Strategy-proof Stochastic Assignment

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Abstract

We study strategy-proof assignment mechanisms where the agents are asked to reveal preference rankings over the available objects. A stochastic mechanism returns lotteries over deterministic assignments, and we compare mechanisms according to first order stochastic dominance.

We show that a strategy-proof non-wasteful mechanism cannot be dominated by a strategy-proof mechanism. Secondly, if a wasteful mechanism can be dominated, the strategy-proof improvement must involve assigning more objects. We find that the widely used Random Priority mechanism (i.e., random serial dictatorship), and the recently adopted school choice mechanism RDA (Random tie-breaking followed by Deferred Acceptance) are not on the efficient frontier of strategy-proof mechanisms, by explicitly constructing mechanisms that improve upon them.

Keywords: random assignment, strategy-proofness, priority based assignment, ordinal efficiency, non-wastefulness, school choice. (JEL: C78, D61, D63.)


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1 Introduction

Consider a simple assignment problem, where each individual has a preference ranking over the indivisible objects and can consume at most one object. Many applications such as school choice, course assignment at universities, or office allocation typically rule out monetary transfers, and rely on agents’ preference rankings over the objects. Three properties are crucial for an assignment procedure: efficiency, strategy-proofness, and fairness. Given the ordinal nature of preference elicitation, Pareto optimality and strategy-proofness are natural notions of efficiency and incentive compatibility, respectively. Indivisibilities make it impossible to treat the agents equally, and thus challenges most notions of fairness in the ex post assignment. However, one can achieve ex-ante fairness of the mechanism by introducing randomization. For example, the random priority mechanism orders the agents randomly, and lets them pick their objects in this order. So while incentive constraints restrict what’s available to the designer, randomization enriches the set of feasible mechanisms.

This paper is about the efficient frontier of strategy-proof mechanisms, i.e., strategy-proof mechanisms which are not Pareto dominated by some other strategy-proof mechanism. First, we find that a non-wasteful strategy-proof mechanism cannot be dominated by another strategy-proof mechanism. Secondly, if a strategy-proof mechanism is dominated by another, such Pareto improvement cannot be achieved by merely re-allocating the objects, but must involve assigning more objects. And as two important applications, we show that the widely used random priority mechanism and the randomized deferred acceptance mechanism admit strategy-proof improvement. Hence, we are able to give explicit strategy-proof mechanisms which dominate them.

The multi-dimensional nature of the preferences makes it very difficult to characterize the incentive constraints and the strategy-proof mechanisms. While there is a number of strategy-proof mechanisms popular in practice, and widely studied in the literature, the class of strategy-proof mechanisms is not well understood. We find that non-wastefulness is crucial for the efficient frontier of this class. A deterministic assignment is called non-wasteful, if whenever an object is not assigned, it cannot be that some agent would rather take that object instead of what she received. We extend this notion to stochastic assignments in a natural way: if the total probability share of an
object is less than 1 at an assignment, no agent must be preferring to have more of this object instead of some other objects he received. The reshuffling lemma implies that a non-wasteful assignment cannot be Pareto dominated. Combining this lemma with incentive constraints, we conclude that if a strategy-proof mechanism is non-wasteful, then it is not dominated by a strategy-proof mechanism. Although most deterministic mechanisms used in practice are non-wasteful, we observe that randomizations over such mechanisms might be wasteful. For instance the widely used random priority mechanism (also known as the random serial dictatorship) is wasteful. Thus, a natural question is whether this mechanism can be improved upon without violating the incentive constraints. We show that if a mechanism is dominated within the strategy-proof class, then the improvement has to involve assigning (in ex ante probability) more objects. On the other hand, we find that being on the efficient frontier does not imply being non-wasteful. For example, any strategy-proof mechanism which dominates the random priority mechanism must be wasteful. Let us now briefly discuss our two main applications as we put our results in the context of the related literature.

Random Priority

One popular solution to our assignment problem is to order the agents from first to last, and let them choose, following this priority order, their favorite object from the pool of remaining objects. This solution, serial dictatorship\(^1\), is strategy-proof in the sense that we can implement it in dominant strategies by asking the agents to reveal their preference rankings over the objects and run the procedure on their behalf. Moreover, the outcome is Pareto optimal. On the other hand, this mechanism seems rather unfair. Indivisibilities make it impossible to treat the agents equally. Randomization, on the other hand, allows “symmetric treatment” of the agents in an ex-ante sense. The random priority mechanism (also called the random serial dictatorship\(^2\)) draws uniformly from all possible orders over agents, and runs the above procedure according to the chosen order. This exogenous randomization over strategy-proof mechanisms preserves strategy-proofness. Moreover, once the lottery realizes, the outcome will be Pareto efficient. However, the mechanism is actually stochastic. Each individual is re-

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1Sonnenschein and Satterthwaite (1981).
ceiving a lottery over the available objects. Bogomolnaia and Moulin (2001) make the critical observation that this stochastic assignment may be *ordinally inefficient*. That is, there may be a re-allocation of the assignment probabilities so that each agent receives a better lottery in the sense of first order stochastic domination. On the other hand, they also show that there is no strategy-proof mechanism which ensures equal treatment of equals and ordinal efficiency. It has been an open question (see, e.g., Zhou, 1990) whether the random priority mechanism is on the efficient frontier of the strategy-proof mechanisms. We find that it is not.

*Randomized Deferred Acceptance*

A question of the same nature recently emerged from another practical market design issue. In a typical school admissions program, each school has an exogenous priority ranking over the students. The education authority would like to match the students to schools *respecting these priorities* in the sense that if a student prefers a school $x$ to her own, it must be that only students of equal or higher priority for $x$ are assigned to school $x$.\(^3\) When these priority rankings are strict, the deferred acceptance mechanism of Gale and Shapley (1962) gives the student-optimal matching among those assignments which respect the priorities. Moreover, the mechanism is strategy-proof. (Dubins and Freedman, 1981; Roth, 1982.) However, priorities are typically *coarse*, i.e., large groups of students are considered to have equal priority. As in the random priority mechanism (where each school has one seat, and all students have equal priority at each school), we can *randomly pick* an order over the students, and use this order to break the ties in the priorities before using the deferred acceptance mechanism. But the *arbitrariness* of the tie-breaking rule might lead to inefficiency even at the *ex post* stage. Some tie-breaking rules suffer from this welfare loss, and this stochastic mechanism, which we call *randomized deferred acceptance (RDA)*, puts positive probability on each tie-breaking rule. Once the preferences are revealed, Erdil and Ergin (2008) describe how to find a “good tie-breaking rule.” That is one which would not lead to such ex post inefficiency. But they also show that any mechanism which always returns a student

\(^3\)For example, if students living within the walk-zone of the school have higher priority than those who live outside, then a student who live within a school $x$’s walk-zone should never be left envying a student $j$ who lives outside the walk-zone of $x$ and is assigned a seat at $x$. 

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optimal assignment fails to be strategy-proof. Abdulkadiroğlu, Pathak, and Roth (2009) show that there is no strategy-proof mechanism which Pareto dominates the deferred acceptance mechanism with strict priorities. Hence, RDA is a uniform randomization over undominated strategy-proof mechanisms. Yet, we are able to construct explicitly a strategy-proof mechanism that dominates the RDA.

2 Deterministic Assignments

Let $N$ denote a finite set of agents, and $X$ a finite set of distinct objects. There are $q_x \geq 1$ copies of type $x$ for $x \in X$. For agent $i$, the object that can be interpreted as his outside option (or staying unassigned) is also denoted by $i$. Each agent has a strict preference ranking $R_i$ over $X_i := X \cup \{i\}$. That is, $R_i$ is a complete, transitive and antisymmetric relation over $X_i$, with $P_i$ denoting its asymmetric part. An object $x$ is acceptable to $i$ if $xP_i i$. A preference profile is a vector $R = (R_i)_{i \in N}$ of individual preference relations. A deterministic assignment, also to be called a matching, is a function $\mu : N \to X \cup N$ satisfying: (i) $\forall i \in N : \mu(i) \in X_i$, and (ii) $\forall x \in X : |\mu^{-1}(x)| \leq q_x$. A deterministic mechanism $f$ is a function that associates a deterministic assignment to every preference profile.

