consumption is richer in our model compared to the basic model without storage. In particular, the amnesia and persistence properties do not hold in general, which brings limited commitment models closer to the data (Broer, 2012). Further, in our model agents’ Euler inequalities hold, which is consistent with empirical evidence from developed countries (Attanasio, 1999).

The effects of the availability of both public and private storage on asset accumulation, consumption dispersion, and welfare depend on its return. In the long run, (i) for low $r$ access to storage technology is welfare neutral, because it is not used, thus we are back to the basic model of Kocherlakota (1996); (ii) for higher $r$ storage happens only in autarky, therefore, consumption dispersion increases and welfare decreases, but storage does not matter otherwise; (iii) for yet higher $r$, hidden storage matters in equilibrium in the basic model, public storage is sometimes positive and stochastic, and many consumption values occur (this third case only occurs under certain conditions); (iv) for yet higher $r$, public storage is positive
and constant, and only two consumption levels occur; (v) if $r = 1/\beta - 1$, public storage is positive and constant, perfect risk sharing occurs, and welfare goes up. Welfare goes up above some threshold return $\tilde{r} < 1/\beta - 1$ in the long run. However, there are short-run costs to accumulation assets, so only above a higher threshold access to storage improves welfare if we take the transition into account as well. However, access to private storage given, public asset accumulation always reduces consumption dispersion and improves welfare.

The literature on incomplete markets either exogenously restricts asset trade, most prominently by allowing only a risk-free bond to be traded (Huggett, 1993; Aiyagari, 1994), or considers a deep friction that limits risk sharing endogenously. The literature has focused on two such frictions, namely limited commitment and private information. This paper merges these two strands of the incomplete markets literature by allowing for state-contingent trading subject to a deep friction and a self-insurance opportunity at the same time. This has been done when the deep friction is hidden income or effort (Allen, 1985; Cole and Kocherlakota, 2001; Ábrahám, Koehne, and Pavoni, 2011), but not in the case where the deep friction is lack of commitment, to our knowledge.

Comparing our model with limited commitment and storage to models with hidden income or effort and storage points to remarkable differences. In our model, if the same storage technology is available to the planner and the agents, storage is used in equilibrium and welfare improves if its return is sufficiently high. In contrast, in models with hidden income/effort, public storage is never used and the presence of the hidden storage opportunity reduces welfare. This is because in our model storage by the planner both improves insurance and relaxes the incentive problem, by relaxing future participation constraints; while in the hidden income/effort context aggregate asset accumulation makes incentive provision more expensive.

Our model could be applied in several economic contexts. The model predicts that risk sharing among households in villages can be improved by a public grain storage facility. Cooperation among partners in a law firm, for example, should be facilitated by common assets that someone quitting the partnership has no access to. Our model also provides a rationale for marriage contracts to specify that some commonly held assets are lost by the spouse who files for divorce. Finally, supranational organizations may help international risk sharing by simply having a jointly held stock of assets. The European Stability Mechanism may serve this purpose. Future work should study the quantitative implications of storage using some of these applications.
References