Allocation on the basis of priorities

When the demand for an object exceeds the supply, a common method to decide who should get the objects is to appeal to an exogenous priority ranking over the agents. A priority structure is a profile $\succsim = (\succsim_x)_{x \in X}$ of weak orders (complete and transitive relations) on $N$, where for each $x \in X$, $\succsim_x$ ranks agents with respect to their priority for $x$. Let $\succsim_x$ denote the asymmetric part of $\succsim_x$. We say that $\succsim$ is strict if $\succsim_x$ is antisymmetric for each $x \in X$. A strict priority structure $\succsim'$ is called a strict resolution of $\succsim$ if $i \succsim_x' j \Rightarrow i \succsim_x j$ for all $i, j \in N$ and $x \in X$.

Given $\succsim$ and $R$, an assignment $\mu$ violates the priority of $i$ for $x$, if there is an agent $j$ such that $j$ is assigned to $x$ whereas $i$ both prefers $x$ to his assigned object and has strictly higher priority for it, i.e., $\mu(j) = x$, $xP_i \mu(i)$, and $i \succsim_j x$. An assignment $\mu$ respects priorities, or is said to be stable, if

(i) it does not violate any priorities,
(ii) $\mu(i)R_i i$ for any $i$, i.e., it is individually rational, and

(iii) there do not exist $i$ and $x$ such that $xP_i\mu(i)$ and $q_x > |\mu^{-1}(x)|$.

The last two properties together can be referred to as non-wastefulness, since they mean that neither the outside options nor the objects in $X$ are wasted. A mechanism is called non-wasteful if it always returns a non-wasteful assignment. An assignment $\mu'$ Pareto dominates another assignment $\mu$ if $\mu'(i)R_i\mu(i)$ for every $i \in N$, and $\mu'(j)P_j\mu(j)$ for some $j \in N$. An assignment is efficient if it is not Pareto dominated by another assignment; an efficient mechanism is one that returns an efficient assignment for every preference profile. An assignment $\mu$ is constrained efficient if it is stable, and is not Pareto dominated by any other stable assignment.

**Example 1** A commonly used priority based mechanism is the so-called *Serial Dictatorship (SD)*. The mechanism enumerates the agents as $i_1, \ldots, i_n$ and sets the priority ranking $i_1 \succ_x i_2 \succ_x \cdots \succ_x i_n$ for each object $x$. The agent $i_1$ receives her favorite object; $i_2$ gets her favorite from among the remaining objects; and so on. SD is the only $\succeq$-stable mechanism. It is Pareto efficient and strategy-proof.

For an arbitrary priority structure $\succ$, there may be a multitude of $\succeq$-stable mechanisms. When priorities are strict orders for each object, Gale and Shapley’s (1962) *Deferred Acceptance (DA)* mechanism returns the optimal stable outcome. We can implement the mechanism by simulating the following procedure: Given strict $\succ$ and $R$, at step 1, every $i$ applies to her favorite acceptable object. For each $x$, $q_x$ applicants who have the highest priority for $x$ are placed on the waiting list of $x$, and the others are rejected. At step $r$, the applicants who were rejected at step $r-1$ apply to their next best acceptable objects. For each $x$, the highest priority $q_x$ agents among the new applicants and those in the waiting list are placed on the new waiting list, and the rest are rejected. The algorithm terminates when every agent is either on a waiting list or has been rejected by every object that is acceptable to her. At the end, objects are assigned to agents on their waiting lists.

**Theorem** (Gale & Shapley, 1962) For any strict $\succeq$ and $R$, the outcome of the deferred acceptance algorithm is stable and is Pareto superior to any other stable assignment.
On the other hand, the outcome of the DA might be Pareto inefficient. Suppose, for instance, that there are three agents $1, 2, 3$, and three objects $a, b, c$ with a single copy each. Let

\[
\begin{array}{cccc|ccc}
\succsim_a & \succsim_b & \succsim_c & R_1 & R_2 & R_3 \\
1 & 2 & 3 & b & a & a \\
3 & 1 & 2 & a & b & c \\
2 & 3 & 1 & c & c & b \\
\end{array}
\]

Denoting by $DA^{\succsim}$ the deferred acceptance mechanism associated with priorities $\succsim$, the optimal stable (i.e., constrained efficient) assignment is $DA^{\succsim}(R) = (1, 2, 3)$. But it is not Pareto efficient as it is dominated by $(1, 2, 3)$.

Can we come up with a Pareto efficient mechanism which everyone likes better than the deferred acceptance? We say that a mechanism $g$ dominates $f$ if for every preference profile $R$, the assignment $g(R)$ weakly Pareto dominates $f(R)$ with strict domination for at least one preference profile.\(^4\) Kesten (2010) shows that there is no strategy-proof and Pareto efficient mechanism which dominates the deferred acceptance mechanism. Abdulkadiroğlu, Pathak, and Roth (2009) strengthen the impossibility result and show that DA is un-dominated within the class of strategy-proof mechanisms. One interpretation that this tension between efficiency and strategy-proofness hinges on stability, i.e., on the requirement that the outcome respects priorities. However, it turns out that the nature of the impossibility result is even more general: within the class of strategy-proof mechanisms, a deterministic mechanism is un-dominated if it is non-wasteful.

**Proposition 1** If a deterministic strategy-proof mechanism is non-wasteful, then it is not dominated by a strategy-proof mechanism.

Hence, the impossibility of strategy-proof improvement over the DA mechanism is due to its being non-wasteful.\(^5\) We should emphasize that so far we only dealt with deterministic mechanisms. Many applications, however, involve assignment mechanisms

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\(^4\)Holmstrom and Myerson (1983) argue that efficiency should be considered as a property of the mechanism. For the particular deterministic environment of school assignment, Abdulkadiroğlu, Pathak, and Roth (2009) study this notion of domination between mechanism.

\(^5\)The converse of the above proposition does not hold. That is, there exist deterministic, un-dominated mechanisms which are wasteful.
which involve some form of randomization. For example, in New York City High School Match, large classes of students have equal priorities for various schools, and the ties are broken randomly to ensure fairness between such students. Thus we turn now to \textit{stochastic mechanisms}.

\section*{3 Stochastic Assignments}

Randomization is a standard method to appeal to various notions of justice, equity, or fairness, especially in allocation of discrete and scarce resources. Consider, for example, the serial dictatorship mechanism we discussed earlier. If we pick randomly (and uniformly) the exogenous order in which the agents get to pick their objects, we turn our procedure into a \textit{stochastic} one which treats agents symmetrically. Moreover, strategy-proofness is preserved, and the \textit{ex post} outcome is Pareto efficient. But is this stochastic mechanism efficient? Could the agents agree on \textit{ex ante} efficiency improvements? If so, would any such agreement be incentive compatible?

Take, for another important application, the school admissions model, where schools typically have different priority rankings with large indifference classes. Now, the arbitrariness of the tie-breaking rule might lead to even \textit{ex post} inefficiency. For instance, when the ties are resolved alphabetically,

\begin{center}
\begin{tabular}{c|c|c|c}
\hline
$\succeq_a$ & $\succeq_b$ & $\succeq_c$ \\
\hline
1 & 2 & 3 \\
2, 3 & 1, 3 & 1, 2 \\
\hline
\end{tabular}
leads to
\begin{tabular}{c|c|c|c}
\hline
$\succeq'_a$ & $\succeq'_b$ & $\succeq'_c$ \\
\hline
1 & 2 & 3 \\
2 & 1 & 1 \\
3 & 3 & 2 \\
\hline
\end{tabular}
\end{center}

And if

\begin{center}
\begin{tabular}{c|c|c|c}
\hline
$R_1$ & $R_2$ & $R_3$ \\
\hline
c & a & a \\
a & b & c \\
b & c & b \\
\hline
\end{tabular}
then \quad \text{DA}$\succeq'(R)$ = \begin{pmatrix} 1 & 2 & 3 \\ a & b & c \end{pmatrix},
\end{center}

which is dominated by $\begin{pmatrix} 1 & 2 & 3 \\ c & b & a \end{pmatrix}$, which actually respects the original priorities $\succeq$. On the other hand, Erdil and Ergin (2008) show that no strategy-proof mechanism, deterministic or stochastic, ensures constrained efficient assignment.
Abdulkadiroğlu, Pathak, and Roth (2009), in their study of the New York City school choice system, emphasize that for strict priorities DA is not dominated by a strategy-proof mechanism. The mechanism used in NYC is of course not the DA with strict priorities. Instead it is a uniform randomization over the mechanisms DA$^\tau(\succ)$, where $\tau(\succ)$ varies over all strict priority structures derived from $\succ$ according to an order on students. It remains to be answered whether this stochastic mechanism is on the efficient frontier of strategy-proof mechanisms.

As we noted earlier, uniform randomization ensures *equal treatment of equal agents*, while preserving strategy-proofness. Do randomizations preserve the property of “being un-dominated” along with satisfying the other two attractive qualities? The first special case of this question which regards the random priority mechanism is attributed to Gale. The second special case arose from the recent work on redesign of the NYC school choice system. In order to make precise what we mean by the general question we pose, we will now turn to stochastic mechanisms.

Let $N, X$ and $R_i$ for $i \in N$ are as they were defined in the previous section. A **stochastic assignment** assigns each agent $i$ a lottery over the available objects (including her outside option). We will also require that scarcity constraints are met, so that at most $q_x$ copies of $x$ can be assigned. In other words, a stochastic assignment is a vector $\gamma \in \mathbb{R}^N_+ \times (|X| + |N|)$ such that

(a) $\gamma_{ix} \neq 0 \implies x \in X_i = X \cup \{i\}$ for every agent $i \in N$,

(b) $\sum_{i \in N} \gamma_{ix} \leq q_x$ for each object $x \in X$,

(c) $\sum_{x \in X_i} \gamma_{ix} = 1$ for each agent $i \in N$.

We know from Birkhoff-von Neumann theorem that any stochastic assignment is a convex combination of deterministic assignments, i.e., is of the form $\sum \alpha_\mu \mu$, where $\alpha_\mu \geq 0$, $\sum \alpha_\mu = 1$ and $\mu_{ix} \in \{0, 1\}$ for all $\mu, i, x$. Hence, any stochastic assignment can be interpreted as a running a lottery over deterministic assignments.

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6This special case is discussed in Gale (1987) and Zhou (1990). Specifically, Zhou (1990) writes that “it is not known whether random priority is optimal among symmetric, ex post Pareto optimal, strategy-proof mechanisms.” We will answer this question in Proposition 5.
Total amount of objects allocated by $\gamma$ will be called the **size of an assignment** $\gamma$, and is formally defined as $|\gamma| = \sum_{x \in X} \sum_{i \in N} \gamma_{ix}$.

Given a preference profile $R$, we say a stochastic assignment $\gamma$ is **wasteful** if there exist $j \in N$ and $y, z \in X_j$ such that

$$zP_jy, \quad \gamma_{jy} > 0, \quad \sum_{i \in N} \gamma_{iz} < 1.$$ 

If $\gamma$ is not wasteful, it is called **non-wasteful**. Note that non-wastefulness implies **individual rationality**, i.e., no agent is assigned an object she likes less than her outside option:

$$iP_i x \implies \gamma_{ix} = 0.$$ 

A **stochastic assignment mechanism** is a function which associates to each preference profile a stochastic assignment.

Given an agent $i$’s ordinal preference relation $R_i$ over $X$, one can easily extend it to a relation $\tilde{R}_i$ over lotteries over $X$ via the first-order stochastic domination (FOSD) so that if $\omega = \sum_{x \in X_i} \omega_x x$ and $\gamma = \sum_{x \in X_i} \gamma_x x$, then

$$\gamma \tilde{R}_i \omega \iff \text{for any } y \in X_i, \sum_{x : x \tilde{R}_i y} \gamma_x \geq \sum_{x : x \tilde{R}_i y} \omega_x.$$ 

$\gamma$ is said to FOSD $\omega$, denoted $\gamma \tilde{P}_i \omega$, if in addition to the above, the inequality is strict for at least one $y$.

Since this relation $\tilde{R}_i$ is anti-symmetric and transitive, one can speak of strategy-proofness of stochastic mechanisms. We will say that a stochastic mechanism $f$ is **strategy-proof** if for any $R$ and $R'_i$, one has

$$f_i(R) \tilde{R}_i f_i(R'_i, R_{-i}).$$

Note that this condition is equivalent to the requirement that for any vNM utility function $u_i$ representing $R_i$,

$$E_{u_i}[f_i(R)] \geq E_{u_i}[f(R'_i, R_{-i})].$$

An assignment $\mu$ **weakly Pareto dominates** another assignment $\nu$ if for each agent $i$, his assignment under $\mu$ weakly stochastically dominates his assignment under $\nu$, i.e.,
\(\mu(i)\tilde{R}_i\nu(i)\). If we also have \(\mu(j)\tilde{P}_j\nu(j)\) for some \(j\), we say that \(\mu\) Pareto dominates \(\nu\). If an assignment is not Pareto dominated by any other assignment, then it is called **ordinally efficient.**\(^7\) A mechanism \(f\) is dominated by another mechanism \(g\), if for any preference profile, the outcome of the first is weakly Pareto dominated by that of the latter, and domination is strict for at least one preference profile. In other words \(g_i(R)\tilde{R}_if_i(R)\) for each \(R\), and for each \(i\); and secondly \(g_j(R')\tilde{P}_jf_j(R')\) for some \(j\) and \(R'\).

A common method to allocate objects is random assignment. For example, when there are \(n\) agents and \(n\) objects, there are \(n!\) possible assignments. *Random Assignment* mechanism is nothing but a random variable over these assignments, with each assignment having equal probability. Obviously this mechanism satisfies equal treatment of equals. As it disregards preferences, it is trivially strategy-proof. On other hand, the **Random Priority (RP)** mechanism, also called the **Random Serial Dictatorship (RSD)**, is defined as a uniform lottery over all possible serial dictatorships. Therefore it always leads to a stochastic assignment that is a convex combination of efficient deterministic assignments.

**Remark 1** *RP dominates random assignment.*

To see this, note that for any agent \(i\), and any integer \(k \leq n\), her total share of her top \(k\) objects under the equal distribution is \(k/n\). Under the RP, with probability \(k/n\), she receives one of her top \(k\) choices. Hence, the RP weakly stochastically dominates the equal distribution. An obvious preference profile at which the domination is strict is where \(i\)’s top choice is bottom ranked by everyone else.

Note also that RA is not individually rational: each agent receives some fraction of each object even when they do not find them acceptable. If we modify RA by allowing free disposal, and thus recover individual rationality, we make the mechanism “more wasteful” than the RP.

A mechanism whose outcome is always deterministic and efficient is not dominated by another mechanism, because a deterministic and Pareto efficient assignment cannot be dominated by stochastic assignments either. On the other hand, a stochastic assignment may be dominated even when it is a convex combination of efficient deterministic assignments as illustrated in the following example from Bogomolnaia and

\(^7\)Also called *sd-efficient.*

**Example 2** Let there be four objects $a, b, c, d$, and four agents 1, 2, 3, 4 with preferences

<table>
<thead>
<tr>
<th>$R_1$</th>
<th>$R_2$</th>
<th>$R_3$</th>
<th>$R_4$</th>
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<tbody>
<tr>
<td>$a$</td>
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<td>$d$</td>
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In particular in our example, say the outcome of RP is denoted by $\mu$, and compare it with another assignment $\nu$ below:

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
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<td>4</td>
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<table>
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<tr>
<th>$\nu$</th>
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<td>0</td>
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<td>0</td>
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Thus, *ex post efficiency* does not imply *ordinal efficiency*.\(^8\)

In assignment problems with no exogenous priority orders, the random priority (i.e., random serial dictatorship), a strategy-proof mechanism, is not ordinally efficient. On the other hand, as Bogomolnaia and Moulin (2001) show, ordinal efficiency, equal treatment of equals and strategy-proofness are incompatible. Given the prominence of the random priority mechanism in practice, it is important to know whether this mechanism is on the efficient frontier of strategy-proof class.

In general, a **tie-breaking rule** is a function $\tau$ from the set of weak priority structures to the set of strict priority structures such that the image of any priority structure respects the first, i.e.,

\[ i \succeq_x j \implies i\tau(x) \succeq_j. \]

According to this definition, a tie-breaking rule is independent of agents’ preferences.

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\(^8\)The example also shows that randomly assigning objects (each allocation is equally likely), and then letting the agents trade is not ex ante efficient (first best), because doing so is equivalent to implementing core from random endowment, and the latter is equivalent to Random Serial Dictatorship as shown by Abdulkadiroğlu and Sönmez (1998).
Let $\mathcal{T}$ be a set of tie-breaking rules, and $\pi$ be a probability distribution over $\mathcal{T}$. Given a priority structure $\preceq$, let the mechanism $\text{DA}^\tau$ stand for running the DA algorithm after breaking the ties in priorities according to the tie-breaking rule $\tau$. Define the stochastic DA mechanism associated with $\mathcal{T}$ and $\pi$ as

$$\text{RDA}^{\mathcal{T},\pi} = \sum_{\tau \in \mathcal{T}} \pi_\tau \text{DA}^\tau.$$ 

It is not hard to see that $\text{DA}^{\mathcal{T},\pi}$ is strategy-proof.

A commonly used tie-breaking rule is the one which fixes a linear order $\tau$ on $N$, and breaks all the ties according to $\tau$. In other words, for all $i, j \in N$ and $x \in X$

$$i \sim_x j \quad \text{and} \quad i \tau j \implies i \tau (x) \tau j.$$

The mechanism used in New York City and Boston is $\text{DA}^{\mathcal{T},\pi}$, where $\mathcal{T}$ is the set of tie-breaking rules each of which follows a linear order on the set of agents $N$, and $\pi$ is the uniform distribution over $\mathcal{T}$. Thus, $|\mathcal{T}| = n!$, and $\pi(\tau) = \frac{1}{n!}$ for any $\tau \in \mathcal{T}$. In analogy with the random priority being denoted by RP (or the random serial dictatorship being denoted by RSD), we will use the notation RDA, so:

$$\text{RDA} = \frac{1}{n!} \sum_{\tau \in \mathcal{T}} \text{DA}^\tau.$$ 

Note that when the original priority structure is “trivial”, i.e., when all agents are of equal priority for all objects, RDA is equivalent to RP.

Proposition 1 implies that a non-wasteful, deterministic, strategy-proof mechanism cannot be dominated by another strategy-proof mechanism. Once the tie-breaking is resolved, RP (i.e., RSD) becomes a serial dictatorship, whereas the RDA becomes the deferred acceptance mechanism. That is, the mechanisms applied after the tie-breaking cannot be dominated. The following proposition provides a stochastic analogue of this result.

**Proposition 2** A non-wasteful, strategy-proof mechanism is not dominated by another strategy-proof mechanism.$^9$

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$^9$Note that there is no reference to equal treatment of equals. We conjecture that there is no strategy-proof mechanism which satisfies non-wastefulness and equal treatment of equals.
Nevertheless, the above proposition does not apply to the particular mechanisms we have highlighted as special cases of our domain of study. This is because a randomization over non-wasteful mechanisms can be wasteful.

**Remark 2**  
*RP* is wasteful (and, hence, so is RDA).

In order to see why, suppose there are four agents 1, 2, 3, 4, and three objects *a, b, c*. Let \( \succsim \) be a trivial priority structure in the sense that \( i \sim x j \) for all agents \( i, j \), and objects \( x \). Say preferences are such that

\[
\begin{array}{c|c|c|c}
R_1 & R_2 & R_3 & R_4 \\
\hline
a & c & a & c \\
b & b & & \\
\end{array}
\]

Now consider two particular tie-breaking rules: one that favors 1 over 2 over 3 over 4, and another that favors 4 over 3 over 2 over 1. After the first tie-breaking rule, agents 1 and 2 get *a* and *c*, respectively; agents 3 and 4 get nothing, and *b* is not assigned. Therefore *b* is left unassigned with positive probability, i.e., not all of *b* is assigned by the RP. On the other hand, after the second tie-breaking rule, agents 4, 3 and 2 get *c, a*, and *b*, respectively, and agent 1 gets nothing, and therefore the stochastic mechanism assigns to agent 1 less than a total of 1. Since not all of *b* is assigned, whereas 1 finds *b* acceptable, and the sum of her assignments is less than 1, *b* is wasted. ☐

What we see from the above remark is that our stochastic analogue of the deterministic non-domination result is inconclusive about the RP or the RDA.

Let us remember that with strict priorities, the DA is not an ex post efficient mechanism. When the priorities have ties (as they do in many school choice applications), randomization in tie-breaking is usually ex-ante. That is, a strict resolution \( \tau(\succsim) \) of ties is used with some probability \( \pi_\tau \).

Suppose that we have a family \( \{g^\tau\} \) of mechanisms such that

\[ g^\tau \text{ weakly dominates } DA^\tau, \]

for all \( \tau \), with strict domination for some \( \tau \). We know that \( g^\tau \) cannot be strategy-proof if \( g^\tau \) strictly dominates \( DA^\tau \). But perhaps, an “appropriate” randomization over various mechanisms, some of which are not strategy-proof, is strategy-proof.
Recall that with strict priorities, the deferred acceptance algorithm is non-wasteful. If an assignment weakly Pareto dominates $\text{DA}^\tau(R)$, then the reshuffling lemma\textsuperscript{10} applies, and

$$|g^\tau(R)| = |\text{DA}^\tau(R)|.$$  

Therefore, for any $R$

$$\left| \sum \pi^\tau g^\tau(R) \right| = \left| \sum \pi^\tau \text{DA}^\tau(R) \right|.$$

The following proposition will be helpful in answering the above question.

**Proposition 3** A strategy-proof mechanism is not dominated by another strategy-proof mechanism of equal size.

Thus, the proposition applies to the question raised: since the randomized mechanism $\sum \pi^\tau g^\tau$ is of equal size as the strategy-proof mechanism $\sum \pi^\tau \text{DA}^\tau$, the first cannot be strategy-proof. If there is going to be any strategy-proof improvement over the RP or the RDA, the improving mechanism has to be allocating more goods at some preference profile.

**Remark 3** Motivated by the fact that the outcome of RDA is not always student optimal stable (i.e., optimal among priority respecting assignments) Erdil and Ergin (2008) introduced stable improvement cycles, and a related algorithm to identify any such Pareto inefficiency, and improve upon it without violating priorities. Given a stable assignment, a stable improvement cycle is a cyclic trade of objects which preserves the stability of the assignment, and improves everyone involved in the cycle. Their mechanism first resolves the ties according to an arbitrary rule, and runs the DA mechanism to find a stable assignment to initiate the stable improvement process. Let $\text{SIC}^\tau$ be the Stable Improvement Cycles Mechanism which begins with the tie-breaking rule $\tau$. As deterministic mechanisms, we know by construction that $\text{SIC}^\tau$ Pareto dominates $\text{DA}^\tau$. If $\mathcal{T}$ is a set of tie-breaking rules and $\pi$ is a probability distribution over $\mathcal{T}$, we have

$$\sum_{\tau \in \mathcal{T}} \pi_\tau \text{SIC}^\tau \quad \text{dominates} \quad \sum_{\tau \in \mathcal{T}} \pi_\tau \text{DA}^\tau.$$  

\textsuperscript{10}The reshuffling lemma for deterministic assignments follows from Lemma 1 and says that if $\nu$ Pareto dominates a non-wasteful assignment $\mu$, the difference between the assignments is merely a reshuffling of the objects.
A stable improvement cycle reshuffles already assigned objects, therefore for any preference profile $R$, and tie-breaking rule $\tau \in \mathcal{T}$, the assignments $\text{DA}^\tau(R)$ and $\text{SIC}^\tau(R)$ assign the same amount of objects. Thus $\text{SIC}^{\mathcal{T},\pi}$ and $\text{DA}^{\mathcal{T},\pi}$ are of equal size, while the first dominates the latter. The second mechanism is strategy-proof, hence the first cannot be, by Proposition 3.

On the other hand, the proposition does not necessarily imply Erdil and Ergin’s result that when priority orders involve ties, no selection (deterministic or stochastic) from the constrained efficient (i.e., agent optimal stable) set is strategy-proof. This is because, not every such selection rule dominates $\text{DA}^{\mathcal{T},\pi}$ for some set $\mathcal{T}$ of tie-breaking rules, and a lottery $\pi$ over $\mathcal{T}$.

**Remark 4** In order to measure the welfare cost of stability (i.e., of priority constraints) and/or random tie-breaking, Abdulkadiroğlu, Pathak and Roth (2009) employ mechanisms that improve upon a particular DA mechanism via trading cycles. Since each trading cycle keeps constant the number of agents assigned some object, the size of the assignments are fixed along the improvement path. Therefore, by Proposition 3 such improvements cannot be achieved by a strategy-proof mechanism. Note that this result does not necessarily follow from Proposition 1.

So far, we have established that RDA is wasteful, and if we will get any strategy-proof improvement, it has to be increasing (ex ante) the amount of the objects assigned. The deferred acceptance with an arbitrary (but random, and thus deterministic) tie breaking is on the efficient frontier of all strategy-proof mechanisms as shown by Abdulkadiroğlu, Pathak, and Roth (2009) in their study of New York City High School Match. While one might suspect that the the actual mechanism in NYC, which is DA with random tie-breaking is also on the efficient frontier of strategy-proof assignment mechanisms. However, we find that

**Proposition 4** *RDA is dominated within the class of strategy-proof mechanisms.*

As we mentioned earlier, whether the RP is optimal within the strategy-proof class has been an open question dating back at least as far as to Zhou (1990). In a logically independent result, he shows that when the message spaces include cardinal preferences, anonymity, Pareto efficiency and strategy-proofness are incompatible as conjectured by
Gale. For $n \leq 3$, Bogomolnaia and Moulin (2001) show that the RP is ordinally efficient. Therefore it is un-dominated. For $n \geq 4$, we find a different conclusion:

**Proposition 5** If $n \geq 4$, then the Random Serial Dictatorship can be dominated by a strategy-proof mechanism.

*Sketch of proof.* Say there are at least three distinct objects $a, b, c$, and four agents 1, 2, 3, 4. If the preferences of the agents are

\[
\begin{array}{c|c|c|c}
R_1 & R_2 & R_3 & R_4 \\
\hline
a & c & c & c \\
b & a & b & \\
\end{array}
\]

the assignment under the RP looks like

\[
\begin{array}{c|ccc}
RP(R) & a & b & c \\
\hline
1 & 3/4 & 1/6 & \\
2 & 1/4 & 1/3 & \\
3 & 5/8 & 1/3 & \\
4 & 1/3 & \\
\end{array}
\]

Not only $b$ is wasted here, but crucially if 1 were to change her preference ranking to $bP_1'a$, the object $a$ would be wasted due to symmetry of $R_{-1}$. This will allow us to “recycle” some of the wasted $b$ back to agent 1, without skewing her incentives. Namely, we will also be able to recycle an equal amount of $a$ back to agent 1 when her preference ranking is $bP_1'a$, so that she does not want to misreport her preferences to increase her total consumption.

By Proposition 3, any strategy-proof mechanism that dominates RP is necessarily *less wasteful* than the RP, i.e., for some realization of preferences, such a mechanism assigns more objects than the RP. Can we recycle the wasted objects successively to finally reach a non-wasteful assignment? The following proposition says no, if we do not want to skew incentives.

**Proposition 6** If a strategy-proof $g$ dominates the RP, then $g$ is wasteful.

Thus, if we take the RP as a benchmark to improve upon, within the class of strategy-proof mechanisms, it is impossible to avoid ex ante waste.
Corollary 1  The converse of Proposition 2 does not hold. In other words, being undominated in the strategy-proof class does not imply being non-wasteful.

The corollary is an immediate consequence of the preceding proposition, because the if we look at strategy-proof mechanisms which dominate RP, there must be one which is un-dominated, and hence is on the efficient frontier of strategy-proof mechanisms. Now the proposition implies that it is necessarily wasteful.

4 Conclusion

To be written.

A Appendix

Lemma 1 If an assignment \( \nu \) dominates a non-wasteful assignment \( \mu \), and \( \mu(i) = i \), then \( \nu(i) = i \).

Proof of Lemma 1. Note that for any \( x \in X \), not more agents are assigned to \( x \) under \( \nu \) than under \( \mu \), for otherwise, there would be a copy of \( x \) not assigned to anyone under \( \mu \), i.e., \( |\mu^{-1}(x)| < q_x \), and there would be an agent \( i \) with \( \mu(i) \neq \nu(i) = x \). Since \( \nu \) dominates \( \mu \), and \( i \)'s assignment changes from \( \mu \) to \( \nu \), we necessarily have \( xP_i \mu(i) \). This combined with \( |\mu^{-1}(x)| < q_x \) gives a contradiction with non-wastefulness of \( \mu \).

Since for any \( x \in X \) the number of agents assigned to \( x \) cannot increase from \( \mu \) to \( \nu \), the number of agents assigned an object in \( X \) cannot increase from \( \mu \) to \( \nu \):

\[
|\mu^{-1}(X)| \geq |\nu^{-1}(X)|
\]

Secondly, individual rationality of \( \mu \) and the fact that \( \nu \) dominates \( \mu \) imply that whoever is assigned an object under \( \mu \) must be assigned an object under \( \nu \), implying

\[
|\mu^{-1}(X)| \leq |\nu^{-1}(X)|,
\]

and therefore the same set of agents are assigned objects in \( X \) under both \( \mu \) and \( \nu \). In particular, if \( \mu(i) = i \), then \( \nu(i) = i \).

\[\text{Note that the set of mechanisms which dominate RP is closed and bounded in a natural topology.}\]
Proof of Proposition 1. Suppose that a strategy-proof, non-wasteful mechanism \( f \) is dominated by a strategy-proof mechanism \( g \). Then there must be a preference profile \( R \) such that

\[
y = g_i(R)P_i f_i(R) = x.
\]

Let \( R' = (R'_i, R_{-i}) \), where \( R'_i \) is specified such that only \( y \) is acceptable to \( i \). \( f \) is strategy-proof, so \( f_i(R)R_i f_i(R') \). But according to \( R' \), only \( y \) is acceptable, and since \( f \) is individually rational, we have \( f_i(R') \in \{y, i\} \). However \( f_i(R') \) cannot be \( y \) since \( yP_i x = f_i(R)R_i f_i(R') \), and hence \( f_i(R') = i \). Since \( g \) dominates \( f \), by Lemma 1 we have \( g_i(R') = i \).

When \( R' = (R'_i, R_{-i}) \) is the actual preference profile, \( i \) could announce \( R_i \) and get \( g_i(R) = y \) which she prefers to \( i \) at \( R' \). Hence \( g \) cannot be strategy-proof.

Lemma 2. Given \( R_i \) over \( X \cup \{i\} \), let \( \gamma \) and \( \omega \) be two vectors in \( \Delta(X \cup \{i\}) \) such that \( \gamma \) first order stochastically dominates \( \omega \).

Say there are \( m \) elements of \( X \) acceptable to agent \( i \) and enumerate them such that \( x_1 \succsim_i x_2 \succsim_i \cdots \succsim_i x_m \), where \( x_1, \ldots, x_m \) are the only objects acceptable to \( i \). If the stochastic assignments above are written explicitly as

\[
\sum_{j=1}^{m} \gamma_j x_j \quad \text{and} \quad \sum_{j=1}^{m} \omega_j x_j,
\]

then we get the following inequalities

\[
\begin{align*}
\gamma_1 &\geq \omega_1 \\
\gamma_1 + \gamma_2 &\geq \omega_1 + \omega_2 \\
\vdots \\
\gamma_1 + \gamma_2 + \cdots + \gamma_m &\geq \omega_1 + \omega_2 + \cdots + \omega_m
\end{align*}
\]

with at least one of the inequalities strict.

Let \( k \) be the smallest positive integer for which \( \gamma_k > \omega_k \). If \( k > 1 \), then \( \gamma_1 = \omega_1 \), since we already know that \( \gamma_1 \geq \omega_1 \). But then, we have \( \gamma_2 \geq \omega_2 \) from the second inequality.
above, and if \( k > 2 \), then \( \gamma_2 = \omega_2 \). Arguing inductively in the same fashion, we conclude that \( \gamma_j = \omega_j \) for \( j < k \). \( \square \)

For ease of exposition, we will say that an agent \( i \) consumes an amount \( a \) of object \( x \) under assignment \( \gamma \) if \( \gamma_{ix} = a \).

**Lemma 3**

(i) If \( |\omega| = |\gamma| \), and \( \gamma \) dominates \( \omega \), then each agent consumes as much under assignment \( \omega \) as under \( \gamma \), i.e., for each \( i \)

\[
\sum_{x:xP_i} \gamma_{ix} = \sum_{x:xP_i} \omega_{ix}.
\]

(ii) If \( \omega \) is non-wasteful, and \( \gamma \) dominates \( \omega \), then each agent consumes as much under assignment \( \omega \) as under \( \gamma \), i.e., for each \( i \)

\[
\sum_{x:xP_i} \gamma_{ix} = \sum_{x:xP_i} \omega_{ix}.
\]

**Proof of Lemma 3.** Part (i): Since \( \gamma \) dominates \( \omega \), each agent’s consumption under \( \gamma \) is at least as much as is under \( \omega \):

\[
\sum_{x:xP_i} \gamma_{ix} \geq \sum_{x:xP_i} \omega_{ix} \quad \text{for each } i.
\]

Adding up across \( i \):

\[
|\gamma| = \sum_i \sum_{x:xP_i} \gamma_{ix} \geq \sum_i \sum_{x:xP_i} \omega_{ix} = |\omega|.
\]

Since we assumed that \( |\gamma| = |\omega| \), each weak inequality (1) must indeed be equality, rendering the desired result. \( \diamond \)

Part (ii): Let \( N' \) be the set of agents whose match improved from \( \omega \) to \( \gamma \), and let \( X' \) be the set of objects such that there is some agent in \( N' \) whose share of such an object increased from \( \omega \) to \( \gamma \).

First, let us note that no object is consumed less at \( \omega \) than at \( \gamma \), for otherwise, there would be an object \( x \) assigned less than \( q_x \) at \( \omega \), and an agent \( i \) who gets more of \( x \) at \( \gamma \) than \( \omega \). But since \( \gamma \) Pareto dominates \( \omega \), \( i \) prefers her match at \( \gamma \) to that at \( \omega \), and in particular demands more of \( x \) at \( \omega \), whereas not all of \( x \) is assigned under \( \mu \). This
would imply that $x$ is wasted at $\omega$, a contradiction with $\omega$ being non-wasteful. Hence for each $x \in X$

$$\sum_{i \in N} \omega_{ix} \geq \sum_{i \in N} \gamma_{ix}. \quad (2)$$

Adding up across $x$, we get

$$\sum_{x \in X} \sum_{i \in N} \omega_{ix} \geq \sum_{x \in X} \sum_{i \in N} \gamma_{ix}. \quad (3)$$

Secondly, since each agent is weakly better off at $\gamma$, and $\omega$ is individually rational, each agent consumes at least as much at $\gamma$ as she did at $\omega$, i.e., for each $i \in N$:

$$\sum_{x} \omega_{ix} = \sum_{x : x \succeq_p i} \omega_{ix} \leq \sum_{x : x \succeq_p i} \gamma_{ix} = \sum_{x} \gamma_{ix}, \quad (4)$$

which, adding up across $i$, lead to

$$\sum_{i \in N} \sum_{x \in X} \omega_{ix} \leq \sum_{i \in N} \sum_{x \in X} \gamma_{ix}. \quad (5)$$

Hence the inequalities (3) and (5) both must be equality. This, in turn, implies that for each $i$, the weak inequality (4) is indeed equality, rendering the desired conclusion. \(\square\)

**Proof of Proposition 2.** Let $f$ be non-wasteful and strategy-proof, and suppose, for a contradiction, that a strategy-proof mechanism $g$ dominates $f$. Then there must be an agent $i$ and a preference profile $R$ such that $g_i(R) \bar{P}_i f_i(R)$.

Enumerate elements of $X$ so that $x_1 P_i x_2 P_i \cdots P_i x_m$, where $x_1, \ldots, x_m$ are the only objects acceptable to $i$. If the stochastic assignments above are written explicitly as

$$g_i(R) = \sum_{j=1}^{m} \gamma_j x_j \quad \text{and} \quad f_i(R) = \sum_{j=1}^{m} \omega_j x_j,$$

then we know from Lemma 2 that there exists $k$ with $\gamma_k > \omega_k$ such that $\gamma_j = \omega_j$ for $j < k$.

Let the preference relation $R_i'$ be the linear order derived from $R_i$ by declaring everything less preferable than $x_k$ according to $R_i$ unacceptable in $R_i'$ and keeping the order the same for else\(^{12}\), i.e., $x_1 P_i' x_2 P_i' \cdots P_i' x_k$, and $i P_i' x$ if $x \neq x_j$ for some $j = 1, \ldots, k$.

\(^{12}\)If $R_i' = R_i$, then $k = m$, and $\sum_{j=1}^{m} \gamma_j > \sum_{j=1}^{m} \omega_j$, contradicting Lemma 3.
Strategy-proofness of $f$ implies that $f_i(R)\tilde{R}_i f_i(R_i, R_{-i})$. Since $f$ is individually rational, $f_i(R_i, R_{-i})$ does not involve any $x_j$ for $j > k$, and hence can be written explicitly as $\sum_{j=1}^{k} \omega_j x_j$, to get

$$\sum_{j=1}^{m} \omega_j x_j \tilde{R}_i \sum_{j=1}^{k} \omega_j x_j,$$

which implies

$$\omega_1 \geq \omega'_1$$
$$\omega_1 + \omega_2 \geq \omega'_1 + \omega'_2$$
$$\vdots$$
$$\omega_1 + \omega_2 + \cdots + \omega_k \geq \omega'_1 + \omega'_2 + \cdots + \omega'_k$$

On the other hand, strategy-proofness of $g$ implies that $g(R')\tilde{R}'_i g(R)$. Writing $g_i(R_i, R_{-i})$ explicitly as $\sum_{j=1}^{m} \gamma'_j x_j$, we get

$$\sum_{j=1}^{m} \gamma'_j x_j \tilde{R}'_i \sum_{j=1}^{m} \gamma_j x_j,$$

which implies

$$\gamma'_1 \geq \gamma_1$$
$$\gamma'_1 + \gamma'_2 \geq \gamma_1 + \gamma_2$$
$$\vdots$$
$$\gamma'_1 + \gamma'_2 + \cdots + \gamma'_k \geq \gamma_1 + \gamma_2 + \cdots + \gamma_k$$

Combining the sets of inequalities above, we reach the following:

$$\gamma'_1 \geq \gamma_1 \geq \omega_1 \geq \omega'_1$$
$$\gamma'_1 + \gamma'_2 \geq \gamma_1 + \gamma_2 \geq \omega_1 + \omega_2 \geq \omega'_1 + \omega'_2$$
$$\vdots$$
$$\gamma'_1 + \cdots + \gamma'_k \geq \gamma_1 + \cdots + \gamma_k \geq \omega_1 + \cdots + \omega_k \geq \omega'_1 + \cdots + \omega'_k$$

$g$ dominates $f$, so it must be that $g(R'_i, R_{-i})$ weakly Pareto dominates $f(R'_i, R_{-i})$. Since $f$ is non-wasteful, applying Part (ii) of Lemma 3, we get

$$\omega'_1 + \omega'_2 + \cdots + \omega'_k = \gamma'_1 + \gamma'_2 + \cdots + \gamma'_k.$$ (6)
This equation combined with the last inequality above imply that
\[ \gamma_1 + \gamma_2 + \cdots + \gamma_k = \omega_1 + \omega_2 + \cdots + \omega_k. \]
Since \( \gamma_j = \omega_j \) for \( j = 1, \ldots, k - 1 \), the above equation implies that we also have \( \gamma_k = \omega_k \), which contradicts with the earlier choice of \( k \) such that \( \gamma_k > \omega_k \). Hence \( g \) cannot be strategy-proof.

**Proof of Proposition 3.** Proof of Proposition 2 above can simply be modified to refer to the first part of Lemma 3 instead of its second part.

**Proof of Proposition 4.** In order to show that \( f = \frac{1}{n!} \sum_{\tau \in T} DA^\tau \) is not un-dominated, we will explicitly construct a strategy-proof mechanism \( g \) which strictly dominates \( f \). That is,

- everyone is weakly better off: \( g_i(R) \tilde{R} \tilde{f}_i(R) \) for all \( i, \preceq \) and \( R \),
- someone is strictly better off: \( g_j(R) \tilde{P} \tilde{f}_j(R) \) for some \( j, \preceq \), and \( R \), and
- strategy-proofness: \( g_i(R_i, R_{-i}) \tilde{R} g_i(R_i', R_{-i}) \) for all \( \preceq, i, R, \) and \( R' \).

Now suppose that \( \preceq, R \) and \( R' \) are

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<th>( \preceq_b )</th>
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Note that for any preference ranking \( Q_1 \), if \( \hat{Q}_1 \) is its restriction to \( \{a, b\} \), then,

\[ f_1(Q_1, R_{-1}) = f_1(\hat{Q}_1, R_{-1}). \]

In other words, under \( f \), 1’s preferences regarding \( c \) and \( d \) are irrelevant at \( R_{-1} \).

For any \( Q_{-1} \neq R_{-1} \), define \( g(Q) = f(Q) \). At \( R \) and \( R' = (R'_1, R_{-1}) \), define \( g \) as

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<th>( g(R') )</th>
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Compared with $f$ below, $g$ clearly dominates $f$:

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<th>$f(R)$</th>
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In order to show that $g$ is ex post stable, we need to show that $g(R)$ can decomposed into stable deterministic assignments. For any preference profile $Q$ different from $R$ and $R'$, $g$ is identical with $f$, which is defined as a randomization over deterministic stable mechanisms. Hence, it suffices to show that $g(R)$ and $g(R')$ are convex combinations of stable assignments. It is not hard to verify that for any $\lambda \in [0, 1/3]$

$$g(R) = \frac{1}{30} \begin{pmatrix} 1 & b \\ 2 & a \\ 3 & c \\ 4 & - \\ 5 & d \end{pmatrix} + \lambda \begin{pmatrix} 1 & a \\ 2 & - \\ 3 & c \\ 4 & b \\ 5 & - \end{pmatrix} + \left( \frac{1}{3} - \lambda \right) \begin{pmatrix} 1 & a \\ 2 & - \\ 3 & c \\ 4 & d \\ 5 & - \end{pmatrix} +$$

$$\frac{2}{15} \begin{pmatrix} 1 & b \\ 2 & a \\ 3 & c \\ 4 & d \\ 5 & - \end{pmatrix} + \left( \frac{7}{15} - \lambda \right) \begin{pmatrix} 1 & a \\ 2 & c \\ 3 & - \\ 4 & b \\ 5 & - \end{pmatrix} + \left( \lambda + \frac{1}{30} \right) \begin{pmatrix} 1 & a \\ 2 & c \\ 3 & - \\ 4 & d \\ 5 & - \end{pmatrix},$$

where each assignment respects priorities. A similar decomposition holds for $g(R')$ via symmetry.

What remains to show is that $g$ is indeed strategy-proof. We need to check the incentive compatibility constraints for 1 only, because for other agents $f$ is identical with $g$, and $f$ is strategy-proof. Again at preferences other than $R_{-1}$, the mechanism is identical to $f$, therefore it is sufficient to look at $(Q_1, R_{-1})$. At such profiles, 1’s preferences over $c$ and $d$ do not matter. And finally, as can be seen by comparing $g(R)$ and $g(R')$, he does not prefer to switch the ranks of $a$ and $b$. Thus, $g$ is strategy-proof.

$\square$
Proof of Proposition 5. Recall the environment with four agents 1, 2, 3, 4, and three distinct objects \(a, b, c\). Say \(R_{-1}\) is as

\[
\begin{array}{c|c|c|c}
R_2 & R_3 & R_4 \\
\hline
\cdot & c & c \\
a & \cdot & b \\
\end{array}
\]

As for the preferences of the agent 1, let

\[
\mathcal{R}_1^a = \{ R_1 \mid aR_1b \} \quad \text{and} \quad \mathcal{R}_1^b = \{ R_1 \mid bR_1a \}.
\]

Define the mechanism \(g\) to be

\[
g(Q) = \text{RP}(Q) \quad \text{if} \quad Q_{-1} \neq R_{-1}
\]

and

\[
g_i(Q) = f_i(Q) \quad \text{for all} \quad i \neq 1 \quad \text{and} \quad \text{for all} \quad Q.
\]

Agent 1’s consumption however is “topped up” whenever others’ preferences are \(R_{-1}\). For an \(\epsilon > 0\) small enough, we define

\[
g_1(R_1, R_{-1}) = \text{RP}(R) + \epsilon b \quad \text{if} \quad R_1 \in \mathcal{R}_1^a,
\]

and

\[
g_1(R_1, R_{-1}) = \text{RP}(R) + \epsilon a \quad \text{if} \quad R_1 \in \mathcal{R}_1^b.
\]

Incentive compatibility of \(g\) is verified as done in the proof of the previous proposition. What remains to be shown is that the RP indeed wastes \(b\) at \((R_1, R_{-1})\) for all \(R_1 \in \mathcal{R}_1^a\), and wastes \(a\) at \((R_1, R_{-1})\) for all \(R_1 \in \mathcal{R}_1^b\).

Note that, if the serial dictatorship realizes as 2 \(\succ\) 1 \(\succ\) 3 \(\succ\) 4, for preferences \((R_1, R_{-1})\) with \(R_1 \in \mathcal{R}_1^a\), the object \(b\) is not consumed. This is because, agent 2 gets \(c\), then agent 1 gets her favorite object from \(\{a, b\}\), which is not \(b\), as \(aP_1b\). And \(b\) is not acceptable to anyone else. Thus \(b\) is wasted with probability at least \(1/24\).

Likewise, the object \(a\) is not consumed when preferences are \((R_1, R_{-1})\) with \(R_1 \in \mathcal{R}_1^b\), and the serial dictatorship realizes as 3 \(\succ\) 1 \(\succ\) 2 \(\succ\) 4. Therefore \(b\) is wasted with probability at least \(1/24\).
Agent 1’s total consumption at \((R_1, R_{-1})\) is \(11/12\) for all \(R_1 \in R_1^a \cup R_1^b\). Thus choosing \(\epsilon = 1/24\) above is feasible, and the agent 1 is strictly better off under the mechanism \(g\) whenever she finds both objects \(a\) and \(b\) acceptable.

Proof of Proposition 6. Let \(f\) stand for the RP. First, note that if an object \(x\) is ranked first by someone at \(R\), then that object cannot be wasted at \(f(R)\), i.e., \(\sum_i f_i(x)(R) = 1\). This is because, for any realization of the tie-breaking lottery, we implement a serial dictatorship, which is Pareto optimal, and therefore \(x\) is assigned to someone. Thus, if at \(R\), each object is top-ranked by someone, then \(f(R)\) is a non-wasteful assignment.

Let \(R, R_1',\) and \(\hat{R}_2\) be given as

\[
\begin{array}{cccc|ccc}
R_1 & R_2 & R_3 & R_4 & R_1' & \hat{R}_2 \\
a & c & a & c & b & b \\
b & b & & & a & c \\
\end{array}
\]

\(f\) is wasteful at \(R\) since \(f(R)\) assigns the objects as

1: \(\frac{1}{2}a + \frac{5}{12}b\)

2: \(\frac{1}{2}c + \frac{5}{12}b\)

3: \(\frac{1}{2}a\)

4: \(\frac{1}{2}c\)

and it wastes \(b\).

On the other hand, note that \(f\) is not wasteful at \(R' = (R_1', R_{-1})\) nor at \(\hat{R} = (\hat{R}_2, R_{-2})\), because at both profiles, each object is top-ranked by someone. If \(g\) dominates \(f\), then \(g(R')\) and \(g(\hat{R})\) weakly dominate \(f(R')\) and \(f(\hat{R})\), respectively. Then, by Lemma 2, part (ii), we have

\[|g_i(R')| = |f_i(R')|\quad \text{and}\quad |g_i(\hat{R})| = |f_i(\hat{R})|,\]

\(R_1'\) and \(\hat{R}_2\) are derived from \(R_1\) and \(R_2\), respectively, by permuting acceptable objects, so strategy-proofness of \(f\) implies that

\[|f_1(R)| = |f_1(R')|\quad \text{and}\quad |f_2(R)| = |f_2(\hat{R})|,
\]

whereas strategy-proofness of \(g\) implies that

\[|g_1(R)| = |g_1(R')|\quad \text{and}\quad |g_2(R)| = |g_2(\hat{R})|.
\]
Combining the three sets of equalities above, we get
\[ |f_1(R)| = |g_1(R)| \quad \text{and} \quad |f_2(R)| = |g_2(R)|. \]

Since \( g \) dominates \( f \), we know that \( g(R) \) weakly dominates \( f(R) \), and in particular, \( |g_i(R)| \geq |f_i(R)| \) for each \( i \). Note that all agents get 1/2 of their top choices at \( f(R) \), and these objects, namely \( a \) and \( c \), are completely assigned. Therefore \( g(R) \) has to assign the same fractions of these top choices as did \( f(R) \):

\[ f_{ix}(R) = g_{ix}(R) \quad \text{for all } i = 1, 2, 3, 4 \quad \text{and} \quad x = a, c. \]

The object \( b \) is not acceptable to agents 3 and 4, implying that \( g_3(R) = f_3(R) \) and \( g_4(R) = f_4(R) \). On the other hand \( |f_1(R)| = |g_1(R)| \) and \( |f_2(R)| = |g_2(R)| \), and hence agents 1 and 2 consume the same amount of \( b \) under \( g(R) \) as they did under \( f(R) \). Therefore \( g(R) = f(R) \), and thus \( g \) is wasteful at \( R \).

Before proceeding further, we need introduce some notation. Given a preference ranking \( R_i \), denote by \( b(R_i) \) the least favorite acceptable object with respect to \( i \):

\[ b(R_i)P_i \quad \text{and} \quad xR_i b(R_i) \quad \text{for all } x \text{ such that } xP_i. \]

The length of a preference ranking \( R_i \in \mathcal{R}_i \) is defined to be

\[ |R_i| = |\{x \in X \mid xP_i\}|. \]

The length of a preference profile is

\[ |R| = \sum_{i \in N} |R_i|. \]

**Lemma 4** Suppose that a strategy-proof mechanism \( g \) dominates another strategy-proof mechanism \( f \). Let \( R \) be a shortest preference profile at which the allocation \( g(R) \) Pareto dominates \( f(R) \). If \( f_i(R) \neq g_i(R) \), then there exists a \( \delta > 0 \) such that for any acceptable permutation \( R'_i \) of \( R_i \),

- \( g_i(R'_i, R_{-i}) = f_i(R'_i, R_{-i}) + \delta b(R'_i) \),

- \( b(R'_i) \) is wasted by \( f \) at \( (R'_i, R_{-i}) \).
Proof of Lemma 4. We will first prove that \( g_i(R) = f_i(R) + \delta b(R_i) \) for some \( \delta \).

Suppose the contrary. Enumerating the objects acceptable to \( i \) as \( x_1 P_i x_2 P_i \cdots P_i x_m \), we have

\[
b(R_i) = x_m.
\]

Denote

\[g_i(R) = \gamma = \sum_{j=1}^{m} \gamma_j x_m \quad \text{and} \quad f_i(R) = \omega = \sum_{j=1}^{m} \omega_j x_m.\]

Let \( k \) be the smallest integer for which \( \gamma_k > \omega_k \). Then \( \omega_j = \gamma_j \) for \( j < k \). Since we assumed, for a contradiction, that \( g_i(R) \neq f_i(R) + \delta b(R_i) \) for any \( \delta \), we must have \( k < m \).

Now truncate \( R_i \) to get \( R'_i \) by declaring unacceptable any object less preferred than \( x_k \). In other words, \( R'_i \) is the preference order for which only \( x_1, \ldots, x_k \) are acceptable, and

\[x_1 P'_i x_2 P'_i \cdots P'_i x_k.\]

The fact that \( f \) is strategy-proof implies that

\[f_i(R'_i, R_{-i}) = \sum_{j=1}^{k} \omega_j x_j.\]

Likewise, strategy-proofness of \( g \) implies that

\[g_i(R'_i, R_{-i}) = \sum_{j=1}^{k} \gamma_j x_j.\]

But then agent \( i \) strictly prefers \( f(R'_i, R_{-i}) \) to \( f(R'_i, R_{-i}) \), and therefore \( g(R'_i, R_{-i}) \) Pareto dominates \( f(R'_i, R_{-i}) \). Hence we have a preference profile \( (R'_i, R_{-i}) \) at which \( g \) dominates \( f \). On the other hand, \( R'_i \) is shorter than \( R_i \), therefore \( (R'_i, R_{-i}) \) is shorter than \( R \), contradicting with the assumption that \( R \) was the shortest preference profile at which \( g \) dominates \( f \).

Thus we have \( g_i(R) = f_i(R) + \delta_i b(R_i) \) for some \( \delta_i > 0 \).

Now, if \( R'_i \) is an acceptable permutation of \( R_i \), strategy-proofness of \( g \) and \( f \) implies

\[|g_i(R'_i, R_{-i})| = |g_i(R)| \quad \text{and} \quad |f_i(R'_i, R_{-i})| = |f_i(R)|. \] (♣)

Since \( |f_i(R)| + \delta = |g_i(R)| \), we in particular get \( g_i(R'_i, R_{-i}) \neq f_i(R'_i, R_{-i}) \). Repeating the argument above, we conclude that

\[g_i(R'_i, R_{-i}) = f_i(R'_i, R_{-i}) + \delta'_i b_i(R'_i)\]

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By (♠), we have $\delta = \delta'$.

So if agent’s assignment is different under $g$, the only difference is that she has more of her bottom choice. Hence no one is consuming less of any of the objects they were assigned under $f$. Therefore she cannot be getting that extra bit from someone else, and $f$ must be wasting that bottom choice of hers. 

References


